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ANNOUNCEMENT

An agreement has been effected between the Johns Hopkins University and the American Mathematical Society for coöperative publication of the AMERICAN JOURNAL OF MATHEMATICS. The Journal will be enlarged so as to help to meet the rapidly growing demands for the reasonably prompt publication of memoirs. There is no intention of changing the policy.

Substantial theories have, of course, the first place. As far as possible illustrations of such theories should be included, as they facilitate enormously the comprehension of the reader and thereby the spreading of the theories.

In the second place the Journal will welcome the converse process of elucidating a specific problem either with the aid of available theories or as pointing to further theories.

In the third place the Journal will gladly accept the applications of theories; i. e., the illumination of methods however general by specific and easily comprehended cases.

It is hoped that Applied Mathematics may be a valued feature in the next lap of the Journal's career.

G. D. BIRKHOFF,
A. B. COBLE,
ABRAHAM COHEN,
G. C. EVANS,
FRANK MORLEY, *Chairman.*

Stability and the Equations of Dynamics.*

BY GEORGE D. BIRKHOFF.

Introduction.

For a long time attention has been directed towards the equations of dynamics, and more and more there has been the tendency to take these as being derivable from a special type of variational principle and as consequently possessing 'Hamiltonian' or 'canonical' form. In consequence of the fact that the mathematical treatment of these equations involves difficulties due to the lack of flexibility of the underlying group of transformations, a very large formal literature has grown up. On the other hand, physicists, realizing the sufficiency of such equations for most applications and impressed with the purely mathematical developments, have yielded a position of the very first importance to them, just as have the mathematicians because of the physical applications.

For these reasons there has been a disposition to attribute an almost mysterious significance to the Hamiltonian equations.

Now there is one property of these equations, first explicitly and adequately recognized by Poincaré, which is of obvious significance. If there is given an equilibrium position or a periodic motion of a dynamical system of this type, which satisfies the simplest necessary conditions for stability, and if there are no relations of commensurability between the underlying periods, then there will be complete formal trigonometric stability. This fact is of course of great physical interest, and throws great light upon the inward nature of the stability existing in the solar system for instance.

It is the main contention of this paper that essentially the *only* significance of the Hamiltonian (and thus the Lagrangian) equations is that they possess this characteristic property of complete formal stability. In other words, any set of n equations possessing complete formal stability at an equilibrium point, for example, may be given Hamiltonian form by appropriate formal change of variables.

In the present paper attention is primarily directed to the case of a system of equations of even order, although it is proved that the odd order case essentially reduces to that of even order. Furthermore, the treatment is restricted to the typical case of the equilibrium point, although I expect to treat the analogous important case of a periodic motion elsewhere, in so far as the generalization is not an obvious one.

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* See a preliminary note in the *Comptes rendus* for September 20, 1926. The present paper was read at the Philadelphia meeting of the American Mathematical Society on December 29, 1926.

The precise nature of the Hamiltonian equations is not adequately expressed in the mere statement that the property of stability is their one characteristic feature. Indeed, except from a purely formal point of view, these equations form a particular kind of completely stable systems, because of their convergence properties. The interrelation between the various kinds of stability of systems of ordinary differential equations, the nature of the trigonometric series employed and their degree of accuracy, the formal reduction to Hamiltonian or Pfaffian form in the case of complete stability, must be studied in detail before the very interesting situation which exists can be rightly comprehended. It is my purpose in this paper to clarify this situation as far as possible.

Many other problems are immediately suggested. I will only refer to one. In recent work on the theory of relativity there have been notable attempts made by Eddington, Einstein, Hilbert and Weyl to derive the general partial differential equations of physics from a 'Hamiltonian principle.' There arises now the question: Is the significance of the principle the same here as in classical dynamics, namely to insure the complete trigonometric stability of equilibrium or periodic motion?

In concluding these introductory remarks, I wish to express the hope that these important and interesting questions will receive the further attention which they certainly deserve. In the present paper only a beginning is made.

§ 1. *Definition of Stability.*

We propose to consider a system of n ordinary differential equations of the first order

$$(1) \qquad dx_i/dt = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the right-hand members are real analytic functions of $x_1, \cdots, x_n$ in the open connected continuum under consideration. In particular we shall study the properties of the solutions of (1) in the neighborhood of an 'equilibrium point' of this continuum, i. e., a point for which all of the functions

X_i vanish. It is especially questions concerning the stability of nearby motions that will occcupy our attention.

There are two quite distinct properties of stability of equilibrium: the first is that the nearby motions must continue near to equilibrium for a certain time; the second is that these nearby motions have a quasi-periodic character. We propose to incorporate both of these conditions in our definition of stability.

Definition of Stability. An equilibrium point of the differential system (1), taken at the origin in $x_1, \cdots, x_n$ space, will be said to be *stable of the mth order* if it possesses the two following properties: (A) all points lying at a distance ϵ from the origin at time t_0 remain at a distance not exceeding $K\epsilon$ for an interval $| t - t_0 | \leqq L\epsilon^{-m}$ (K, L, definite positive constants); (B) for any fixed interval $| t - t_0 | \leqq T$ the coordinates $x_1, \cdots, x_n$ of any such motion and polynomials with lowest terms of degree $s > 0$ in the coordinates can be represented by trigonometric sums

$$A_0 + \sum (A_i \cos p_i t + B_i \sin p_i t) \qquad (| p_i - p_j | \geqq P > 0)$$

of not more than $N + 1$ terms with an error not exceeding $Q\epsilon^{m+s}$ (P, N, Q, definite positive constants) during the fixed interval.

If the equilibrium point is stable of all orders it will be said to be *completely stable.*

It is obvious that stability of any order necessitates stability of all lower orders.

The equations (1) are invariant in form under any point transformation

$$x_i = \phi_i(\bar{x}_1, \cdots, \bar{x}_n) \qquad (i = 1, \cdots, n)$$

where the functions ϕ_i are analytic with $| \partial\phi_i / \partial\bar{x}_j | \neq 0$ at the transformed equilibrium point. The order of stability is clearly invariant under such a change of variables. In employing these transformations we shall always take the transformed point to be at the origin also, so that ϕ_i are given by power series in $\bar{x}_1, \cdots, \bar{x}_n$ without constant terms.

It will be one of our principal purposes to show the relation between these properties (A) and (B), and we shall find that 'perturbative stability' (A) does not necessitate 'trigonometric stability' (B), though it turns out that the requirement of trigonometric stability suffices to insure the stability of perturbations. We find it convenient first to go as far as possible on the basis of property (A) alone, and then employ the more fundamental requirement (B).

It is extremely instructive to consider a few characteristic examples in this connection.

The simplest possible example of a point of equilibrium is furnished by the equation

$$dx/dt = ax,$$

in which the point lies at $x = 0$. Here the exponential form of solution shows that there can not be first order stability even in the sense (A).

A more instructive example is furnished by the equations

$$dx/dt = y + \tfrac{1}{2}x(x^2 + y^2), \qquad dy/dt = -x + \tfrac{1}{2}y(x^2 + y^2)$$

with equilibrium point at (0, 0). The general solution is easily determined to be

$$x = \frac{r_0 \cos(\theta_0 + t - t_0)}{\sqrt{1 - r_0{}^2(t - t_0)}}, \quad y = \frac{r_0 \sin(\theta_0 + t - t_0)}{\sqrt{1 - r_0{}^2(t - t_0)}},$$

where r_0, θ_0 are the polar coordinates of (x, y) at t_0. Thus r_0 is the distance ϵ which enters into the above definition of stability.

In this case the expressions for x and y remain of the order of ϵ for $| t - t_0 | \leqq L\epsilon^{-2}$ $(0 < L < 1)$, so that the equilibrium point is stable of the second order in the sense (A). But since the denominator in the expressions for x and y vanishes for $t - t_0 = \epsilon^{-2}$, it is obvious that the point is not stable of the third order. The approximate trigonometric form of the solutions with an error of order 3 in ϵ for any fixed interval $| t - t_0 | \leqq T$ shows that the situation is the same for stability in the sense (B).

On the other hand, for the equations

$$dx/dt = y + y(x^2 + y^2), \qquad dy/dt = -x - x(x^2 + y^2),$$

the general solution is readily found to be

$$\begin{cases} x = x_0 \cos a(t - t_0) - y_0 \sin a(t - t_0) & (a = 1 + x_0{}^2 + y_0{}^2), \\ y = x_0 \sin a(t - t_0) + y_0 \cos a(t - t_0) . \end{cases}$$

Hence the origin is completely stable in this case in the full sense (A), (B).

All of the preceding examples seem to indicate that trigonometric stability is invariably associated with perturbative stability. The following example is of the simplest type which illustrates that this is not the case.

We shall take $n = 4$ with two pairs of variables x_1, y_1, x_2, y_2, and the equations

$$\begin{cases} dx_1/dt = -p_1 y_1 + \alpha x_1 (x_2^2 + y_2^2), & dy_1/dt = p_1 x_1 + \alpha y_1 (x_2^2 + y_2^2), \\ dx_2/dt = -p_2 y_2 + \beta x_2 (x_1^2 + y_1^2), & dy_2/dt = p_2 x_2 + \beta y_2 (x_1^2 + y_1^2), \end{cases}$$

with α, β of opposite signs. For convenience in dealing with the equations we shall introduce conjugate variables

$$u_i = x_i + y_i \sqrt{-1}, \quad v_i = x_i - y_i \sqrt{-1} \qquad (i = 1, 2),$$

and write the equations in the form

$$\begin{cases} du_1/dt = (p_1 \sqrt{-1} + \alpha u_2 v_2) u_1, & dv_1/dt = (-p_1 \sqrt{-1} + \alpha u_2 v_2) v_1, \\ du_2/dt = (p_2 \sqrt{-1} + \beta u_1 v_1) u_2, & dv_2/dt = (-p_2 \sqrt{-1} + \beta u_1 v_1) v_2, \end{cases}$$

where α, β are real constants with $\alpha\beta < 0$. It is to be noted that the equations themselves are of conjugate type: that is, when, for instance u_1 and v_1 are interchanged throughout and $\sqrt{-1}$ changed to $-\sqrt{-1}$ in the first pair of equations, the equations are unaltered as a set.

Now multiply the first and second equations of the set just written by v_1, u_1 respectively and add; operate similarly with the second set. There is obtained the pair of equations

$$dw_1/dt = 2\alpha w_1 w_2, \qquad dw_2/dt = 2\beta w_1 w_2,$$

where $w = u_1 v_1$ and $w_2 = u_2 v_2$ are positive or zero for all real solutions x_1, y_1, x_2, y_2 of course.

If w_1, for instance, vanishes for any t, these equations show that it will vanish identically, with w_2 a constant.* This gives $u_1 = v_1 = 0$, and the differential equations in u_2, v_2 enable us to write the corresponding solution

$$u_1 = 0, \quad v_1 = 0, \quad u_2 = u_2^{(0)} e^{\sqrt{-1}\, p_2 (t - t_0)}, \quad v_2 = v_2^{(0)} e^{-\sqrt{-1}\, p_2 (t - t_0)}.$$

Consequently for this type of solution the squared distance from the origin, measured by $w_1 + w_2$ is constant, and the representation is trigonometric.

But let us consider the general solution at distance ϵ from the origin at $t = t_0$ so that

$$w_1^{(0)} + w_2^{(0)} = \epsilon^2, \quad w_1^{(0)} > 0, \quad w_2^{(0)} > 0,$$

---

* This follows from the fact that the initial values of w_1, w_2 at time t_0 determine the solution uniquely, in virtue of the elementary existence theorems, for $w_1 = 0$; $w_2 =$ const. is such a solution in the case at hand.

thus laying aside the possibility just considered. Introduce a parameter τ for which $d\tau = 2w_1w_2dt$. The equations for w_1, w_2 show that we have

$$w_1 = \alpha(\tau - \tau_0) + w_1^{(0)}, \qquad w_2 = \beta(\tau - \tau_0) + w_2^{(0)},$$

so that w_1 and w_2 vary monotonically and in opposite senses.

Because of the integral relation

$$\beta w_1 - \alpha w_2 = \beta w_1^{(0)} - \alpha w_2^{(0)} = \text{const.},$$

in which β and α are of opposite sign by hypothesis, the positive quantities w_1, w_2, and consequently τ also, vary in a finite interval as t passes from $-\infty$ to $+\infty$.

Let us observe the relation between t and τ further. Taking $t_0 = \tau_0 = 0$ for simplicity of notation, we find

$$2t = \int_0^\tau \frac{d\tau}{(\alpha\tau + w_1^{(0)})(\beta\tau + w_2^{(0)})} \, .$$

Here the lower limit of integration lies between the two zeros of the denominator of the integrand. Thus τ ranges between these values τ' and τ'' only.

The integral relation above demonstrates that w_1 and w_2 remain less than a fixed multiple of ϵ for all time. Hence we have stability of every order m, in the sense (A).

On the other hand we have from the original equations

$$\begin{cases} du_1/dt = (p_1\sqrt{-1} + \alpha w_2)u_1, & dv_1/dt = (-p_1\sqrt{-1} + \alpha w_2)v_1, \\ du_2/dt = (p_2\sqrt{-1} + \beta w_1)u_2, & dv_2/dt = (-p_2\sqrt{-1} + \beta w_1)v_2, \end{cases}$$

whence we obtain by integration

$$\begin{cases} u_1 = u_1^{(0)} e^{p_1\sqrt{-1}\, t + \alpha \int w_2 dt}, & v_1 = v_1^{(0)} e^{-p_1\sqrt{-1}\, t + \alpha \int w_2 dt}, \\ u_2 = u_2^{(0)} e^{p_2\sqrt{-1}\, t + \beta \int w_1 dt}, & v_2 = v_2^{(0)} e^{-p_2\sqrt{-1}\, t + \beta \int w_1 dt}. \end{cases}$$

Consequently the coordinates u_1, v_1, u_2, v_2, have amplitudes which vary monotonically, but they are essentially trigonometric in the angle variables.

It is obvious (see § 3) that x_1, y_1, x_2, y_2 and u_1, v_1, u_2, v_2 cannot be representable by trigonometric sums as demanded with an error of the second order in ϵ. Consequently property (B) fails to hold for $m = 1$.

Inasmuch as we are not following out very systematically the conditions for stability of type (A), it may be remarked at this juncture that in general the property (A) will follow if there is an integral in which the initial terms give a definite quadratic form, just as they did in the above example. One is naturally led to conjecture that, in general, condition (A) can only obtain for some order m if there exists a polynomial P in the variables $x_1, \cdots, x_n$ with initial terms yielding a homogeneous form of degree k of definite sign, such that the series on the right in the equation

$$dP/dt = \sum_{j=1}^{n} \partial P/\partial x_j \, X_j$$

will begin with terms of degree $m + k$ at lowest. At any rate the type of argument employed in the following paragraph shows readily that this condition is sufficient for (A).*

Another example illustrating the possibility that (A) may be true for $m = 1$ without (B) holding for $m = 1$ is furnished by the equations

$$dx/dt = xy^2, \qquad dy/dt = -x^2y.$$

Here $x^2 + y^2 = \text{const.}$ is an integral, so that (A) holds for any order m. The general solution is

$$x = \frac{\pm ce^{c^2(t-k)}}{(1 + c^2e^{2c^2(t-k)})^{\frac{1}{2}}}, \qquad y = \frac{\pm c}{(1 + c^2e^{2c^2(t-k)})^{\frac{1}{2}}},$$

which is not trigonometric in type. However, here the origin is not an isolated point of equilibrium, since all of the points of $x = 0$ or $y = 0$ are of this type. On account of its degenerate character the example is not as illuminating as the preceding one.

§ 2. *First Order Stability.*

The condition (A) for stability has reference to solutions of the given differential equations (1) lying for $t = t_0$ at a distance ϵ (less than some definite quantity) from the equilibrium point, which will always be taken to be at the origin. Now suppose that we write $x_i = \epsilon y_i$, thus making a linear transformation of the dependent variables. The equations (1) then take the form

---

* In this connection the reader may be referred to Picard, *Traité d'Analyse*, Vol. 3, chaps. 8, 9, where the fundamental papers of Poincaré, Liapounoff and Levi-Civita are also cited.

$$dy_i/dt = 1/\epsilon \; X_i(\epsilon y_1, \cdots, \epsilon y_n) = Y_i(y_1, \cdots, y_n, \epsilon) \quad (i = 1, \cdots, n).$$

Since the X_i's vanish at the origin in $x_1, \cdots, x_n$ space, Y_i will be given as power series in $y_1, \cdots, y_n, \epsilon$ without constant term. Moreover the domain of uniform convergence of these new series will certainly extend beyond a distance unity from the origin in $y_1, \cdots, y_n$ space if ϵ is sufficiently small.

Corresponding to any solution of (1) at distance ϵ from the origin to the $x_1, \cdots, x_n$ space, we have a unique solution y_i of the transformed equations at distance unity from the origin in $y_1, \cdots, y_n$ space. Let us write $y_i = a_i$ at $t = t_0$ and regard these initial values a_i as fixed, while ϵ is a parameter which approaches 0. There will be defined a set of solutions y_i varying analytically with ϵ, according to the fundamental existence theorem for ordinary differential equations containing a parameter. For $\epsilon = 0$ the set of functions will satisfy the corresponding equations of variation

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} \Big|_0 \, y_j \tag{2}$$

obtained by setting $\epsilon = 0$ in the differential equations for y_i. These are n ordinary linear differential equations with constant coefficients.

Suppose now that we make the hypothesis that the equilibrium point of (1) under consideration is stable in the sense of (A) for $m = 1$. It follows that $y_i = x_i/\epsilon$ is uniformly bounded for $| t - t_0 |$ of order ϵ^{-1}. Consequently, by allowing ϵ to approach 0, we infer that the limiting functions y_i are bounded for all t. Thus for stability of type (A), $m = 1$, it is proved that the equations of variation (2) have all such solutions y_i bounded for all t. But the most general solution of (2) is derived from solutions of this type by multiplication by a constant factor, since the only restriction made was that the sum of the squares $a_1, \cdots, a_n$ is 1. Hence it is a necessary condition for stability of type (A), $m = 1$, that every solution of the equations of variations (2) be bounded for all t.

The known nature of the solutions of (2) enables us immediately to determine the precise significance of this condition.

Suppose first that the roots α of the characteristic equation

$$\left| \frac{\partial X_i}{\partial x_j} \Big|_0 - \alpha \, \delta_{ij} \right| = 0 \qquad (\delta_{ij} = 0, i \neq j; \; \delta_{ii} = 1)$$

are distinct from one another. Then there are n particular solutions of the equations of variation which may be written in the form

$$y_1 = c_{1j}e^{a_jt}, \cdots, y_n = c_{nj}e^{a_jt} \qquad (j = 1, \cdots, n).$$

The most general solution is obtained by linear combination of these particular solutions. Here $a_1, \cdots, a_n$ may be real or complex of course, but corresponding real solutions may be obtained by combination from the pair of solutions corresponding to a complex root and its conjugate. It is apparent that such solutions are bounded only in case the expressions e^{a_jt} are bounded. Hence all of the a_i's must be pure imaginary quantities or zero. In this way we see that in the general case of distinct roots of the characteristic equation, it is necessary that the roots occur in pure imaginary pairs except for possible zero roots.

On the other hand in the more complicated case where the roots of the characteristic equation are not distinct, there may arise degenerate solutions not of the above type, but of the form

$$x_1 = c_{1j}t^{l_j}e^{a_jt}, \quad \cdots, \quad x_n = c_{nj}t^{l_j}e^{a_jt} \qquad (l_j > 0),$$

corresponding to the case in which the elementary divisors corresponding to a multiple root a_j are not simple. Evidently these are not bounded. In this way we are led to the following first conclusion:*

For first order stability of type (A) at an equilibrium point it is necessary that all the roots of the characteristic equation belonging to the point be grouped in pure imaginary pairs except for possible zero roots, and that the elementary divisors of the corresponding a *matrix be distinct.*

We are led at once to ask whether or not this necessary condition for (A) is sufficient. It is easily proved to be sufficient as follows.

Let us effect the linear transformation of (1) (in general complex) which reduces the linear terms in X_i to the canonical form a_ix_i. This may always be done in the case of distinct elementary divisors.

In accordance with the well-known theory of real linear transformations this may be accomplished in such a way that pairs of variables x_i and x_j are conjugate which correspond to conjugate roots a_i and a_j of the characteristic equation. We shall take pains to preserve this conjugacy at every subsequent transformation. It should be observed that for such conjugate variables the squared 'distance' from the equilibrium point is properly measured by the sum of the squares of the quantities $| x_i |$.

---

* The main facts here are of course well known.

The transformed equations take the form in the new variables

$$(3_1) \qquad dx_i/dt = \alpha_i x_i + \mathrm{X}_i \qquad (i = 1, \cdots, n),$$

where the power series X_i in the variables $x_1, \cdots; x_n$ begin with terms of the second degree, while the α_i's are either pure imaginaries or zero. The series X_i, X_j corresponding to conjugate variables x_i, x_j will be conjugate of course, while the series X_i corresponding to zero roots of the characteristic equation will be real. If we make the further change of dependent variables

$$x_i = e^{\alpha_i t} z_i \qquad (i = 1, \cdots, n),$$

we find the equations for z_i to be

$$dz_i/dt = \mathrm{X}_i(e^{\alpha_1 t} z_1, \cdots, e^{\alpha_n t} z_n).$$

Since the exponential factors occurring here have absolute value 1, we have clearly for some constant X

$$| \mathrm{X}_i | \leqq \mathrm{X}\,(| z_1 |^2 + \cdots + | z_n |^2) = \mathrm{X}u^2,$$

so long as $y_1, \cdots, y_n$ or $z_1, \cdots, z_n$ are sufficiently small in absolute value. Hence we have

$$\left| u \frac{du}{dt} \right| \leqq \sum_{j=1}^{n} \left| z_j \frac{d\,| z_j |}{dt} \right| \leqq (| z_1 | + \cdots + | z_n |)\, \mathrm{X}u^2 \leqq \sqrt{n}\, \mathrm{X}u^3;$$

the elementary inequality for real quantities $b_1, \cdots, b_n$,

$$(b_1 + \cdots + b_n)^2 \leqq \sum_{i,j=1}^{n} | b_i b_j | \leqq \sum_{i,j=1}^{n} \tfrac{1}{2}(b_i^2 + b_j^2) = n \sum_{j=1}^{n} b_j^2$$

is employed in deducing this result, as well as the obvious relation

$$\left| \frac{d\,| z_j |}{dt} \right| \leqq \left| \frac{dz_j}{dt} \right| \leqq \mathrm{X}u^2.$$

Since $u \neq 0$ we infer

$$-\sqrt{n}\, \mathrm{X}dt \leqq du/u^2 \leqq +\sqrt{n}\, \mathrm{X}dt,$$

whence by integration

$$-\sqrt{n}\, \mathrm{X}(t - t_0) \leqq 1/u_0 - 1/u \leqq \sqrt{n}\, \mathrm{X}(t - t_0) \qquad (t > t_0)$$

with a like inequality for $t < t_0$.

But at $t = t_0$ we have $u = \epsilon$ of course. It appears then that u will not increase to 2ϵ or diminish to $\frac{1}{2}\epsilon$ within an interval

$$| t - t_0 | \leqq \frac{\epsilon^{-1}}{2\sqrt{n}\,\mathrm{X}} .$$

This is what we desired to prove.

The stated necessary condition is sufficient for first order stability of type (*A*).

A slight extension of the preceding proof leads us to the fact that if this condition (A) is satisfied, so that the general solution of the equations of variation is composed of trigonometric sums, then (B) will hold for $m = 1$, i. e., the coordinates can be suitably approximated to by such sums.

Within the same t interval we have $| \mathrm{X}_i | \leqq 4\mathrm{X}u_0^2$ so that from the differential system in $z_1, \cdots, z_n$, the inequalities

$$| z_i - z_i^{(0)} | \leqq 4\mathrm{X}u_0^2 \, | t - t_0 |$$

result by integration, or, in the variables x_i,

$$| x_i - x_i^{(0)} e^{a_i(t-t_0)} | \leqq 4\mathrm{X}u_0^2 \, | t - t_0 | .$$

In other words the n trigonometric sums which constitute the solution of the equations of variation with the same initial values as x_i at t_0 represent x_i with an error not exceeding $4\mathrm{X}\epsilon^2 | t - t_0 |$ during an interval $| t - t_0 | \leqq \epsilon^{-1}/2\sqrt{n}\,\mathrm{X}$. A like property must then hold for the solutions of (1) which are related to the solutions of the prepared equations (3_1) by a linear transformation with constant coefficients. Hence we may state the following result:

The stated necessary condition for first order stability is sufficient. If then we let

$$c_{1j}e^{a_j t}, \cdots, c_{nj}e^{a_j t} \qquad (j = 1, \cdots, n)$$

be n linearly independent solutions of the equations of variation (2) *so that a_j are pure imaginaries or zero and the determinant $| c_{ij} | \neq 0$, and if we form the unique linear combination of these solutions taking on the same values as the solution of* (1) *at $t = t_0$, the real trigonometric sums so obtained are of the form*

$$A_0 + \sum_{j=1}^{s} (A_j \cos p_j t + B_j \sin p_j t)$$

where A_j, B_j are of the first order in ϵ, and represent x_i with an error not *exceeding* $K\epsilon^2 \mid t - t_0 \mid$ *during an interval* $\mid t - t_0 \mid \leqq L\epsilon^{-1}$ *(K, L, definite positive constants). Here* $p_1, \cdots, p_s$ *are the* s *absolute values of the* $2s$ *pure imaginary quantities* a_i, *so that* $2s \leqq n$.

§ 3. *Supplementary Results for First Order Stability.*

It was stated in § 1 that the condition (B) for stability necessitated the condition (A). We propose at first to establish the fact in the case of first order stability. In order to do so we prove a lemma concerning trigonometric sums of the type referred to in the statement of (B).

Lemma Concerning Trigonometric Sums. If a sequence of trigonometric sums

$$A_0 + \sum_{j=1}^{N} (A_j \cos p_j t + B_j \sin p_j t) \qquad (\mid p_i - p_j \mid \geqq P > 0)$$

where P, N are fixed positive constants, approaches a limiting function $\phi(t)$ uniformly in the fixed interval $\mid t - t_0 \mid \leqq T$ then ϕ is itself a trigonometric sum of the same kind.

Proof. The lemma may be proved by induction. Let us first establish it in the case $N = 1$, and let us take $t_0 = 0$. Here it is given that

$$\phi(t) = \lim_{l=\infty} [A_0^{(l)} + (A_1^{(l)} \cos p_1^{(l)} t + B_1^{(l)} \sin p_1^{(l)} t)]$$

where the limit is approached uniformly by the sequence $\phi_l(t)$.

Now the double integration of the term of trigonometric type here occurring simply divides it by $-p_1^{(l)2}$. Hence the sequence of functions

$$\int_0^t \int_0^t \phi_l(t)\, dt^2 + \phi_l(t)/p_1^{(l)2}$$

consists of polynomials of the second degree in t.

Suppose now that $p_1^{(l)}$ has some limit p_1 ($\mid p_1 \mid \geqq P$ since $p_0^{(l)} = 0$) other than ∞ as l becomes infinite. Since the uniform limit of a polynomial of any degree can only be a polynomial of that degree, it is necessary that

$$\int_0^t \int_0^t \phi(t)\, dt^2 + \phi(t)/p_1^2$$

is a polynomial of the second degree in t. Consequently, by double differentiation, we derive

$$d^2\phi/dt^2 + p_1^2 \phi = \text{const.},$$

so that ϕ has the form

$$\phi = A_0 + (A_1 \cos p_1 t + B_1 \sin p_1 t) \qquad (\mid p_1 \mid \geqq P)$$

in agreement with the lemma.

If, however, $p_1^{(l)}$ has only the limit ∞ we conclude immediately by a similar method

$$\phi = A_0,$$

in agreement with the lemma.

We remark also that if additional polynomials of the second degree are added to the trigonometric sums, $N = 1$, a similar proof shows that the limit $\phi(t)$ necessarily consists of such a sum augmented by a polynomial of the second degree.

To prove the lemma for $N = 2$ we need only isolate one of the two trigonometric terms with period $2\pi/p_2^{(l)}$ in the approximating trigonometric sums. Proceeding as above we are led to consider

$$\int_0^t \int_0^t \phi_l(t)\, dt^2 + \phi_l(t)/p_2^{(l)2}.$$

This expression lacks the trigonometric terms of the type isolated so that if $p_2^{(l)}$ has a finite limit p_2,

$$\int_0^t \int_0^t \phi(t)\, dt^2 + \phi(t)/p_2^2$$

can be approached uniformly by trigonometric sums of the above type of order $N = 1$ augmented by polynomials of the second degree, and so is itself similarly constituted by the previous remark. By double differentiation, we conclude that ϕ is a trigonometric sum of the second order and of the stated type in this case. When $p_2^{(l)}$ has only the limit ∞ the method can be carried to a conclusion as before.

Evidently we have here the basis of a method of proof by means of induction for $N = 1, 2, \cdots$. Hence the lemma must hold for all values of N.

We may prove that the condition (B) implies (A) for $m = 1$ by making use of this lemma. By hypothesis x_i/ϵ can be approximated to by a trigonometric sum of the prescribed type in any fixed interval $|t - t_0| \leqq T$ with an error of the first order at most in ϵ. But by choosing x_i properly, x_i/ϵ can be made to approach any solution y_i of the equations of variation having $a_1^2 + \cdots + a_n^2 = 1$. By the lemma then since the functions y_i can be approached uniformly by such trigonometric sums they must be such sums themselves. This means of course that the general solution of the equation of variation is trigonometric, which was seen to be a necessary and sufficient condition for stability of the first order.

It follows that the condition (B) *alone in the case* $m = 1$ *implies condition* (A), *and thus stability of the first order.*

The second result to which I wish to call attention is implicitly contained in the results for trigonometric sums specified in § 2 and has to do with certain amplitudes. Let us formulate the simple notions involved.

The amplitudes are defined to be the absolute values of the variables x_i under consideration, taken as a solution of the prepared equations (3_1). If there are s pairs of conjugate imaginary roots α_i and k zero roots ($n = 2s + k$) then there are only the amplitudes

$$C_1, \cdots, C_s, \quad C_{s+1}, \cdots, C_{s+k}$$

since conjugate imaginary x_i's have the same absolute value.

It is a fundamental property of the $s + k$ amplitudes that the sum

$$C_1^2 + \cdots C_{s+k}^2$$

serves to measure essentially the squared distance of a point in the original $x_1, \cdots, x_n$ space from the position of equilibrium. In fact the linear transformation relating (1) and (3_1) evidently takes the above sum into a positive definite quadratic form in the values of $x_1, \cdots, x_n$. Any such quadratic form serves to measure the squared distance from the origin as well as the sum of the squares used earlier.

The fact which we desire to establish directly may now be formulated in the following way:

In the case of first order stability the $s + k$ amplitude variables $C_1, \cdots, C_{s+k}$ change only slightly in comparison with the initial total amplitude,

$$\epsilon = (C_1^2 + \cdots + C_{s+k}^2)^{\frac{1}{2}} \qquad (t = t_0)$$

during a time interval $|t - t_0| \leqq L^* \epsilon^{-1}$ where L^* is to be suitably chosen.

For conjugate x_i and x_j the prepared form (3_1) gives

$$d(x_i x_j)/dt = x_i \mathrm{X}_j + x_j \mathrm{X}_i,$$

while for real x_i we have

$$dx_i^2/dt = 2x_i \mathrm{X}_i.$$

Hence if we write u^2 for the squared amplitude and recall that X_i commence with terms of at least the second degree, we find, as in § 2,

$$du/dt \leqq \mathrm{X} u^2,$$

where X is an appropriately chosen positive constant. Now in § 2 it was shown how from such an inequality we may infer that u will vary between

$\frac{1}{2}u_0$ and $2u_0$ in a suitably taken interval $|t - t_0| \leqq L\epsilon^{-1}$. The same argument permits one to infer that u will vary between $(1 - \eta)u_0$ and $(1 + \eta)u_0$, η arbitrarily small, if L is chosen small enough. In other words the sum of the squares of the individual amplitudes only varies slightly as compared with the total initial amplitude under these conditions.

But the variables $x_1, \cdots, x_n$ of the prepared equations (3_1) are not uniquely determined, for we may replace $x_1, \cdots, x_n$ respectively by $c_1x_1, \cdots, c_nx_n$ respectively, where $c_1, \cdots, c_n$ are arbitrary positive constants except that c_i and c_j associated with a conjugate pair x_i and x_j are to be equal. In so doing the definition of the amplitudes is changed in that these are replaced by arbitrary constant multiples of themselves. Thus we conclude that, for any arbitrary set of $s + k$ positive constants $\lambda_1, \cdots, \lambda_{s+k}$, the sum

$$\lambda_1C_1^2 + \cdots + \lambda_{k+s}C_{k+s}^2$$

changes only varies slightly as compared with its initial value for an interval of time of order ϵ^{-1}.

Now choose $s + k$ linearly independent sets $\lambda_1, \cdots, \lambda_{s+k}$ and an arbitrary small value of $\eta > 0$. It will then be possible to choose for each corresponding sum and therefore for all, a constant L^* so small that the ratio of the corresponding sum of squares to the initial sum u_0^2 at $t = t_0$ will vary as little as may be desired for $|t - t_0| \leqq L^*\epsilon^{-1}$.

The theory of linear algebraic equations shows then at once that all of the amplitudes can vary only slightly as compared with the initial total amplitude, as first measured. This yields the result stated.

It will be proved in the following section that when the amplitudes are more carefully defined in the case of first order stability, n even, the variation in these amplitudes remains slight for an interval of time, $|t - t_0| \leqq L^*\epsilon^{-2}$.

§ 4. *Second Order Stability.*

In dealing with second and higher orders of stability, it is convenient to treat separately the cases when the order of the differential system (1) is even $(n = 2s)$ and odd $(n = 2s + 1)$. Furthermore we have now to take account of commensurability relations between the characteristic frequencies $p_i/2\pi$.

For the case of even order we confine attention to the case where the inequality

$$i_1p_1 + \cdots i_sp_s \neq 0$$

holds for sets of integers $i_1, \cdots, i_s$ not all zero. This inequality excludes in particular any zero or equal roots of the characteristic equation. For the purpose of the discussion of second order stability here made, only these special restrictions are necessary.

In the case of odd order there is always one zero root which we take to belong to the coordinate x_n; we designate the $2s$ other pure imaginary or zero roots as in the even order case, and again make the assumption stated above, although only the case of further zero roots or of equal roots needs to be excluded as far as the argument of this section is concerned.

It is with the intention of avoiding undue complexity that restriction to the 'general case' is thus made.

We take up first the case $n = 2s$, and suppose that pairs of conjugate variables are called

$$x_1 = \xi_1, \quad x_2 = \eta_1, \quad \cdots, \quad x_{n-1} = \xi_s, \quad x_n = \eta_s.$$

The prepared form of equations (3_1) may then be written

$$(3) \qquad d\xi_i/dt = p_i \sqrt{-1}\, \xi_i + \Phi_i, \quad d\eta_i/dt = -p_i \sqrt{-1}\, \eta_i + \Psi_i \qquad (i = 1, \cdots, s)$$

where Φ_i, Ψ_i contain no terms of lower than the second degree.

Let us attempt to proceed with the preparation of these second degree terms by means of a further transformation of the form

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s),$$

where both of the functions $\bar{F}_i$, $\bar{G}_i$ are conjugate homogeneous polynomials of the second degree in $\bar{\xi}_1, \cdots, \bar{\eta}_s$, so that if $\bar{\xi}_i$, $\bar{\eta}_i$ are conjugate imaginaries, ξ_i, η_i $(i = 1, \cdots, s)$ will be, and conversely. The inverse transformation has the form

$$\bar{\xi}_i = \xi_i - F_i + \cdots, \quad \bar{\eta}_i = \eta_i - G_i + \cdots \qquad (i = 1, \cdots, s),$$

where F_i, G_i are the same polynomials in ξ_i, η_i as $\bar{F}_i$, $\bar{G}_i$ in $\bar{\xi}_i$, $\bar{\eta}_i$, and where higher order terms are only indicated. Such a transformation has already been observed not to affect the order of stability (§ 1).

On making this change of variables in the prepared equations (3), we obtain the specific modified equations

$$
\begin{aligned}
\frac{d\xi_i}{dt} &= p_i\sqrt{-1}\,\xi_i + \Phi_i^{(2)} + p_i\sqrt{-1}\,F_i \\
&\quad - \sum_{j=1}^{s} p_j\sqrt{-1}\left(\xi_j\frac{\partial F_i}{\partial \xi_j} - \eta_j\frac{\partial F_i}{\partial \eta_j}\right) + \cdots \\
\frac{d\eta_i}{dt} &= -p_i\sqrt{-1}\,\eta_i + \Psi_i^{(2)} - p_i\sqrt{-1}\,G_i \\
&\quad - \sum_{j=1}^{s} p_j\sqrt{-1}\left(\xi_j\frac{\partial G_i}{\partial \xi_j} - \eta_j\frac{\partial G_i}{\partial \eta_j}\right) + \cdots
\end{aligned}
\tag{4}
$$

where dashes over the ξ_i's and η_i's have been dropped, and where $\Phi_i^{(2)}$, $\Psi_i^{(2)}$ are the second degree components of Φ_i, Ψ_i respectively. The displayed parts contain the complete second degree term in the new equations. Hence this change of variables does not affect the character of the prepared form.

If we let c_i stand for the complete coefficient of a definite term of the second degree

$$c_i\xi_i^{l_1}\eta_1^{m_1}\cdots\xi_s^{l_s}\eta_s^{m_s} \qquad (l_1 + m_1 + \cdots + m_s = 2)$$

in Φ_i, while q_i is the analogous coefficient in F_i, then the corresponding coefficient in the second degree term of the first equation above is

$$c_i - [p_1(l_1 - m_1) + \cdots + (l_i - m_i - 1)p_i + \cdots + (l_s - m_s)p_s]\sqrt{-1}\,q_i.$$

It is at this point that the hypothesis concerning the general case enters. The coefficient of q_i cannot vanish unless

$$l_1 = m_1, \cdots, l_i = m_i + 1, \cdots, l_s = m_s,$$

Now the argument just stated has not made use of the fact that we are dealing with the simple case $l_1 + m_1 + \cdots + m_s = 2$ when there is evidently no possibility for the coefficient of q_i to vanish. Hence we conclude that without any assumption of second order stability, we may further prepare the equations (3) so that no terms $\Phi_i^{(2)}$, $\Psi_i^{(2)}$ remain.

We may now prove the following result:

In the general case of first order stability, $n = 2s$, there will also be second order stability. Let the combined linear and quadratic transformation from ξ_i, η_i to $\bar{\xi}_i$, $\bar{\eta}_i$ be made which takes (1) *into the prepared form* (3) *lacking second degree terms in Φ_i, Ψ_i. If we substitute*

$$\bar{\xi}_i^{(0)}e^{p_i\sqrt{-1}(t-t_0)}, \quad \bar{\eta}_i^{(0)}e^{-p_i\sqrt{-1}(t-t_0)} \qquad (i = 1, \cdots, s)$$

for $\bar{\xi}_i$, $\bar{\eta}_i$ in the formulas, the trigonometric sums obtained will represent

$x_1, \cdots, x_n$ *with an error not exceeding* $K\epsilon^3 \mid t - t_0 \mid$ *during an interval of time* $\mid t - t_0 \mid \leqq L\epsilon^{-2}$.

The details of the proof are very much like those for the analogous result concerning first order stability (§ 2). To begin with, we note that in the prepared form the non-linear terms on the right are of at least the third degree.

Now, using exactly the argument of § 2 except that we can assume $\mid \mathrm{X}_i \mid \leqq \mathrm{X} u^3$, we are able to establish the relation stated.

Suppose now that we define the amplitudes $C_1, \cdots, C_{k+s}$ more precisely by the aid of this especially prepared equation (3). Following a natural modification of the argument made in the preceding section we are at once led to infer the following additional result:

In the case of first, and thus second, order stability ($n = 2s$) *the* s *individual amplitudes* $C_1, \cdots, C_s$ *may be defined by means of prepared equations* (3) *in which* Φ_i, Ψ_i *contain no second degree terms. With this definition, these amplitudes change only slightly in comparison with the total amplitude during a time interval* $\mid t - t_0 \mid \leqq L^* \epsilon^{-2}$.

We turn next to the general case of odd order $n = 2s + 1$ for the system of equations (1), and propose to briefly indicate some analogous results.

In this case, besides the $2s$ equations like (3), we have a last equation

$$dx_n/dt = \mathrm{X}$$

where X begins with second degree terms or higher. We attempt to prepare further these equations by a transformation

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i, \quad x_n = \bar{x}_n + \bar{H} \qquad (i = 1, \cdots, s),$$

where $\bar{F}_i, \bar{G}_i, \bar{H}$ are homogeneous polynomials of the second degree. We find precisely as before that we can make $\Phi_i^{(2)} = \Psi_i^{(2)} = 0$ by a proper choice of F_i, G_i, while the second degree terms in X have the form

$$\mathrm{X}^{(2)} - \sum_{j=1}^{s} p_j \sqrt{-1} \left(\xi_j \frac{\partial H}{\partial \xi_j} - \eta_j \frac{\partial H}{\partial \eta_j} \right)$$

If we take typical corresponding terms of the second degree in $\mathrm{X}^{(2)}$ and H, both with exponents $l_1, \cdots, m_s, r$ in $\xi_1, \cdots, \eta_s, x_n$ respectively, and with coefficients c and q, the new analogous coefficient is

$$c - [p_1(l_1 - m_1) + \cdots + p_s(l_s - m_s)] \sqrt{-1}\, q.$$

In the case at hand the only possibility that the coefficient of q vanishes is that some $l_i = m_i = 1$ while the other l_j, m_j must then vanish. Thus the terms $\xi_i\eta_i$ and x_n^2 in $X^{(2)}$ cannot be removed by this method.

In consequence of this failure in removing certain second degree terms it seems probable in advance that for the case $n = 2s + 1$, first order stability does not imply that of the second order.

A simple example can be devised to show the truth of such a surmise. Take $n = 3$ and write the equations

$$d\xi_1/dt = (p_1\sqrt{-1} + cx_3^2)\xi_1, \qquad d\eta_1/dt = (-p_1\sqrt{-1} + cx_3^2)\eta_1,$$
$$dx_3/dt = e\xi_1\eta_1 \qquad\qquad (ce > 0),$$

containing one such non-removable term in the last equation.

On writing $u = \xi_1\eta_1$, $v = x_3$ so that $u_0 + v_0^2 = \epsilon^2$, we obtain immediately

$$du/dt = 2cv^2u, \qquad dv/dt = eu,$$

whence

$$d^2v/dt^2 = 2cv^2\,dv/dt.$$

This is immediately integrable, and yields

$$t - t_0 = 3/2c \int_{v0}^{v} \frac{dv}{v^3 - v_0^3 + (3e/2c)u_0} .$$

Suppose now that we take $v_0 = 0$ and substitute

$$v = \left(\frac{3eu_0}{2c}\right)^{1/3} w.$$

We find

$$t - t_0 = (3/2c)^{1/3} u_0^{-2/3} e^{-2/3} \int_0^{w} \frac{dw}{w^3 + 1} .$$

Taking the upper limit $w = +\infty$ we conclude that the corresponding interval of time is of order $\epsilon^{-4/3}$, since the integral converges. Thus v does not remain finite in an interval of order $\epsilon^{-4/3}$, and property (A) does not hold for $m = 2$. Hence in this special example the presence of non-removable terms is associated with instability of the second order.

We proceed now to prove that (B) can only hold if there are no such non-removable terms in the completely prepared form. Here we use once more the lemma on trigonometric sums of the preceding section.

Let

$$\xi_i = \epsilon\xi_i^* + \epsilon^2\xi_i^{**} + \cdots, \qquad \eta_i = \epsilon\eta_i^* + \epsilon^2\eta_i^{**} + \cdots,$$
$$x_n = \epsilon x_n^* + \epsilon^2 x_n^{**} + \cdots$$

be that solution of the prepared equations for which

$$\xi_i = a_i^{(0)}\epsilon, \quad \eta_i = b_i^{(0)}\epsilon, \quad x_n = c^{(0)}\epsilon \qquad (t = t_0).$$

Here $a_i^{(0)}$ and $b_i^{(0)}$ are conjugate imaginary quantities, and $c^{(0)}$ real, with

$$a_1^{(0)}b_1^{(0)} + \cdots + a_s^{(0)}b_s^{(0)} + c^{(0)2} = 1.$$

Also ξ_i^*, η_i^*, x_n^* are solutions of the equations of variation

$$\xi_i^* = a_i^{(0)}e^{p_i\sqrt{-1}(t-t_0)}, \quad \eta_i^* = b_i^{(0)}e^{-p_i\sqrt{-1}(t-t_0)}, \quad x_n^* = c^{(0)}.$$

We consider now the second terms in these series.

By differentiating the prepared equations twice as to ϵ and setting $\epsilon = 0$, we find

$$\frac{d\xi_i^{**}}{dt} = p_i\sqrt{-1}\,\xi_i^{**}, \quad \frac{d\eta_i^{**}}{dt} = -p_i\sqrt{-1}\,\eta_i^{**},$$
$$\frac{dx_n^{**}}{dt} = c_1\xi_1^*\eta_1^* + \cdots + c_s\xi_s^*\eta_s^* + bx_n^{*2},$$

where the terms on the right in the last equation arise from the non-removable terms of X. Furthermore for $t = t_0$, $\xi_i^{**} = \eta_i^{**} = x_n^{**} = 0$, since, for example, ξ_i^{**} is the second partial derivative of ξ_i as to ϵ, and ξ_i is constantly equal to $a_i^{(0)}\epsilon$ at $t = t_0$.

Thus we infer that ξ_i^{**}, η_i^{**} are identically zero, while from the last equation

$$x_n^{**} = [c_1a_1^{(0)}b_1^{(0)} + \cdots + c_sa_s^{(0)}b_s^{(0)} + bc^{(0)2}]\,(t - t_0).$$

But x_n can be approximated by a trigonometric sum of the stated type to the third order in ϵ, by hypothesis. Thus $(x_n - c^{(0)}\epsilon)/\epsilon^2$ can be similarly approximated to the first order in ϵ. Consequently the limit of this quotient, x_n^{**}, is such a sum by the lemma. This can only be a trigonometric sum for all choices of $a_i^{(0)}$, $b_i^{(0)}$, $c^{(0)}$ if $c_i = d = 0$ $(i = 1, \cdots, s)$, which we wished to prove.

In the general case $n = 2s + 1$ *a necessary and sufficient condition for second order stability is that a prepared form* (3) *exists with a supplementary equation*

$$dx_n/dt = \mathrm{X},$$

where Φ_i, Ψ_i, X *begin with terms of at least the third degree. Thence follow results concerning the representation by trigonometric sums, and concerning the variation of the* $s + 1$ *suitably defined amplitudes, which are entirely analogous to the results in the case* $n = 2s$.

In fact, we have reached a prepared form in the case $n = 2s + 1$ with the same characteristics as that obtained in the case $n = 2s$ earlier, and can apply the same arguments with obvious modifications.

§ 5. *Third and Higher Order Stability.*

We propose now to assume stability of the third order, taking $n = 2s$ first, and then, by making use of condition (B) only, to obtain a necessary prepared form of equations, which can readily be proved sufficient for stability of the third order. An entirely analogous treatment can be made for $n = 2s + 1$.

By passing thus from the case of second to third order stability as well as from the first to the second, the general results can be formulated.

Starting with the prepared form (3) we now attempt to remove as many of the third order terms as possible by writing

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s)$$

where now $\bar{F}_i$, $\bar{G}_i$ are homogeneous polynomials of the *third* degree in $\bar{\xi}_1, \cdots, \bar{\eta}_s$. By exactly the same formal work as in our treatment of second order stability, it turns out that all such terms may be removed except for terms of the form $U_i\xi_i$, $V_i\eta_i$ in the ith pair of equations, where U_i, V_i are linear combinations of the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$. Thus we have

$$\begin{aligned} d\xi_i/dt &= (p_i\sqrt{-1} + U_i)\xi_i + \cdots, \\ d\eta_i/dt &= (-p_i\sqrt{-1} + V_i)\eta_i + \cdots \qquad (i = 1, \cdots, s), \end{aligned}$$

where the terms not explicitly written are at least of the fourth degree. From these equations we derive

$$\frac{d(\xi_i\eta_i)}{dt} = (U_i + V_i)\xi_i\eta_i + \cdots \qquad (i = 1, \cdots, s),$$

where the omitted terms are at least of the fifth degree.

Now we expand ξ_i, η_i in power series in ϵ as before. To determine the coefficients in this series we proceed precisely as in the case of second order stability and find series in ϵ which we write for $i = 1, \cdots, s$,

$$\begin{aligned} \xi_i &= \epsilon a_i^{(0)} e^{p_i\sqrt{-1}(t-t_0)} + * + \epsilon^3\xi_i^{**} + \cdots, \\ \eta_i &= \epsilon b_i^{(0)} e^{-p_i\sqrt{-1}(t-t_0)} + * + \epsilon^3\eta_i^{**} + \cdots. \end{aligned}$$

1.✓

In virtue of the hypothesis of third order stability and the use of our lemma on trigonometric sums, we conclude that ϕ_i^*, where

$$\phi_i = \xi_i \eta_i = \epsilon^2 a_i^{(0)} b_i^{(0)} + \epsilon^4 \phi_i^* + \cdots,$$

are trigonometric sums.

If now we substitute these different series in the equations above in ϕ_i, and compare terms in ϵ^4 we find simply

$$d\phi_i^*/dt = [U_i(a_1^{(0)} b_1^{(0)}, \cdots,) + V_i(a_1^{(0)} b_1^{(0)}, \cdots)]\, a_i^{(0)} b_i^{(0)}.$$

Consequently unless $U_i + V_i$ vanishes identically for $i = 1, \cdots, s$, the functions ϕ_i^* will not be trigonometric sums but will be linear functions of t, which is not possible.

This gives us the necessary form of prepared equations in the case of third order stability, $n = 2s$,

$$(5) \qquad \begin{cases} d\xi_i/dt = M_i \xi_i + \Phi_i, \\ d\eta_i/dt = -M_i \eta_i + \Psi_i, \qquad (i = 1, \cdots, s), \end{cases}$$

where M_i has a constant term $p_i\sqrt{-1}$ and linear terms in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$, and where Φ_i, Ψ_i begin with of at least fourth degree.

Our first general conclusion, based on a step-by-step process to higher and higher orders, is thus the following.

For $n = 2s$ *a necessary condition that* (B) *may hold for any odd order* m *of stability is that the equations may be transformed to the form* (5) *in which the functions* M_i *are polynomials in the* s *products* $\xi_1\eta_1, \cdots, \xi_s\eta_s$ *of degree* $\frac{1}{2}(m-1)$ *at most, with initial terms* $\sqrt{-1}\,p_i$, *and having pure imaginary coefficients throughout,** *and where* Φ_i, Ψ_i *commence with terms of degree at least* $m+1$. *In this case* Φ_i, Ψ_i *may be made to commence with terms of degree* $m+2$ *at lowest without further assumption.*

To show that this form (5) induces $(m+1)$st order stability (m odd) we obtain first of course

$$\left| \frac{d(\xi_i\eta_i)}{dt} \right| \leqq X u^{m+3},$$

where $u^2 = \xi_1\eta_1 + \cdots + \xi_s\eta_s$ and thence

$$\left| \frac{du}{dt} \right| \leqq \tfrac{1}{2} s X u^{m+2}.$$

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* These are pure imaginary quantities since the variables ξ_i, η_i are self-conjugate, and $M_i\xi_i$ must be conjugate to $-M_i\eta_i$.

Here X is a fixed constant. From this inequality we infer that u remains of the first order in ϵ for the interval $|t-t_0| \leqq L\epsilon^{-(m+1)}$ by the same method already employed for $m=1$.

This gives us the property (A) as a consequence of the prepared form. Now we deduce also from the same inequality

$$|\xi_i\eta_i - \xi_i^{(0)}\eta_i^{(0)}| \leqq kX\epsilon^{m+3}\,|t-t_0|$$

in this same interval. On going back to the prepared form this yields at once

$$|d\xi_i/dt - M_i^{(0)}\xi_i| \leqq K\epsilon^{m+3}\,|t-t_0| + N\epsilon^{m+2}$$

in which $M_i^{(0)}$ designates what M_i becomes when ξ_i, η_i are replaced by $\xi_i^{(0)}$, $\eta_i^{(0)}$. Hence we infer

$$|d(\xi_i e^{-M_i^{(0)}(t-t_0)})/dt| \leqq K\epsilon^{m+3}\,|t-t_0| + N\epsilon^{m+2},$$

and finally

$$|\xi_i - \xi_i^{(0)} e^{M_i^{(0)}(t-t_0)}| \leqq K\epsilon^{m+3}\,\frac{(t-t_0)^2}{2} + N\epsilon^{m+2}\,|t-t_0|$$

in an interval of the order of $\epsilon^{-(m+1)}$.

Hence we infer in passing that property (B) holds as well, and obtain additional properties of the approximate trigonometric representation.

The necessary condition of the prepared form (5) *insures stability in the complete sense that* (A) *and* (B) *hold, and so is sufficient. The explicit solution of the prepared form with terms* Φ_i, Ψ_i *omitted, and with initial values* $\xi_i^{(0)}$, $\eta_i^{(0)}$ *chosen to correspond to the initial values* $x_1^{(0)}, \cdots, x_n^{(0)}$ *of* $x_1, \cdots, x_n$ *yields trigonometric sums by means of the composite transformation which represent* x_i *with an error not exceeding*

$$K\epsilon^{m+3}\,\frac{(t-t_0)^2}{2} + N\epsilon^{m+2}\,|t-t_0|$$

during a time interval

$$|t-t_0| \leqq L\epsilon^{-(m+1)}$$

where ϵ *denotes the initial distance from the origin and where* K, L, N *are positive constants.*

If we define the amplitudes $C_1, \cdots, C_s$ of a solution by means of the prepared form it is clear that we can establish the following result:

The amplitudes $C_1, \cdots, C_s$ *vary slightly in comparison with the total* *initial amplitude* ϵ *during an interval* $|t - t_0| \leqq L^* \epsilon^{-(m+1)}$, L^* *being a suitably chosen positive constant.*

There remains now to refer briefly to the case n odd. It is only the handling of the last equation in x_n that needs to be referred to. In passing from the second order case to the third it is seen at once that the third order terms in this equation may be entirely removed without any assumption concerning third order stability. The third order terms in the other equations can be treated as in the case n even.

In the case of mth order stability, $n = 2s + 1$, *the prepared form is*

$$d\xi_i/dt = M_i \xi_i + \Phi_i, \qquad d\eta_i/dt = -M_i \eta_i + \Psi_i, \qquad (i = 1, \cdots, s) \tag{6}$$
$$dx_n/dt = \mathrm{X},$$

where M_i *are functions of the products* $\xi_i \eta_i$ *and of* x_n *of degree* m *at most in* $\xi_1, \cdots, \eta_s, x_n$, and $\Phi_i, \Psi_i, \mathrm{X}$ *have no terms of degree lower than* $m + 1$. *From this form, the properties* (A) *and* (B), *suitable trigonometric sums, and the slow variation of the amplitudes follow just as in the case* $n = 2s$.

§ 6. *Formal Theory of Complete Stability.*

In this section we shall consider equations (1) of even order $n = 2s$, and for convenience we shall write the equations in the form

$$d\xi_i/dt = \Phi_i(\xi_1, \cdots, \eta_s), \qquad d\eta_i/dt = \Psi_i(\xi_1, \cdots, \eta_s) \qquad (i = 1, \cdots, s) \tag{7}$$

where the conjugate power series Φ_i, Ψ_i are without constant terms and where the first degree terms in Φ_i and Ψ_i are $p_i\sqrt{-1}$ and $-p_i\sqrt{-1}$ respectively. We suppose furthermore that we are confronted by the general case as defined in § 3.

In other words, of all possible equations (1) we are dealing with those typical of first order stability.

It is obvious that any formal power series transformation

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \qquad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s), \tag{8}$$

with $\bar{F}_i$ and $\bar{G}_i$ conjugate series in $\bar{\xi}_i, \bar{\eta}_i$, carries the system into another formal system of the same type, in which $\bar{\Phi}_i, \bar{\Psi}_i$ may not be convergent series, if $\bar{F}_i, \bar{G}_i$ are not. It is apparent that we have here a formal group leaving

(7) invariant, which will be perfectly consistent with itself in the sense that the result of carrying out two successive transformations is the same as that derived by the composite transformation.*

Now as already proved we can remove the second degree terms in Φ_i, Ψ_i by using $\bar{F}_i$, $\bar{G}_i$ which are homogeneous quadratic polynomials. Following this with a transformation in which $\bar{F}_i$, $\bar{G}_i$ are cubic, we can remove all terms of the third degree save those in Φ_i and Ψ_i of the respective types $U_i\xi_i$ and $V_i\xi_i$ where U_i, V_i are linear in the s variables $\xi_1\eta_1, \cdots, \xi_s\eta_s$. All of this goes just as in our discussion of stability in the earlier sections, and we arrive at the following first conclusion by an infinite succession of steps.

Under the formal group the equations (7) *may be given the form*

$$(9) \qquad d\xi_i/dt = U_i\xi_i, \qquad d\eta_i/dt = V_i\eta_i \qquad (i = 1, \cdots, s),$$

where U_i, V_i *are power series in the s products* $\xi_1\eta_1, \cdots, \xi_s\eta_s$.

This result is extremely interesting from the formal point of view for the following reason:

If we write $u_i = \xi_i\eta_i$ *then we have the formal equations in* $u_1, \cdots, u_s$

$$(10) \qquad du_i/dt = [U_i(u_1, \cdots) + V_i(u_1, \cdots)]u_i \qquad (i = 1, \cdots, s),$$

thus yielding an associated formal differential system of one half the order of the original system (7).

We propose next to ask what is the most general transformation of the group (8) carrying the normal form (9) into the normal form. Let us consider terms of the second order in $\bar{F}_i$, $\bar{G}_i$ which alone can modify those of the second order in U_i, V_i. According to the discussion carried out in the treatment of second order stability, these are uniquely determined by the initial and final forms of (9). But in neither are there terms of the second degree. Hence $\bar{F}_i$, $\bar{G}_i$ lack second degree terms.

Pass next to third degree terms in U_i, V_i which are only affected by the terms of equal or less degree in $\bar{F}_i$, $\bar{G}_i$. Here we find as in dealing with third order stability that since $U_i\xi_i$ and $V_i\eta_i$ have only terms of the third degree which involve a factor ξ_i and η_i together with a product $\xi_i\eta_i$ linearly, $\bar{F}_i$, $\bar{G}_i$ contain only terms of the third degree of the same character. Thus we may continue indefinitely, and we infer

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* It is to be observed this group admits of a slight extension in that the initial terms $\bar{\xi}_i, \bar{\eta}_i$ in the series for ξ_i, η_i might be replaced by $c_i\xi_i$, $d_i\eta_i$ respectively ($c_i d_i$ being self-conjugate of course).

The most general transformation (8) *which carries the normal form into itself has the form*

$$(11) \qquad \xi_i = \bar{\xi}_i \bar{F}_i^*, \qquad \eta_i = \bar{\eta}_i \bar{G}_i^* \qquad (i = 1, \cdots, s)$$

where F_i^*, G_i^* *are series in the products* $\xi_1\eta_1, \cdots, \xi_s\eta_s$ *with constant terms* 1. *The form of this transformation indicates that the family of formal 'surfaces'* $Q(u_1, \cdots, u_s) = \text{const.}$ *is invariant under the group.*

The general invariant theory of equations (7) is less simple than that of the special class of equations possessing complete formal stability.

A necessary and sufficient condition for complete stability of a system of equations (7) *is that* $U_i + V_i = 0$ $(i = 1, \cdots, s)$ *so that the equations take a normal form*

$$(12) \qquad d\xi_i/dt = M_i\xi_i, \qquad d\eta_i/dt = -M_i\eta_i \qquad (i = 1, \cdots, s)$$

where $M_1, \cdots, M_s$ *are pure imaginary formal power series in the s products* $\xi_i\eta_i, \cdots, \xi_s\eta_s$ *with initial terms* $p_1\sqrt{-1}, \cdots, p_s\sqrt{-1}$ *respectively.*

For it is precisely a result of the earlier sections that stability of the mth order yields this normal form to the terms of order m. Also the new transformations employed at successive stages $(m = 1, 2, 3, \cdots)$ do not affect terms of lesser degree. Hence we obtain a limiting formal transformation which takes the equations into the given form. Conversely, if there is such a form, the actual transformation obtained from the formal transformation by breaking off at the term of $(m+1)$st degree will yield the same form of equations out to terms of the same degree so that the prepared form is actually obtained, and thus stability of any arbitrary order m is ensured.

These series M_i must be pure imaginary since the products $\xi_i\eta_i$ involved are self-conjugate.

For this system (12) we have the integrals $\xi_i\eta_i = \text{const.}$ Hence in the most general transformation (11) which keeps the normal form (12) unaltered we have $\bar{\xi}_i\bar{\eta}_i = \text{const.}$ also. On this account it is possible to determine the transformed equations at once. We find

$$\frac{d\bar{\xi}_i}{dt} = \frac{1}{\bar{F}_i^*}\,\frac{d\xi_i}{dt} = \frac{M_i\xi_i}{\bar{F}_i^*} = M_i\bar{\xi}_i$$

with like equations in $\bar{\eta}_i$.

The most general transformation (11) *leaves the normal equations* (12) *unaffected except that in each* M_i *the products* $\xi_i\eta_i$ *are replaced by* $\xi_i\eta_i\bar{F}_i^*G_i^*$. *Thus the subgroup for which* $G_i^* = 1/F_i^*$ *leaves the equations unaffected in every respect.*

The question naturally presents itself as to what are the invariants of the equations of completely stable type. The reply is obviously the following:

Reduce the given equations with the property of complete stability to the normal form (12), *thus obtaining* s *associated series*

$$M_1(u_1, \cdots, u_s), \cdots, M_s(u_1, \cdots, u_s).$$

The invariants of the completely stable type of equations will be the invariants of these forms under arbitrary transformations

$$u_i = \bar{u}_i\bar{f}_i \qquad (i = 1, \cdots, s)$$

where the $\bar{f}_i$ *are arbitrary real power series except that they must have a constant term* 1.

In this normal form the equations can be at once integrated precisely because $\xi_i\eta_i = \xi_i^{(0)}\eta_i^{(0)}$ give s integrals for $i = 1, \cdots, s$.

The integrated form of (12) *is*

$$\xi_i = \xi_i^{(0)}e^{M_i^{(0)}(t-t_0)}, \quad \eta_i = \eta_i^{(0)}e^{-M_i^{(0)}(t-t_0)}, \quad (i = 1, \cdots, s). \tag{13}$$

When these expressions are substituted in the formal series giving $x_1, \cdots, x_n$ *in terms of* $\xi_1, \cdots, \eta_s$ *the general formal trigonometric solution is obtained, involving* $\xi_1^{(0)}, \cdots, \eta_s^{(0)}$ *as the* n *arbitrary constants of integration.*

The products $\xi_i\eta_i$ give real power series in the s variables $x_1, \cdots, x_n$ evidently beginning with a quadratic form which is the product of two conjugate linear factors. This result may be stated as follows:

If the equations (1) *have the property of complete stability there exist* s *first integrals* $\phi_i = c_i$ $(i = 1, \cdots, s)$, *given by formal series beginning with a second degree term which is the product of two conjugate linear factors, forming in all* $n = 2s$ *linearly independent linear expressions.*

Of course an arbitrary $Q(u_1, \cdots, u_s) =$ const. also yields a first integral, for any real power series Q.

It remains to say a word about the case of odd order $n = 2s + 1$. It is the normal form that is particularly to be noted. This may be written

$$(12') \qquad d\xi_i/dt = M_i\xi_i, \qquad d\eta_i/dt = -M_i\eta_i, \qquad dx_n/dt = 0 \qquad (i = 1, \cdots, s).$$

Here M_i is a power series in the s products $\xi_1\eta_1, \cdots, \xi_s\eta_s$ and in x_n only. Thus the equations separate into a system of the type obtained in the case of even order together with an equation yielding $x_n = \text{const.}$ for the remaining variable.

The normal form of equation for the completely stable case, $n = 2s + 1$, consists of the equations (12′). These equations display the slight difference between the odd and even order case. In the odd order case there exists a real formal series integral $\phi = \text{const.}$ (corresponding to $x_n = \text{const.}$) containing a first degree term, as well as s integrals $\phi_i = \text{const.}$

§ 7. *Variational Principles and the Equations of Dynamics.*

Let $P_1, \cdots, P_n, Q$ be $n + 1$ real analytic functions $(n = 2s)$ of $x_1, \cdots, x_n$ in a certain open connected continuum under consideration, and let C^* be an arc of a curve with continuously turning tangent lying within this continuum given by

$$x_i = x_i^*(t) \qquad (t_0 \leqq t \leqq t_1)$$

for $i = 1, \cdots, n$, where x_i^* are continuous functions of t with continuous derivatives. In this case the integral

$$(14) \qquad I = \int_{t_0}^{t_1} \{P_1 dx_1 + \cdots + P_n dx_n + Q dt\}$$

has a certain numerical value I^* along the curve C^*. Now suppose that

$$x_i = x_i^*(t, \lambda) \qquad (t_0 \leqq t \leqq t_1)$$

for $i = 1, \cdots, n$ represents any arbitrary one-parameter family of such curves which gives C^* for $\lambda = 0$, and has the same end points as C^* for all values of λ. Suppose furthermore that the functions $x_i^*(t, \lambda)$ are restricted to have continuous partial derivatives as to t and λ. A family

$$x_i = x_i^*(t) + \lambda\, \delta x_i(t) \qquad (i = 1, \cdots, n)$$

will fulfill these conditions if the δx_i are continuous together with their derivatives and vanish at $t = t_0$ and $t = t_1$, while λ is taken sufficiently small. Under these circumstances I becomes a function $I(\lambda)$ when $x_i^*(t, \lambda)$ are substituted for x_i under the sign of integration, and this function $I(\lambda)$ will have a continuous derivative as to λ whose value for $\lambda = 0$ we denote by δI. This variation δI may vanish identically along a curve C^*, i. e., $\delta I^* = 0$, and it turns out by direct inspection of the expression for δI that the condition is precisely that the coordinates $x_1^*, \cdots, x_n^*$ satisfy the set of n Pfaffin differential equations,

$$\sum_{j=1}^{n}\left(\frac{\partial P_i}{\partial x_j}-\frac{\partial P_j}{\partial x_i}\right)\frac{dx_j}{dt}-\frac{\partial Q}{\partial x_i}=0 \qquad (i=1,\cdots,n). \tag{15}$$

All this amounts merely to the statement of some elementary principles in the calculus of variations as applied to a particular and very important type of integral.

On account of the fact that the vanishing of δI along a curve in this sense is obviously invariant of the particular coordinate system, it follows that if we make a change of variables

$$x_i = \phi_i(\bar{x}_1, \cdots, \bar{x}_n) \qquad (i=1,\cdots,n), \tag{16}$$

where the correspondence between $x_1, \cdots, x_n$ and $\bar{x}_1, \cdots, \bar{x}_n$ space is one-to-one and analytic along the curve, the new equations can be obtained by determining $P_1, \cdots, P_n, Q$ by substitution under the integral sign, and then writing down the new corresponding Pfaffian equations. But the formal relation so established can of course be stated and proved directly, without intervention of the principle of variation.

Suppose now that we consider formal (i. e. convergent or divergent) series for P_i, Q in the neighborhood of the origin in $x_1, \cdots, x_n$ space, and also the corresponding formal equations (15), and suppose that we effect the formal transformation (16) in which ϕ_i are given by similar series without constant terms for which the formal Jacobian has a constant term not zero. The new equations can be obtained by the same rule of transformation as before, and in general the result of more than one such transformation is of course the same as that for the composite formal transformation. These statements can be proved by first taking the actual equations obtained by breaking off the series for P_i, Q at terms of high degree m, while actual transformations are obtained by breaking off the series for ϕ_i in the same way. By considering the relations involved as m becomes infinite, the stated formal results follow at once.

I state these facts at some length since it is important to realize that in this way the condition $\delta I = 0$ yields a perfectly definite system of formal differential equations (15) subject to the same formal rules as in the case of actual convergence.

This point of view admits of a slight extension which will come into play in our first theorem. Suppose that P_i, Q are generalized to the respective forms P_i/D, Q/D, where P_i, Q, D are formal power series, but where D need not contain a constant term. However, nevertheless we obtain an extended Pfaffian system

$$(15') \qquad \sum_{j=1}^{n} \left\{ \left[D\frac{\partial P_i}{\partial x_j} - P_i\frac{\partial D}{\partial x_j} \right] - \left[D\frac{\partial P_j}{\partial x_i} - P_j\frac{\partial D}{\partial x_i} \right] \right\} \frac{dx_j}{dt} - D\frac{\partial Q}{\partial x_i} + Q\frac{\partial D}{\partial x_i} = 0.$$

The equations obtained will be independent of any formal factor μ introduced in all of the numerators and denominators of P_i/D, Q/D. Thus the formal significance of the condition $\delta I = 0$ is maintained even in this case.

If the equations (1) *are of completely stable type at the origin, and if* ξ_i, η_i, M_i $(i = 1, \cdots, s)$ *are the series in* $x_1, \cdots, x_n$ *which appear in the normal form* (12), *then the equations* (1) *may be expressed in the extended Pfaffian form*

$$(17) \qquad \delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} \frac{M_j}{\xi_j\eta_j} (\eta_j d\xi_j - \xi_j d\eta_j) - \sum_{j=1}^{s} M_j^2\, dt \right\} = 0.$$

According to what has been said above, it is only necessary to verify that (17) leads to the equations (12) when $\xi_1, \cdots, \eta_s$ are taken as the dependent variables. This may be immediately verified if the conditions $dM_i = 0$ $(i = 1, \cdots, s)$ are equivalent to $d\xi_i\eta_i = 0$, as we shall take them to be. A still more formal attack is to change variables in (12), writing $u_i = M_i$, $v_i = \log \xi_i/\eta_i$ $(i = 1, \cdots, s)$. The condition (17) becomes

$$(17') \qquad \delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} u_j^2\, dt - \sum_{j=1}^{s} u_j^2\, dt \right\} = 0,$$

giving the desired equations

$$du_i/dt = 0, \quad dv_i/dt = 2u_i, \qquad (i = 1, \cdots, s)$$

if we grant the legitimacy of the formal processes involved.

The equations given by (17') are of Hamiltonian form with u_i, v_i as conjugate variables and with principal function $H = \sum_{j=1}^{s} u_j^2$. Or, better still, we can remain within the non-extended type of Hamiltonian equations in case M_i, aside from its constant term which does not effectively enter in (17), is divisible by $\xi_i\eta_i$. In this case $(M_j - p_j\sqrt{-1})/\xi_j\eta_j$ is an admissible power series, and (17) may be modified to the equivalent form

$$\delta \int_{t_0}^{t_1} \left\{ \sum_{j=1}^{s} \sqrt{-1}\,(\bar{\eta}_j d\xi_j - \bar{\xi}_j d\bar{\eta}_j) - \sum_{j=1}^{s} M_j{}^2 dt \right\} = 0,$$

giving equations of the Hamiltonian form, if we take

$$(18) \qquad \bar{\xi}_i = \xi_i \left(\frac{M_i - p_i\sqrt{-1}}{\xi_i\eta_i\sqrt{-1}} \right)^{\frac{1}{2}}, \qquad \bar{\eta}_i = \eta_i \left(\frac{M_i - p_i\sqrt{-1}}{\xi_i\eta_i\sqrt{-1}} \right)^{\frac{1}{2}}$$

as conjugate variables and $\frac{1}{2}\sum_{j=1}^{s} M_j{}^2/\sqrt{-1}$ as principal function. Also, this choice of variables will lie in the original formal group in case M_i contains a term $c_i\xi_i\eta_i$, $c_i \neq 0$, of the first degree. Of course it can contain no other such terms in this case. These results may be summarized as follows.

The equations (1) *of completely stable type can be made to assume the Hamiltonian form, at least if we admit the legitimacy of certain extended formal transformations. In any case if* $M_i - p_i\sqrt{-1}$ *is divisible by* $\xi_i\eta_i$ *for* $i = 1, \cdots, s$, *with linear term* $c_i\xi_i\eta_i$ $(c_i \neq 0)$ *then the equations* (1) *can be made to take Hamiltonian form by means of* (18), *without any use of such extended transformations.*

The last part of this result is obvious from the normal form (12) which becomes of the very special type

$$d\xi_i/dt = (p_i + \xi_i\eta_i)\sqrt{-1}\,\xi_i, \quad d\eta_i/dt = (-p_i - \xi_i\eta_i)\sqrt{-1}\,\eta_i \qquad (i = 1, \cdots, s)$$

if use be made of the special variables (18).

Consider now a Hamiltonian system (1) of general type with equilibrium point at the origin, which can be written

$$\frac{du_i}{dt} = \frac{\partial H}{\partial v_i}, \quad \frac{dv_i}{dt} = -\frac{\partial H}{\partial u_i} \qquad (i = 1, \cdots, s)$$

in which the lowest degree terms in H are

$$\sqrt{-1}\,(p_1u_1v_1 + \cdots + p_su_sv_s)$$

where the numbers p_i arise in the same way as for any equilibrium point with first order stability. The variables u_i, v_i are conjugate imaginary variables of course so that a preliminary linear transformation of the real variables is presupposed.

It is well-known that the transformation for arbitrary K,

$$u_i = \frac{\partial K}{\partial v_i}, \qquad \bar{v}_i = \frac{\partial K}{\partial \bar{u}_i} \qquad (i = 1, \cdots, s)$$

preserves the Hamiltonian form, i. e. that the transformed equations are obtained by replacing u_i, v_i by their series in $\bar{u}_i$, $\bar{v}_i$ in H to obtain $\bar{H}$. We shall assume that

$$K = \bar{u}_1 v_1 + \cdots + \bar{u}_s v_s + \cdots$$

where the terms not explicitly written are of higher than the second degree. If K has only the first term, the transformation becomes the identity

$$u_i = \bar{u}_i, \qquad \bar{v}_i = v_i \qquad (i = 1, \cdots, s).$$

These transformations form a group.

We propose first to take K of the third degree and thus get rid of the third degree terms in H; next to take K as the sum of the first quadratic term and a homogeneous fourth degree term and thus eliminate all the fourth degree terms of H except those quadratic in the s products $u_1 v_1, \cdots, u_s v_s$; and so on. Thus by an indefinite sequence of such transformations we propose to make H a function of the s products $u_1 v_1, \cdots, u_s v_s$.

Now the first of these transformations is

$$u_i = \bar{u}_i + \frac{\partial K^*}{\partial v_i}, \quad \bar{v}_i = v_i + \frac{\partial K^*}{\partial \bar{u}_i} \qquad (i = 1, \cdots, s),$$

in which the variables on the right of each equation are $\bar{u}_i$, v_i, and K^* is a homogeneous cubic in these variables. If we express u_i, v_i in terms of $\bar{u}_i$, $\bar{v}_i$ they take the form

$$u_i = \bar{u}_i + \frac{\partial \bar{K}^*}{\partial \bar{v}_i} + \cdots, \qquad v_i = \bar{v}_i - \frac{\partial \bar{K}^*}{\partial \bar{u}_i} + \cdots,$$

where $\bar{K}^*$ is K^* with the variables $\bar{u}_i$, v_i replaced by $\bar{u}_i$, $\bar{v}_i$, and the complete first and second degree terms are explicitly written.

Consequently $\bar{H}^{(3)}$ as obtained by the usual expansion in Taylor's formula is

$$H^{(3)} - \sum_{j=1}^{s} p_j \sqrt{-1} \left(u_j \frac{\partial K^*}{\partial u_j} - v_j \frac{\partial K^*}{\partial v_j} \right)$$

where $H^{(3)}$ designates the third degree terms in H, and the dashes over the letters are omitted.

Evidently then the homogeneous cubic polynomial K^* can be so determined, and uniquely, that all of the third degree terms disappear in H. In fact a term in K^*

$$qu_1^{l_1}v_1^{m_1}\cdots u_s^{l_s}v_s^{m_s}$$

leads to a similar new term in $\bar{H}$ with coefficient

$$-q\,[\sum_{j=1}^{s} p_j\sqrt{-1}\,(l_j-m_j)]\,,$$

where the term in brackets is not zero for

$$l_1+m_1+\cdots+l_s+m_s=3.$$

This is in accordance with our statement since there can be no third degree term in the products $u_1v_1, \cdots, u_sv_s$.

Another transformation will lead to a modified fourth degree term in H of the same type,

$$H^{(4)}-\sum_{j=1}^{s} p_j\sqrt{-1}\left(u_j\frac{\partial K^*}{\partial u_j}-v_j\frac{\partial K^*}{\partial v_j}\right)$$

where K^* is now the arbitrary homogeneous polynomial of the fourth degree at our disposal. By the aid of it we can eliminate all the fourth degree terms in $H^{(4)}$ except those for which $l_i=m_i$, i. e. the very terms we do not attempt to remove.

By continuing in this way, we are led to the conclusion:

For the general Hamiltonian equations (1) *with first order stability, H can be taken in the form $H(u_1v_1, \cdots, u_sv_s)$. Inasmuch as the Hamiltonian equations then take the form for $i=1,\cdots s$,*

$$\frac{du_i}{dt}=\frac{\partial H}{\partial \pi_i}\,u_i, \qquad \frac{dv_i}{dt}=-\frac{\partial H}{\partial \pi_i}\,v_i \qquad (\pi_i=u_iv_i)$$

of the completely prepared type (12), *it is clear that first order stability necessitates complete stability in the case of the Hamiltonian equations.*

Of course it is well known that the Hamiltonian equations admit a complete formal trigonometric solution in the case of first order stability.

We can now readily attack the question as to the possibility of reducing a completely stable system (1) to Hamiltonian form by means of a transformation belonging to the restricted formal group.

It was proved in the preceding section that the most general transformation of the formal group which kept preserved the completely prepared form (12) was given by (11). It was furthermore seen that $M_1, \cdots, M_s$ are not altered, i. e. that $\overline{M}_i$ is obtained directly from M_i by substitution. But such a substitution changes the variables in $M_1, \cdots, M_s$ from $\pi_1, \cdots, \pi_s$ to $\overline{\pi}_1, \cdots, \overline{\pi}_s$ in accordance with the formulas

$$\pi_i = \overline{\pi}_i \bar{F}_i^* \bar{G}_i^*.$$

Imagine now the particular transformation performed which reduces the completely stable prepared equations (1) to Hamiltonian form, if that be possible. Since $M_1, \cdots, M_s$ are not altered, there must obtain

$$M_i = \partial H / \partial \overline{\pi}_i,$$

or, more briefly,

$$dH = \sum_{i=1}^{s} M_i d\overline{\pi}_i.$$

A necessary condition that such a transformation to Hamiltonian form be possible is therefore that

$$\sum_{j=1}^{s} M_j(\pi_1, \cdots, \pi_s) d\overline{\pi}_j$$

becomes an exact differential for suitable real power series

$$\overline{\pi}_i = \pi_i f_i(\pi_1, \cdots, \pi_s) \qquad (i = 1, \cdots, s)$$

where the constant terms c_i in f_i are not zero.

In particular this requires that the first degree terms in

$$\sum_{j=1}^{s} c_j M_j d\pi_j$$

yield an exact differential. More precisely if this term in M_i is

$$a_{i1}\pi_1 + \cdots + a_{is}\pi_s$$

we must have

$$\frac{a_{ij}}{a_{ji}} = \frac{c_j}{c_i} \qquad (i, j = 1, \cdots, s)$$

which is not in general true for $s > 2$.

The special case first pointed out when transformation to Hamiltonian form is possible, namely when $M_i - p_i\sqrt{-1}$ is divisible by $\xi_i\eta_i$, falls under the type treated with $d\overline{\pi}_i = dM_i/\sqrt{-1}$.

The necessary condition stated is also sufficient for transformation to Hamiltonian form.

In fact choose F_i^*, G_i^* in (11) so that $\overline{\pi}_i = \bar{\xi}_i\bar{\eta}_i$, which is obviously possible in an infinite variety of ways. Then $M_1, \cdots, M_s$ are unaltered, and also

$$M_1 d\bar{\pi}_1 + \cdots + M_s d\bar{\pi}_s = dH.$$

It follows that

$$M_i = \frac{\partial H}{\partial \bar{\pi}_i} \qquad (i = 1, \cdots, s)$$

which is what we desire to prove.

It is apparent in this way that while completely stable systems are of equal generality with the Hamiltonian type in a very wide formal sense, they are more general if only transformations of restricted type are admitted.

We turn now to the consideration of the general Pfaffian type of equation derived from the non-specialized variation problem.

At the equilibrium point the equations of variation of a Pfaffian system (15) are evidently of the form

$$\sum_{j=1}^{n} \left(\alpha_{ij} \frac{dx_j}{dt} + \beta_{ij} x_j \right) = 0$$

where α_{ij} is skew-symmetric (i. e. $\alpha_{ij} = -\alpha_{ji}$) and β_{ij} is symmetric (i. e. $\beta_{ij} = \beta_{ji}$). These are linear differential equations with constant coefficients, the nature of whose solutions depends on the roots of the determinant equation in λ

$$| \alpha_{ij} \lambda_j + \beta_{ij} | = 0.$$

In the case of stability the roots must be zero or pure imaginary, of course, while the elementary divisions of this λ matrix must be simple. We shall assume that there are no linear homogeneous combinations of these roots with integral coefficients which vanish. This is the general case.

In order to obtain a preliminary prepared form for the Pfaffian equations, we note first that the constant terms in P_i and Q can be omitted since they do not enter into the differential equations. Moreover, if the origin is an equilibrium point, there are no linear terms in Q. According to the principle of variation, we may directly transform the part of the integral I containing the linear terms so as to obtain the linear terms with the new variables. But it is known that by linear transformation and omission of an exact differential, the linear terms in P_i and P_j become x_j and $-x_i$, where x_i and x_j are conjugate variables in the sense of the Hamiltonian equations, while simultaneously Q takes the form

$$Q = \sum p_i (x_i^2 + x_j^2) + \cdots,$$

where only the lowest degree terms are written. Now, if we note that $-x_j dx_i$ differs from $x_i dx_j$ by a perfect differential we can write the lowest degree terms under the integral sign as

$$2 \sum x_i dx_j - \sum p_j (x_i^2 + x_j^2) dt$$

where the index i runs over s values while j takes the conjugate index values. But these are essentially the first order terms in the Hamiltonian integral.

At this stage we may introduce the conjugate variables $\xi_i \eta_i$ by setting

$$\xi_i = x_i + \sqrt{-1}\, x_j, \quad \eta_i = x_i - \sqrt{-1}\, x_j\,,$$

where we find the linear terms above take essentially the form

$$\sum_{j=1}^{s} \xi_j d\eta_j,$$

while

$$Q = \sum_{j=1}^{s} p_j \sqrt{-1}\, \xi_j \eta_j + \cdots .$$

It is this preparation of the Pfaffian system in its lowest degree terms that we shall assume to be made, and we shall seek to reduce it further to complete Hamiltonian form by a succession of changes of variables

$$\xi_i = \bar{\xi}_i + \bar{F}_i, \quad \eta_i = \bar{\eta}_i + \bar{G}_i \qquad (i = 1, \cdots, s)$$

where $\bar{F}_i$, $\bar{G}_i$ are conjugate series in $\bar{\xi}_1, \cdots, \bar{\eta}_s$ beginning with terms of at least the second degree.

Let us write the prepared linear differential form as

$$R_1 d\xi_1 + \cdots + R_s d\xi_s + S_1 d\eta_1 + \cdots + S_s d\eta_s + Q dt,$$

and let us seek to eliminate the terms of second degree in $R_1, \cdots, R_s$ by a change of variables of the above type in which $\bar{F}_i$, $\bar{G}_i$ are arbitrary homogeneous quadratic polynomials.

We find immediately that the above linear form is preserved in character while we have

$$\bar{R}_i^{(2)} = R_i^{(2)} + \sum_{j=1}^{s} \xi_j \frac{\partial G_j}{\partial \xi_i} \qquad (i = 1, \cdots, s)$$

if $\bar{R}_i^{(2)}$, $R_i^{(2)}$ denote the second degree terms in $\bar{R}_i$ and R_i respectively. Evidently, by the subtraction of an exact differential, we can replace the sums on the right by the more abbreviated expressions $-G_i$.

Hence there exists a set of G_i which removes the remaining terms in $R_i^{(2)}$, and it is clear at the same time that in this way higher and higher degree terms in R_i, S_i can be disposed of by taking F_i, G_i as homogeneous polynomials of corresponding higher and higher degree. Thus a formal limiting composite transformation is set up which reduces $R_1, \cdots, R_s$ to zero, while taking $S_1, \cdots, S_s$ into $\xi_1, \cdots, \xi_s$ respectively, as desired.

In the general case of first order stability the Pfaffian equations can be transformed formally to Hamiltonian form.

§ 8. *Questions of Convergence.*

The methods which we have been following are based on the use of certain series. Their convergence or divergence is immaterial for the properties which we have so far discussed. Nevertheless the questions of convergence are of great importance for the theoretic treatment of a given dynamical problem.

For example, the Hamiltonian and Pfaffian types of equations (1) admit of the convergent series integral $H = \text{const.}$ and $P = \text{const.}$ respectively, whence in the case of first order stability we can immediately infer the *permanent* complete stability in the sense (A). Or again, there may exist a convergent series P for which

$$\partial/\partial x_1\,(PX_1) + \cdots + \partial/\partial x_n\,(PX_n) = 0$$

in which case $\int P dv$ is an invariant volume integral (dv, element of volume); from the existence of this invariant volume integral, various important properties of recurrence can be deduced.

In the most favorable circumstances the normal form of the equations (12) can be obtained by ordinary transformations defined by convergent series. Here the series M_i and the trigonometric series solutions of the original equations will converge. The precise character of the motion can then be specified for all time. Concerning this integrable case we prove the following simple result:

For integrability of the system (1) *it suffices that the formal integrals*

$$\phi_i = \xi_i \eta_i = \text{const.} \qquad (i = 1, \cdots, s)$$

are given by convergent series.

In fact suppose that these real series do converge. Since they are formally factorable into conjugate factors $\xi_i \eta_i$, the separate components are given by convergent series of course. This follows from Weierstrass's preparation theorem. Hence

$$M_i = \frac{1}{\xi_i} \frac{d\xi_i}{dt} = 1/\xi_i \sum_{j=1}^{n} \frac{\partial \xi_i}{\partial x_j} X_j \qquad (i = 1, \cdots, s)$$

are series given as a sum of formal quotients of two convergent series and so themselves are convergent. This demonstrates the complete integrability of (1) in case the series converge.

The existence of 'small divisors' in the series ξ_i, η_i shows that in general there will be no such convergent series ϕ_i, and the series M_i appear to be divergent for the same reason.

The possibilities of classification of completely stable systems (1) on the basis of diverse convergence properties are limitless. Every system of series or of differential equations which may be attached to (1) so as to have invariantive significance under the general formal group evidently affords a means of classification of systems (1) in which convergence or divergence plays a rôle.

It is with these few remarks that we terminate our reference to convergence questions. While on the purely formal side any completely stable system of equations (1) is to be regarded as equivalent to the extended Pfaffian or Hamiltonian type derived from a variational principle, it is clear that these more special types possess at least one ordinary integral and other convergence properties. In this non-formal direction the complete characterization of the completely stable equations (1) which may be reduced to the classical types still remains to be made.

Quaternary Quadratic Forms Representing all Integers.

By L. E. Dickson.

1. For the case in which only squares of the four variables occur, the forms representing all positive integers were determined by Ramanujan.[*] It is a much more difficult problem to determine the forms which involve also products of the variables whose coefficients may be odd. This paper develops a method of solving this problem completely.

The fact that one of these new forms represents all positive integers is equivalent to the fact that a related form involving only squares shall represent all positive multiples of a fixed integer, such as 4, 8, 12, 28, 44. A very few of the latter facts were known, and their importance has been emphasized in the literature.

The first part of the paper proves the necessary auxiliary theorems on ternary forms. They are useful also in other problems in the theory of numbers.

Part 1. Representation by Ternary Quadratic Forms.

2. Dirichlet[†] gave the simplest proof of the fact that $x^2+y^2+z^2$ represents all positive integers except those of the form $4^k(8n+7)$, but represents no integer of that form. This will be expressed briefly as

$$(1)\qquad x^2+y^2+z^2 \quad \text{represents all} \neq 4^k(8n+7).$$

From (1) we readily deduce[‡]

$$(2)\qquad x^2+2y^2+2z^2 \quad \text{represents all} \neq 4^k(8n+7);$$

$$(3)\qquad x^2+2y^2+4z^2 \quad \text{represents all} \neq 4^k(16n+14).$$

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[*] *Proc. Cambridge Phil. Soc.*, 19, 1916-9, pp. 11-21. Proofs of the needed theorems on ternary forms are given in the writer's paper cited in § 2.

[†] *Journal für Mathematik*, Vol. 40 (1850), p. 228; *Werke*, II, p. 91.

[‡] Dickson, *Bulletin of the American Mathematical Society*, Vol. 33 (1927).

The proof is similar to that now given for

(4) $f = x^2 + 2y^2 + 8z^2$ represents all $\neq 8n + 5$ and $\neq 4^k(8n + 7)$.

If f is even, $x = 2X$, $f = 2F$, $F = 2X^2 + y^2 + 4z^2$. Applying (3), we conclude that f represents all even integers except those of the form $4^k(8n + 7)$. If f is odd, $x^2 \equiv 1$, $2y^2 \equiv 0$ or $2 \pmod 8$, whence $f \equiv 1$ or $3 \pmod 8$. But when $m \equiv 1$ or $3 \pmod 8$, (2) shows that m is represented by $x^2 + 2y^2 + 2z^2$. Then $x^2 \equiv 1$, $2(y^2 + z^2) \equiv 0$ or $2 \pmod 8$, whence y and z are not both odd. Permuting them if necessary, we may assume that $z = 2Z$, whence $m = x^2 + 2y^2 + 8Z^2$.

The writer proved (*loc. cit.*)

(5) $g = x^2 + y^2 + 3z^2$ represents all $\neq 9^k(9n + 6)$.

If g represents $3m$, x and y are multiples of 3, $x = 3X$, $y = 3Y$, and m is represented by $3X^2 + 3Y^2 + z^2$. The converse is true since the product of the latter by 3 is of the form g.

(6) $3x^2 + 3y^2 + z^2$ represents all $\neq 9^k(3n + 2)$.

3. We shall need the following results:

(7) $\lambda = x^2 + xy + y^2 + z^2$ represents all $\neq 9^k(9n + 6)$;

(8) $\mu = x^2 + xy + y^2 + 3z^2$ represents all $\neq 9^k(3n + 2)$;

(9) $\rho = x^2 + y^2 + 2z^2 + xz$ represents all $\neq 49^k(49n + 7e)$, $e = 3, 5, 6$;

(10) $\sigma = x^2 + 2y^2 + 7z^2 - xy$ represents all $\neq 49^k(7n + e)$, $e = 3, 5, 6$.

By the Hessian H of a quadratic form whose cross products have even coefficients we shall mean the determinant of the halves of its second partial derivatives. By Eisenstein's* table of improperly primitive reduced positive ternary quadratic forms of Hessian $\leqq 100$, 2λ, 2μ, 2ρ, 2σ are the only reduced forms with all coefficients even of Hessians 6, 18, 14, 98, respectively.

We employ the form, lacking a term in xy,

(11) $2Ax^2 + 2By^2 + 2Cz^2 + 2Ryz + 2xz$.

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* *Journal für Mathematik*, Vol. 41 (1851), pp. 186-9.

Its Hessian will be $2D$ if

$$B = A\Delta - D, \qquad \Delta = 4BC - R^2. \tag{12}$$

LEMMA 1. *If $p = 3$ or 7 and A is any positive integer not divisible by p, $2A$ is represented by the unique reduced form 2λ or 2ρ of Hessian $2p$ with all coefficients even.*

Here $D = p$. We take $\Delta = 4d$, $d = 2pt + 1$, and choose t so that $B = 4dA - p = 8Apt + 4A - p$ is a prime. This choice is possible in infinitely many ways, since the coefficients of the linear function of t are relatively prime. Since $B \equiv -p \equiv 1 \pmod 4$,

$$(-\Delta/B) = (d/B) = (B/d) = (-p/d) = (d/p) = (1/p) = 1.$$

Hence the last equation (12) is satisfied by integers C and $R = 2r$. Thus (11) is equivalent to 2λ or 2ρ. Hence the latter also represent $2A$.

LEMMA 2. *If $p = 3$ or 7 and A is any positive quadratic residue of p, $2A$ is represented by the unique reduced form 2μ or 2σ of Hessian $2p^2$ with all coefficients even.*

Here $D = p^2$. Take $\Delta = 8pl$, where l is prime to $2p$. By (12), $B = pb$, $b = 8lA - p$. Then b is a prime for infinitely many choices of l. Since $b \equiv -p \equiv 1 \pmod 4$,

$$(l/b) = (b/l) = (-p/l) = (l/p), \quad (p/b) = (b/p) = (2lA/p),$$
$$(-2/b) = (2/b) = (2/p), \quad (A/p) = 1,$$
$$(-\Delta/b) = (-2/b)(p/b)(l/b) = 1.$$

To prove (7), note that, by Lemma 1, λ represents every positive integer prime to 3. Since $\lambda \equiv (x-y)^2 + z^2 \pmod 3$, λ is divisible by 3 if and only if $x - y$ and z are divisible by 3. Write $y = x + 3w$, $z = 3Z$. Then $\lambda = 3L$, $L = x^2 + 3xw + 3w^2 + 3Z^2$. Since $L \equiv 0$ or $1 \pmod 3$, λ represents no number of the form $3(3n+2)$. In L replace x by $X - Y$ and w by Y. We get μ written in capital letters. By Lemma 2, μ represents every positive quadratic residue $3n + 1$ of 3. Thus λ represents every $9n + 3$ and hence its product by 9^k. Finally, if $L \equiv 0$, then $x \equiv 0 \pmod 3$, and x, y, z are all multiples of 3, whence λ is the product of a like form by 9. This proves (7).

Next, (8) follows from (7), since we proved that, when $3m$ is represented by λ, m is represented by L and hence by μ. The converse is true since *

$$3(x^2+xy+y^2)=X^2+XY+Y^2,\quad X=x-y,\quad Y=x+2y.$$

To prove (9), note that, by Lemma 1, ρ represents every positive integer prime to 7. Since $\rho\equiv(x-3z)^2+y^2 \pmod 7$, ρ is divisible by 7 if and only if $x=3z+7X$, $y=7Y$. Then $\rho=7R$, $R=2z^2+7zX+7X^2+7Y^2$. Hence ρ represents none of the integers excluded in (9). For $z=Z-2X$, R becomes $r=2Z^2-ZX+X^2+7Y^2$, which is of type σ in (10). By Lemma 2, σ and hence r represents all positive quadratic residues $7n+1, 2, 4$ of 7. Hence ρ represents their products by $7\cdot 49^k$. If $R\equiv 0$, then $z\equiv x\equiv y\equiv 0 \pmod 7$, and ρ is the product of a like form by 49.

Finally, (10) follows from (9), since we proved that, when $7m$ is represented by ρ, m is represented by R and hence by σ. The converse is true since †

$$7(x^2+2y^2-xy)=X^2+2Y^2-XY,\quad X=x-4y,\quad Y=2x-y.$$

The following results are not required in this paper. Lemma 1 holds also if $p=5$ or 13. In the proof take $d=2pt+\nu$, where ν is odd and a quadratic non-residue of p. Lemma 2 holds also if $p=5$ and A is any non-residue of 5. As before, we prove that

(13) $x^2+y^2+2z^2+xy+xz+yz$ represents all $\neq 25^k(25n\pm 5)$;

(14) $2x^2+2y^2+2z^2-xy-xz-yz$ represents all $\neq 25^k(5n\pm 1)$.

These forms are the only reduced ones whose doubles have the Hessian 10 and 50 respectively.

4. *Forms of Hessian* 6. The reduced forms are

$$\tau=x^2+y^2+6z^2,\quad \omega=x^2+2y^2+3z^2,\quad 2\lambda.$$

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* For if ω is an imaginary cube root of unity, the product of the factor $x-\omega y$ of x^2+xy+y^2 by the corresponding factor $1-\omega$ of 3 is $X-\omega Y$. Also, (8) follows directly from Lemmas 1 and 2. If μ is divisible by 3, $x=y+3X$; write $y=Y-X$. Then μ becomes the triple of a form λ.

† For, if $\epsilon^2+2-\epsilon=0$, the product of the factor $x-\epsilon y$ by the corresponding factor $1-2\epsilon$ of $7=1+2\cdot 2^2-2$ is $X-\epsilon Y$. Also, (10) follows from Lemmas 1 and 2. If σ and hence $(x+3y)^2$ is divisible by 7, $y=2x+7Z$ and $\sigma=7S$, $S=x^2+7xZ+14Z^2+z^2$. For $x=X-3Z$, $z=Y$, S becomes ρ written in capital letters.

Suppose that $\omega \equiv 10 \pmod{16}$. Then $\omega \equiv 2 \pmod{4}$. Thus $\epsilon = x^2 + 3z^2$ is even. Whether x and z are both even or both odd, $\epsilon \equiv 0 \pmod{4}$. Hence y is odd and $10 \equiv \omega \equiv \epsilon + 2$, $\epsilon \equiv 8 \pmod{16}$. If x and z are odd, $\epsilon \equiv 4 \pmod{8}$. Hence $x = 2X$, $z = 2Z$, $X^2 + 3Z \equiv 2$ mod 4), contrary to the above. Thus ω represents* no integer of the form $2(8n + 5)$. This is used in proving

(15) $\quad \tau = x^2 + y^2 + 6z^2$ represents all $\neq 9^k(9n + 3)$.

If τ represents $2m$, then $x + y = 2X$, $x - y = 2Y$, $m = X^2 + Y^2 + 3z^2$. Conversely, the latter implies $2m = (X + Y)^2 + (X - Y)^2 + 6z^2$. Hence by (5) the even integers represented by τ are the doubles of the integers represented by g and include all $\neq 9^k(9m + 3)$, where $m = 2n + 1$. We next prove that τ represents every positive odd integer a not divisible by 3. We employ

$$\kappa = ax^2 + by^2 + cz^2 + 2ryz + 2xz,$$

whose Hessian is $H = a\Delta - b$, where $\Delta = bc - r^2$. Take $H = 6$, $\Delta = 16k$, $k \equiv 1 \pmod{6}$. Then $b = 16ka - 6 = 2\beta$, where $\beta = 8ka - 3$ is a prime for infinitely many choices of k. Thus b is not represented by ω. Since b and the odd a are represented by κ, κ is equivalent to neither ω nor 2λ and hence is equivalent to τ. Since

$$(-\Delta/\beta) = (-k/\beta) = (\beta/k) = (-3/k) = +1,$$

$\Delta = bc - r^2$ is satisfied by integers c and r.

If τ is divisible by 3, $x = 3X$, $y = 3Y$, $\tau = 3T$, $T = 3X^2 + 3Y^2 + 2z^2$, whence τ represents no $3(3n + 1)$.

It remains only to prove that τ represents $a = 3A$, where A is a positive odd integer $3n - 1$. We employ

$$F = ax^2 + 2By^2 + cz^2 + 4Ryz + 2sxz,$$
$$s = 1 + A, \quad B = 8kA - 3, \quad k \equiv 1 \pmod{6},$$

with k chosen so that B is a prime. The Hessian of F is 6 if

$$B + 2AB + A^2B + 3 - 3ABc + 6AR^2 = 0.$$

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* It was proved (*Bulletin*, l. c.) that a positive integer is represented by ω if and only if it is not of the form $4^k(16n + 10)$.

Replace the first term B by its value and cancel A; we get

$$8k + 6R^2 - BP = 0, \quad P = 3c - A - 2.$$

Hence $8k(1 + P) \equiv 0$, $P \equiv -1 \pmod 3$. Hence c is an integer. The congruence $48k + (6R)^2 \equiv 0 \pmod B$ holds if and only if

$$(-48k/B) = (-3k/B) = +1.$$

The latter follows from

$$(k/B) = (B/k) = (-3/k) = (k/3),$$
$$(-3/B) = (B/3) = (8kA/3) = (k/3).$$

5. *Forms of Hessian* 12. We shall prove that

(16) $\delta = 2x^2 + 2y^2 + 3z^2$ represents all $\neq 8n + 1$ and $\neq 9^k(9n + 6)$.

If δ is even, it is the double of a form (15). Henceforth, let z be odd. Evidently $\delta \not\equiv 1 \pmod 8$. Let a be a positive integer such that $a \equiv 3 \pmod 4$, $a \neq 9^k(9n + 6)$. Then a is represented by g in (5). If z is even, $g \equiv 0$, 1 or 2 $\pmod 4$, contrary to $a \equiv 3$. Hence z is odd, $x^2 + y^2 \equiv 0 \pmod 4$, $x = 2X$, $y = 2Y$. Hence a is represented by $2(X + Y)^2 + 2(X - Y)^2 + 3z^2$ and therefore by δ.

It remains only to prove that δ represents every positive integer b such that $b \equiv 5 \pmod 8$, $b \neq 9^k(9n + 6)$. Then b is represented by g in (5). First, let z be odd. Then $x^2 + y^2 \equiv 2 \pmod 4$, whence x and y are odd. Thus $x + y = 2X$, $x - y = 2Y$, and $g = 2X^2 + 2Y^2 + 3z^2$, so that b is represented by δ. Second, let $z = 2\zeta$. Thus $x + y$ is odd. Since we may permute x and y, we may take $x = 2\xi$, y odd. Then $5 \equiv g \equiv 4\xi^2 + 1 + 12\zeta^2 \pmod 8$, and $\xi + \zeta$ is odd. Thus

(17) $$X = \tfrac{1}{2}(x - 3z), \quad Z = \tfrac{1}{2}(x + z)$$

are odd integers and

(18) $$X^2 + 3Z^2 = x^2 + 3z^2.$$

Hence we have a representation of b by g in which the unknowns are now all odd, as in the first case.

We shall need the fact that

(19) $\epsilon = x^2 + 3y^2 + 4z^2$ represents all $\neq 4n + 2$ and $\neq 9^k(9n + 6)$.

If ϵ is even, $x = y + 2X$, $\epsilon = 4\lambda$, where λ is of the form (7), and conversely. Thus (19) holds for even integers. Next, let a be a positive odd integer $\neq 9^k(9n+6)$. Then a is represented by g in (5). Then $x + y + z$ is odd. First, let two of x, y, z be even. Permuting x and y if necessary, we may take x even. Then $g = 4X^2 + y^2 + 3z^2$, whence a is represented by ϵ. Next, let x, y, z be all odd. Changing the sign of x or z if necessary, we may take $x \equiv 1$, $z \equiv -1 \pmod 4$. Then X and Y in (17) are even integers for which (18) holds. Hence the second case reduces to the first.

6. *Corollaries.* We require the results

$$\alpha = 3x^2 + 4y^2 + 12z^2 \text{ represents all } \neq 4n+1, 4n+2, 9^k(12n+11), 9^k(12n+8); \tag{20}$$

$$\beta = x^2 + 3y^2 + 12z^2 \text{ represents all } \neq 4n+2, \quad 9^k(3n+2). \tag{21}$$

If m is represented by β, $3m$ is represented by $3x^2 + Y^2 + 4Z^2$ with $Y = 3y$, $Z = 3z$, and conversely, whence (19) implies (21).

If α is even, $x = 2X$, α is the quadruple of a form (6). Next, let α and hence x be odd. Then $\alpha \equiv 3 \pmod 4$. Any integer represented by α is represented by β with x even. Conversely, if $\beta \equiv 3 \pmod 4$, y is odd and x is even, whence β is of the form α. This proves (20). Finally,

$$\gamma = x^2 + 12y^2 + 12z^2 \text{ represents all } \neq 4n+2, 4n+3, 9^k(12n+5), 9^k(12n+8); \tag{22}$$

$$\nu = 3x^2 + 4y^2 + 4z^2 \text{ represents all } \neq 4n+1, 4n+2, 9^k(36n+15), 9^k(36n+24). \tag{23}$$

If ν is even, ν is the quadruple of a form of type (5). Next, let ν be odd. Then $\nu \equiv 3 \pmod 4$. Write $Y = y + z$, $Z = y - z$. Then ν becomes $\delta = 3x^2 + 2Y^2 + 2Z^2$. Conversely, if $\delta \equiv 3 \pmod 4$, x is odd and $Y + Z$ is even $= 2y$. Write $Y - Z = 2z$. Then δ becomes ν. Applying (16), we get (23).

If γ represents m, then $3m$ is represented by ν with y and z multiples of 3. Conversely, the latter follows if $\nu = 3m$, whence $\gamma = m$. Thus (23) implies (22).

PART 2. QUATERNARY QUADRATIC FORMS.

7. Let τ be a positive ternary quadratic form such that $f = u^2 + \tau$ represents all positive integers. Let m be the minimum positive integer repre-

sented by τ. If $m \geqq 3$, f would not represent 2. Postponing the case $m = 2$, let $m = 1$. Let H be the Hessian of the form 2τ all of whose coefficients are even. Since the minimum of 2τ is 2, a known theorem * states that 2τ is equivalent to a form $2T$ such that $4T - X^2$ is a reduced form $C = Ly^2 + 2Myz + Nz^2$ of Hessian $2H$, where $X = 2x + ty + sz$, while the values of t and s are both chosen from 0, 1. Further, we may assume † that $M \geqq 0$ and omit the case $L = 4$, $M = 2$, $s = 0$. Since the coefficients of $X^2 + C = 4T$ are all multiples of 4, we have

$$(24) \qquad L + t^2 \equiv N + s^2 \equiv 0 \pmod 4, \quad M + ts \equiv 0 \pmod 2.$$

The Hessian $2H = LN - M^2$ of C is therefore a multiple of 4, whence H is even, $H = 2h$. By hypothesis, $f = u^2 + T$ represents all positive integers. Hence

$$F = 4f = 4u^2 + X^2 + C$$

represents all positive multiples of 4. Since C is reduced, its minimum > 0 is L. If $L > 12$, $F = 12$ therefore requires that $C = 0$. Since 12 is not a sum of two squares, this proves that $L \leqq 12$. Since $t = 0$ or 1, (24) gives $L \equiv 0$ or $-1 \pmod 4$. Hence the only possible values of L are 4, 8, 12, 3, 7, 11. The least positive multiple of 4 not represented by $4u^2 + X^2 + Ly^2$ is $l = 28$, 56, 24, 24, 84, 88 for the corresponding value 4, $\cdots$, 11 of L. Since F represents l, we have $C \leqq l$ for certain integers y, z, $z \neq 0$. Then

$$(25) \qquad LC \equiv (Ly + Mz)^2 + 4hz^2$$

implies that

$$(26) \qquad 4h \leqq Ll.$$

Hence there is a limited number of values of h and hence of reduced forms C. The number of forms T is therefore finite.

8. The simplest case is $L = 4$. Since C is reduced, $2M \leqq L$, $M = 0$, 1, 2. By (24), $t = 0$, $M \equiv 0 \pmod 2$, and $N + s^2$ is divisible by 4.

First, let $s = 0$. If $M = 2$, our ternary form is known to be equivalent to one with $s = 1$. Hence there remains here the case $M = 0$. By (26),

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* Landau, *Handbuch ... Verteilung der Primzahlen*, 1909, p. 545.

† Eisenstein, *Journal für Mathematik*, 41, 1851, 141-90.

$h \leqq 28$. Since $h = N$ is a multiple $4m$ of 4, $1 \leqq m \leqq 7$, and $f = u^2 + x^2 + y^2 + mz^2$ represents every positive integer (Ramanujan).

Second, let $s = 1$. We may write $N = 4m - 1$. There are two cases.

(i) $M = 0$. Then $h = N$, $h \leqq 28$, $1 \leqq m \leqq 7$,

$$(27_1) \qquad 4f = 4u^2 + X^2 + 4y^2 + Nz^2, \quad X = 2x + z \equiv z \pmod 2.$$

This is of the form $F = U^2 + X^2 + Y^2 + Nz^2$. Conversely, if F is a multiple of 4, either z is even and U, X, Y are all even, or z is odd and just one of them is odd. Permuting them, we may take X odd, whence U and Y are even. In both cases, $U = 2u$, $Y = 2y$, $X \equiv z \pmod 2$, and we have (27_1). To show that F represents all multiples of 4, we apply (1). Let $a = 4^k(8n + 7)$ be a multiple of 4, whence $k \geqq 1$. For $z = 2^{k-1}$, $a - Nz^2 = 4^{k-1}P$, where $P = 32n + 28 - N \equiv 1 \pmod 4$. Thus P is positive and a sum of three squares.

(ii) $M = 2$. Then $h = N - 1$, $h \leqq 28$, $1 \leqq m \leqq 7$,

$$(27_2) \qquad 4f = U^2 + X^2 + W^2 + (4m - 2)z^2, \quad U = 2u, \quad X = 2x + z, \quad W = 2y + z.$$

Conversely, if this form is a multiple of 4, either z, U, X, W are all even, or z is odd, and just one of U, X, W is even. Permuting them, we may take U even, $X \equiv z$, $W \equiv z \pmod 2$, and deduce (27_2). To show that it represents all multiples of 4, take $z = 2^{k-1}$. Then

$$a - (4m - 2)z^2 = 4^{k-1}Q, \quad Q = 32n + 28 - 4m + 2 \equiv 2 \pmod 4, \ Q > 0.$$

Hence Q is a sum of three squares.

THEOREM 1. For $1 \leqq m \leqq 7$, $f_1 = u^2 + y^2 + x^2 + xz + mz^2$ *and* $f_2 = f_1 + yz$ *both represent all integers. Expressed otherwise,* $X^2 + Y^2 + U^2 + (4m - 1)z^2$ *and* $X^2 + W^2 + U^2 + (4m - 2)z^2$ *both represent all positive multiples of* 4.

9. *The case* $L = 8$. By (24), $t = 0$, M is even, and $N = 4k - s$. Write $M = 2\mu$. By $2M \leqq L$, $\mu \leqq 2$. We postpone to § 10 the case $\mu = 1$. Here $\mu = 2\nu$, $\nu = 0$ or 1. Then $4f = F$,

$$(28) \qquad F=\phi+\sigma z^2, \quad \phi\equiv 4u^2+X^2+2W^2, \quad \sigma=4k-s-2\nu^2,$$

where $X=2x+sz$, $W=2y+\nu z$. Thus

$$(29) \qquad X\equiv sz, \quad W\equiv \nu z \pmod 2.$$

Conversely, if F represents a multiple of 4, (29) hold. Evidently $X\equiv\sigma z\equiv sz \pmod 2$. If z is even, X and W are even. If z is odd, $s^2+2W^2+\sigma\equiv 0$, $2W^2-2\nu^2\equiv 0 \pmod 4$, whence $W\equiv\nu\equiv\nu z \pmod 2$.

We have $8N-M^2=4h\leqq 8\cdot 56$ by (26), whence $\sigma=N-2\nu^2\leqq 56$. Also, $N\geqq L$. Hence $2\leqq k\leqq 14$. By 3, ϕ represents exclusively all positive integers except $4^e(16n+14)$. If the latter is a multiple of 4, it is $a=4^p(64n+56)$. For $z=2^p$, F represents a if ϕ represents

$$a-4^p\sigma=4^pq, \quad q=64n+56-\sigma\geqq 0.$$

If $s=1$, σ and q are odd, and 4^pq is represented by ϕ. If $s=\nu=0$, (28) gives $f=u^2+x^2+2y^2+kz^2$, which represents all positive integers (Ramanujan). Finally, let $s=0$, $\nu=1$. Then $q\equiv 14 \pmod{16}$ only when $k\equiv -1 \pmod 4$, whence $k=3, 7, 11$; $\sigma=10, 26, 42$, respectively. Except in these cases 4^pq is represented by ϕ. For $z=2^{p+1}$,

$$B=a-\sigma z^2=4^{p+1}b, \quad b=16n+14-\sigma.$$

If $\sigma=10$, $b=4(4n+1)$, and B is represented by ϕ. If $\sigma=26$, $b=16(n-1)+4$ is the quadruple of an odd integer; if $n>0$, B is represented by ϕ. For $\sigma=26$, $n=0$, $a=4^p\cdot 56$. If $p>0$, take $z=2^{p-1}$; then $a-\sigma z^2$ is the product of 4^{p-1} by $198\equiv 6 \pmod{16}$. Hence unless a is 56, it is represented by F, while 56 is not. Finally, let $\sigma=42$. Then $b=16(n-2)+4>0$ if $n>1$. For $n=0$ or 1 and $p>0$, take $z=2^{p-1}$; then $a-\sigma z^2$ is the product of 4^{p-1} by 182 or 438, each $\equiv 6 \pmod{16}$. Hence unless a is 56 or $64+56$, it is represented by F, while they are not.

THEOREM 2. *For* $2\leqq k\leqq 14$, $s=0$ *or* 1, $\nu=0$ *or* 1,

$$f=u^2+x^2+sxz+2y^2+2\nu yz+kz^2$$

represents all positive integers except 14 *when* $k=7$ *and* 14 *and* 30 *when* $k=11$, *with* $s=0$, $\nu=1$ *for all exceptions. Expressed otherwise, the form* (28) *with only square terms represents all positive multiples of* 4 *except* 56 *and* 120 *in the cases mentioned.*

10. *The case* $L=8$, $M=2$. Now $8f=G$,

(30) $$G=\psi+hz^2,\quad \psi=2X^2+W^2+8u^2,\quad h=2N-1,\quad N=4k-s,$$

where $X=2x+sz$, $W=4y+z$. Hence

(31) $$X\equiv sz \pmod 2,\quad W\equiv z \pmod 4.$$

Conversely, the latter follow if G represents a multiple of 8. Evidently $W\equiv z$ (mod 2). If z is odd, $2X^2+1+h\equiv 0$ (mod 4), whence X^2+N is even and $X\equiv s\equiv sz$ (mod 2). If $W-z=4r+2$, $W+z$ is divisible by 4; after changing the sign of z if necessary, $W-z$ is divisible by 4. Next, let z be even, $z=2\zeta$. Then W and X are even, $W=2\omega$. Then

$$G\equiv 4(\omega^2+h\zeta^2)\equiv 0 \pmod 8,\quad \omega\equiv\zeta \pmod 2.$$

Thus (31) hold.

By (4), any multiple of 8 not represented by ψ is of the form $a=4^e\cdot 16(8n+7)$. Take $z=2^e$. Since $h\leqq 112$ by (26), $a-hz^2$ is the product of 4^e by a positive integer $\equiv 2s+1\equiv 1$ or 3 (mod 8), and hence is represented by ψ.

THEOREM 3. *For* $2\leqq k\leqq 14$, $s=0$ *or* 1,

$$f=u^2+x^2+sxz+2y^2+yz+kz^2$$

represents all positive integers. In other words, (30) *represents all positive multiples of* 8.

11. *The case* $L=3$. Then $M=0$ or 1. By (24), $t=1$, $M=s$, $N=4k-s$. Then $12f=F$,

(32) $$F=\phi+4hz^2,\quad \phi=3U^2+3X^2+W^2,\quad h=3k-s,$$

where $U=2u$, $X=2x+y+sz$, $W=3y+sz$. Conversely, if F represents a multiple of 12, we may write

$$U\equiv 0,\quad X\equiv W \pmod 2,\quad W\equiv sz \pmod 3.$$

By (6), any multiple of 12 not represented by ϕ is of the form $a=9^e\cdot 9(12r+8)$. By (26), $h\leqq 18$. Take $z=3^e$. Then $a-4hz^2$ is the product of 9^e by $g=9\cdot 12r+72-4h\geqq 0$. Thus $g\equiv s$ (mod 3). If

$s=1$, $G=9^e g$ is represented by ϕ. Next, let $s=0$. If k is not divisible by 3, g is divisible by 3, but not by 9, whence G is represented by ϕ. For $k=3$, $g=9(12r+4)$ and G is represented by ϕ. For $k=6$, $g=9\cdot 12r$ and G is repreesnted by ϕ if r is not divisible by 3. When r is divisible by 3, take $z=2\cdot 3^c$, whence $a-4hz^2=9^{e+1}\cdot 12(r-2)$ is represented by ϕ.

THEOREM 4. *For* $1\leqq k\leqq 6$, $s=0$ *or* 1,

$$u^2+x^2+xy+sxz+y^2+syz+kz^2$$

represents all positive integers. In other words, the form (32) *represents all positive multiples of* 12.

12. *L odd.* By (24), $t=1$, $N=4k-s$, and $M+s$ is even. Write

$$U=2u,\quad X=2x+y+sz,\quad W=Ly+Mz.$$

Using (25), we get

$$K=4Lf=LU^2+LX^2+W^2+4hz^2, \tag{33}$$

$$U\equiv 0,\quad W\equiv X \pmod 2),\quad W\equiv Mz \pmod L). \tag{34}$$

Conversely, let K be divisible by $4L$. Since $4h=LN-M^2$, $W^2-M^2z^2\equiv 0 \pmod L$. Changing the sign of z if necessary, and noting that $L=3$, 7, or 11 is a prime, we get (34_3). Since $L\equiv -1 \pmod 4$, $K\equiv 0$ requires that either W, U, X are all even or that W is odd and just one of U and X is even. Permuting the latter if necessary, we may take U even, X odd. In both cases the first two congruences (34) hold. Hence f represents all positive integers if and only if K represents all positive multiples of $4L$. Write

$$W=X+2\xi,\quad L=4\lambda-1.$$

Then we may remove the factor 4 from (33) and get

$$F=Lf=\phi+hz^2,\quad \phi=Lu^2+\lambda X^2+X\xi+\xi^2. \tag{35}$$

Conversely, if F is divisible by L,

$$(X+2\xi)^2\equiv M^2z^2 \pmod L),\quad X+2\xi=Mz+Ly=W,$$

after changing the sign of z if necessary. Hence f represents all positive integers if and only if F represents all positive multiples of L. For $L=3$,

ϕ is of type (8) and we may give a new proof of Theorem 4 by use of (35).

We have

$$(36)\qquad h = Lk - \tau, \quad \tau = \tfrac{1}{4}(Ls + M^2), \quad 0 \leqq M \leqq \tfrac{1}{2}L.$$

13. *The case* $L = 7$. Since ϕ is of type (10), any multiple of 7 not represented by ϕ is of the form $a = 49^e \cdot 49(7n + \nu)$, where $\nu = 3$, 5, or 6 is a non-residue of 7. By (26), $h \leqq 7 \cdot 21$. By (36), $h = 7k - \tau$, where $\tau = 0$, 1, 2, or 4. Thus $2 \leqq k \leqq 21$. Take $z = 7^e$. Then

$$a - hz^2 = G = 49^e g, \quad g = 49(7n + \nu) - 7k + \tau \geqq 0.$$

Hence G is represented by ϕ if $\tau \neq 0$, or if $\tau = 0$ and k is prime to 7. Henceforth, let $\tau = 0$, $k = 7\kappa$, $\kappa = 1$, 2, or 3. Then $G = 49^{e+1}\gamma$, where $\gamma = 7n + \nu - \kappa$. Next, take $z = 2 \cdot 7^e$. Then

$$a - hz^2 = D = 49^{e+1}\delta, \ \delta = 7n + \nu - 4\kappa.$$

If $\nu = 6$, $\gamma = 7n + 2$ for $\kappa = 2$, $\delta = 7n + 2$ for $\kappa = 1$,
$\delta = 7(n-1) + 1$ for $\kappa = 3$.

If $\nu = 5$, $\gamma = 7n + 4$ for $\kappa = 1$, $\gamma = 7n + 2$ for $\kappa = 3$,
$\delta = 7(n-1) + 4$ for $\kappa = 2$.

If $\nu = 3$, $\delta = 7n + 2$ or $7n + 1$ for $\kappa = 1$ or 2.

In all these cases, G or D is represented by ϕ if it is positive. It is negative only when $n = 0$, $\nu = 6$, $\kappa = 3$, or $\nu = 5$, $\kappa = 2$. If $e > 0$ in the former case, $z = 7^{e-1}$ gives $a - hz^2 = 49^e(49 \cdot 6 - 3)$, which is represented by ϕ. But if $e = 0$, $a = 49 \cdot 6$ is not represented by ϕ, since $a - hz^2 = 49(6 - 3z^2)$ is positive only when $z^2 = 0$ or 1 and then the second factor is a non-residue of 7. In the second of the exceptional cases, $a = 5 \cdot 49^{e+1}$, $h = 2 \cdot 49$. We may take $z = \epsilon 7^m$, ϵ prime to 7. According as $m \geqq e$ or $m < e$,

$$a - hz^2 = 49^{e+1}(5 - 2\epsilon^2\, 49^{m-e}), \quad 49^{m+1}(5 \cdot 49^{e-m} - 2\epsilon^2),$$

and the second factors are non-residues of 7 if positive.

Finally, let $\nu = \kappa = 3$. Then G is represented by ϕ unless n is a positive multiple of 7. In that case, take $z = 3 \cdot 7^e$. Then

$$a - hz^2 = 49^{e+1}m, \quad m = 7n + 3 - 27 = 7(n-4) + 4 > 0,$$

and the former is represented by ϕ.

THEOREM 5. *For* $2 \leqq k \leqq 21$; $s = 0, 1$; $M = 0, 1, 2, 3$; $M + s$ *even,*

$$u^2 + x^2 + xy + sxz + 2y^2 + \tfrac{1}{2}(M + s)yz + ks^2$$

represents all positive integers except 42 *when* $k = 21$, $s = M = 0$, *and* $35 \cdot 49^e$ *when* $k = 14$, $s = M = 0$. *In other words, for* h *in* (36) *and* $L = 7$, *the form* (33) *represents all positive multiples of* 28 *except* $28 \cdot 42$ *and* $28 \cdot 35 \cdot 49^e$ *in the cases mentioned.*

14. *The case* $L = 11$. Then ϕ in (35) represents all positive integers except $121^k(11n + \nu)$, where ν ranges over the non-residues 2, 6, 7, 8, 10 of 11. Let it be a multiple $a = 121^e \cdot 121\ (11n + \nu)$ of 11. By (36), $h = 11k - \tau$, where τ is 0 or one of the quadratic residues 1, 3, 4, 5, 9 of 11. By (26), $h \leqq 2 \cdot 11^2$. Thus $3 \leqq k \leqq 22$. Take $z = 11^e$. Then

$$a - hz^2 = G = 121^e g,\quad g = 121(11n + \nu) - 11k + \tau \geqq 0.$$

Thus G is represented by ϕ if $\tau \neq 0$, or if $\tau = 0$ and k is prime to 11. Henceforth, let $\tau = 0$, $k = 11\kappa$, $\kappa = 1$ or 2. Then $G = 121^{e+1}\gamma$, $\gamma = 11n + \nu - \kappa$. Next, take $z = 2 \cdot 11^e$. Then

$$a - hz^2 = D = 121^{e+1}\delta,\quad \delta = 11n + \nu - 4\kappa.$$

First, let $\kappa = 1$. For $\nu = 2$ or 6, $\gamma = 11n + 4$ or $11n + 5$. For $\nu = 7$ or 8, $\delta = 11n + 3$ or $11n + 4$. Hence G or D is represented by ϕ.

Next, let $\kappa = 2$. For $\nu = 6$ or 7, $\gamma = 11n + 1$ or $11n + 5$. For $\nu = 2$, $\gamma = 11n$ and G is represented by ϕ if $n = 0$, while if $n > 0$, $\delta = 11(n - 1) + 5 > 0$ and D is represented. Finally, let $\nu = 8$. If $n = 0$, $D = 0$.

If $n > 0$, take $z = 3 \cdot 11^e$. Then ϕ represents

$$a - hz^2 = 121^{e+1}[11(n - 1) + 1].$$

THEOREM 6. *For* $3 \leqq k \leqq 22$; $s = 0, 1$; $M = 0, 1, 2, 3, 4, 5$; $M + s$ *even,*

$$u^2 + x^2 + xy + sxz + 3y^2 + \tfrac{1}{2}(M + s)yz + kz^2$$

represents all positive integers. In other words, for h *in* (36), $L = 11$, *the form* (33) *represents all multiples of* 44.

15. *The case* $L = 12$. By (24), $t = 0$, $N = 4k - s$, $M = 2\mu$, $0 \leqq \mu \leqq 3$. Also, $h = 3N - \mu^2$.

$$12f = 12u^2 + 3X^2 + W^2 + hz^2,\quad X = 2x + sz,\quad W = 6y + \mu z. \tag{37}$$

However this form is divisible by 4 also when z is even, and X and W are odd. Since the usual converse argument fails here, we apply devices appropriate to the necessary sub-cases.

First, let μ be even, $\mu = 2\nu$, whence $\nu = 0$ or 1. In (37), $W = 2Y$, $Y = 3y + \nu z$, whence

$$F = 12f = a + hz^2, \quad a = 12u^2 + 3X^2 + 4Y^2, \qquad X \equiv sz \pmod{2}, \quad Y \equiv \nu z \pmod{3}. \tag{38}$$

Conversely, the latter follow if F is divisible by 12.

We have $h = 12k - \tau$, where $\tau = 3s + 4\nu^2$ takes the values 0, 3, 4, 7. By (26) and $N \geqq L$, $3 \leqq k \leqq 6$. By (20), a represents all multiples of 12 except $a = 9^e \cdot 9(12n + 8)$. For $z = 3^e$,

$$D = a - hz^2 = 9^e d, \quad d = 9 \cdot 12n + 72 - 12k + \tau \geqq 0.$$

If $\tau = 7$, $d \equiv 7 \pmod{12}$, and D is represented by a.

Let $\tau = 3$. Then d is divisible by 9 only if $k \equiv 1 \pmod{3}$, whence $k = 4$. If $k \neq 4$, $d \equiv 3 \pmod{12}$, and D is represented by a. If $k = 4$, $h = 45$, $d = 9(12n + 3)$, and D is represented unless $12n + 3$ is divisible by 9, whence $n = 3m + 2$. In the latter case, take $z = 2 \cdot 3^e$. Then a represents

$$a - hz^2 = 9^{e+1} \cdot 12(3m + 1).$$

If $\tau = 4$, $d \equiv 4 \pmod{12}$, and D is represented by a.

Let $\tau = 0$. If k is prime to 3, d is divisible by 12, but not by 9, when a represents D. If $k = 3$, $d = 9(12n + 4)$, and D is represented. Finally, let $k = 6$. Then $D = 9^{e+1} \cdot 12n$ is represented unless $n = 3m$, $m > 0$. Take $z = 2 \cdot 3^e$. Then a represents

$$a - hz^2 = 9^{e+1} \cdot 12(3m - 2) > 0.$$

THEOREM 7. *For* $k = 3, 4, 5, 6$; $s = 0, 1$; $\nu = 0, 1$.

$$u^2 + x^2 + sxz + 3y^2 + 2\nu yz + kz^2$$

represents all positive integers. In other words, (38) *represents all positive multiples of* 12.

16. $L = 12$, *μ odd*, $s = 0$. By (37),

$$12f = F = \gamma + hz^2, \quad \gamma = 12u^2 + 12x^2 + W^2, \quad h = 12k - \mu^2, \tag{39}$$

where $W = 6y + \mu z$. Conversely, if F is a multiple of 12, we may write $W \equiv \mu z \pmod{6}$. Since $N = 4k \geqq 12$, $h \leqq 3 \cdot 24$, $k = 3, 4, 5,$ or 6. By (22), γ represents all multiples of 12 except $a = 9^e \cdot 9(12n + 8)$. For $z = 3^e$,

$$D = a - hz^2 = 9^e d, \quad d = 9(12n + 8) - 12k + \mu^2 \geqq 0.$$

If $\mu = 1$, then $d \equiv 1 \pmod{12}$, and γ represents D.

Let $\mu = 3$. If $k = 4$ or 5, then $d \equiv 9 \pmod{12}$ and d is not divisible by 9, whence γ represents D. If $k = 6$, then $d = 9(12n + 1)$ and γ represents D. If $k = 3$, $e > 0$, take $z = 3^{e-1}$; then γ represents $a - hz^2 = 9^e g$, where $g = 9(12n + 8) - 3 \equiv 9 \pmod{12}$. There remains only the case $\mu = k = 3$, $e = 0$. Then $a = 9(12n + 8)$ is not represented by F. For, $a - hz^2 = 9\delta$, $\delta = 12n + 8 - 3z^2$. According as z is odd or even, $\delta \equiv 5$ or $8 \pmod{12}$, whence 9δ is not represented by γ.

THEOREM 8. *For $k = 3, 4, 5, 6$; $\mu = 1, 3$, $u^2 + x^2 + 3y^2 + \mu yz + kz^2$ represents all positive integers except those of the form $9n + 6$ when $k = \mu = 3$. In other words,* (39) *represents all positive multiples of* 12 *except* $12(9n + 6)$ *when $k = \mu = 3$.*

17. $L = 12$, $\mu = s = 1$. Then $4k - 1 = N \geqq 12$, $k \geqq 4$, $h = 12k - 4 \leqq 3 \cdot 24$, $k \leqq 6$. Hence

$$f = u^2 + x^2 + xz + 3y^2 + yz + kz^2 \qquad (k = 4, 5, 6)\ldots \tag{40}$$

We first determine the integers represented by f when z is even, $z = 2Z$. Write $X = x + Z$, $Y = 3y + Z$. Then $3f = F$,

$$F = \phi + hZ^2, \quad \phi = 3u^2 + 3X^2 + Y^2, \quad Y \equiv Z \pmod{3}. \tag{41}$$

Conversely, $F \equiv 0$ implies $Y^2 - Z^2 \equiv 0 \pmod{3}$. Changing the sign of Z if necessary, we have $Y \equiv Z$. We shall prove that F represents all positive multiples of 3 except 18, and, when $k = 5$ or 6, except 45. By (6), ϕ represents exclusively all positive integers except $9^k(3n + 2)$. If the latter be a multiple of 3, it is $a = 9^e \cdot 9(3n + 2)$. If $e > 0$, take $Z = 3^{e-1}$; then

$$a - hZ^2 = D = 9^{e-1} d, \quad d = 9^2(3n + 2) - h > 0, \quad d \equiv 1 \pmod{3}.$$

Hence ϕ represents D, and F represents a if $e > 0$. Next, let $e = 0$. Take $Z = 1$. Then $a - hZ^2 = 27n + 18 - h \equiv 1 \pmod{3}$ and is positive if $n > 0$ when $k = 4$, $n > 1$ when $k = 5$ or 6.

For the exceptional cases we employ (40) with $y=0$, $z=1$, whence $f=u^2+x^2+x+k$. We have

$f=6$ if

k	u	x
4	0	1
5	1	0
6	0	0

$f=15$ if

k	u	x
5	2	2
6	3	0

THEOREM 9. *The forms* (40) *represent all positive integers.*

This and Theorem 10 are equivalent to the fact that (37) subject to the condition $X \equiv z \pmod 2$ represents all multiples of 12 apart from the exceptions noted.

18. $L=12$, $s=1$, $\mu=3$. Then $h=12(k-1) \leqq 3 \cdot 24$, $k \leqq 7$. Thus

$$f=u^2+x^2+xz+3y^2+3yz+kz^2 \qquad (k=4,5,6,7). \tag{42}$$

We first determine the integers represented by f when $z=2Z$. Write $X=x+Z$, $Y=y+Z$. Then f becomes

$$F=\phi+mZ^2, \quad \phi=u^2+X^2+3Y^2, \quad m=4(k-1). \tag{43}$$

By (5), ϕ represents exclusively all positive integers except $a=9^e(9n+6)$. For $Z=3^e$, $D=a-mZ^2=9^e d$; $d=9n+6-m$. When $k=4$, 5, 6, $d\equiv 3, 8, 4 \pmod 9$ and D is represented by ϕ unless it be negative, i. e., unless $n=0$ for $k=4$, $n=0$ or 1 for $k=5$, 6. Next, for $e>0$, take $Z=3^{e-1}$. Then

$$G=a-mZ^2=9^{e-1}g, \quad g=9(9n+6)-m>0.$$

When $k=5, 6, 7$, $g\equiv 2, 7, 3 \pmod 9$, and G is represented by ϕ.

It remains to consider only the following cases:

(i) $k=4$, $a=9^e\cdot 6$. Then* $f=a$ for $x=z=3^e$, $u=y=0$.

(ii) $k=5$, $a=6$ or 15. Neither a is represented by F, since $m=16$. For $y=0$, $z=1$, $f=u^2+x^2+x+5$. Then $f=6$ if $u=1$, $x=0$; $f=15$ if $u=x=2$.

(iii) $k=6$, $a=6$ or 15. Neither a is represented by F, since $m=20$. For $x=y=0$, $z=1$, $f=u^2+6=6$ or 15 if $u=0$ or 3.

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* But F does not represent any such a. For, if $Z=3^m\epsilon$, ϵ prime to 3, $P=a-12Z^2$ is negative if $m \geqq e$. If $m<e$, $P=9^m p$, $p=9^{e-m}\cdot 6-12\epsilon^2 \equiv 6 \pmod 9$.

(iv) $k=7$, $a=9n+6$. For $Z=1$, $a-24Z^2=9(n-2)$ is represented by ϕ if $n \geqq 2$ and $n-2 \not\equiv 0$ or 6 (mod 9) and hence if $n \not\equiv 2$ or 8 (mod 9). Similarly, for $Z=2$, $a-24Z^2=9(n-10)$ is represented by ϕ if $n \geqq 10$ and $n \not\equiv 1$ or 7 (mod 9). Hence a is represented by F unless $n=0$, 1, or 8; whence $a=6$, 15, or 78. We enquire whether $12a$ is represented by (37), viz., $B=\beta+72z^2$, with $X-z$ even, where $\beta=12u^2+3X^2+W^2$.

By (21), $\beta \neq 72$, whence $z^2=1$, $X=0$, and $X-z$ is odd. Thus (42) does not represent 6. Next, $B=12 \cdot 15$ for $u=0$, $z=1$, $X=3$, $W=9$, whence $f=15$ for $u=0$, $x=y=z=1$. Finally, $B=12 \cdot 78$ for $u=3$, $z=1$, $X=3$, $W=27$, whence $f=78$ for $u=3$, $x=z=1$, $y=4$.

THEOREM 10. *The forms* (42) *represent all positive integers with the exception of* 6 *when* $k=7$.

19. By way of summary, we note that Theorems 1-10 give 329 forms which represent all positive integers, and six forms which represent all but one or two small integers. We have not included the known forms u^2+T which represent all positive integers, where T denotes

$$x^2+y^2+sz^2 (s \leqq 7), \qquad x^2+2y^2+sz^2 \ (2 \leqq s \leqq 14),$$
$$x^2+3y^2+sz^2 \ (s=3,\ 4,\ 5,\ 6).$$

Including the latter 24, we obtain

THEOREM 11. *There are exactly* 353 *classes of positive ternary forms* T *whose minimum is* 1 *such that* u^2+T *is a quaternary form representing all positive integers.*

Similar methods apply in the remaining case $m=2$ (§ 7), and the number of reduced forms T is again finite. The results will be published on another occasion.

Reduction Formulas for the Number of Representations of Integers in Certain Quadratic Forms.

By E. T. Bell.

1. *Introduction.* We shall illustrate by its application to three interesting formulas, summarized in § 3, an algebraic method whereby, knowing the number of representations of an arbitrary integer in a given quadratic form in $r \geqq 2$ indeterminates, we can obtain a finite reduction formula for the number of representations in a quadratic form in rs indeterminates, $s > 1$. The formulas in any case can be obtained by strictly arithmetical methods depending on a simple use of the theory of binary quadratic forms of negative determinant $-c$; I prefer however to use an algebraic method, partly on account of its independent interest, but chiefly because such a method can be more easily extended.

Let c, n, $p > 0$ be integers, of which p is prime; write (r, s) for the G.C.D. of r, s, and $N\,[n = f]$ for the number of representations of n in the form f. The three formulas mentioned refer to

$$f \equiv x^2 + cy^2 + p(z^2 + ct^2),$$

so that $N[n = f]$ is the total number of sets of integers $x, y, z, t \geqq 0$ such that, for n fixed,

$$n = x^2 + cy^2 + p(z^2 + ct^2).$$

Any integer $t > 0$ is of the form $p^a n$, where $(p, n) = 1$, $a \geqq 0$. We shall call the expression of $N[p^a n = f]$, $a > 0$, as an algebraic function of $N[n = f]$ the *reduction* of $N[p^a n = f]$, and similarly for any form at least one of whose coefficients is divisible by p.

For $c = 1, 2, 3$ the complete reductions of $N[p^a n = f]$ were stated by Liouville,* who however suppressed all details concerning the only cases in each instance which are not immediately obvious. His interesting and useful formulas therefore stand in need of proof, the more so as a direct use of them greatly abridges Pepin's memoir * on Liouville's numerous theorems concerning special quadratic forms which also were published without proof.

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* *Journal des Mathématiques*, Vol. 10 (1865), pp. 43, 49; Vol. 11, p. 211. In § 3 here we considerably shorten Liouville's statements of his results; also, in §§ 6-8 we derive formulas which unify several of his.

A particular interest attaches to those reduction formulas which express $N[p^a n = f]$, $a > 0$, in terms of $N[n = f]$ and algebraic functions of the divisors of the given integers n, p, c. Liouville's theorems for $c = 1, 2, 3$ are of this type; equally simple results can be found for $c = 5, 7$ and n properly restricted, also in any case where there is but one class of binary quadratic forms of determinant $-c$, or where each order contains but one class. This will appear from the algebra for $c = 1, 2, 3$, although the details are given only for these values of c.

2. *A difference equation.* Let $(m, 2) = 1$, $n = 2^s m$, $s \geqq 0$, $(n, p) = 1$, and write $\sigma(j) =$ the sum of all the divisors of j. Then

$$\sigma(2^s p^a m) = \sigma(p^a)\sigma(n), \quad a \geqq 0.$$

Now if $N[p^a n = f] = \psi(a)\phi(n)$, where ϕ is independent of p, a, and for p fixed $\psi(a)$ depends only on a, we need attend only to $\psi(a)$ in the determination of $N[p^a n = f]$. The difference equation for $\psi(a)$ in all cases with which we shall be concerned is of the type

$$\psi(a+1) + \psi(a) = k\sigma(p^a), \tag{1}$$

where k is independent of p, a. The solution of this equation is

$$\psi(a) = k \sum_{j=0}^{a-1} (-1)^j \sigma(p^{a-j-1}) + (-1)^a \psi(0),$$

and hence, since $\sigma(p^r) = (p^{r+1} - 1)/(p - 1)$, we have

$$\psi(a) = (-1)^a \psi(0) + \frac{k}{p-1}\left[\frac{p\{p^a - (-1)^a\}}{p+1} - \tfrac{1}{2}\{1 - (-1)^a\}\right], \tag{2}$$

or, as it may be written,

$$\psi(a) = \frac{k(2p^{a+1} - p - 1)}{2(p^2 - 1)} + (-1)^a \left[\psi(0) - \frac{k}{2(p+1)}\right]. \tag{3}$$

For m, n, s as stated, k is of the form $\chi(s)\sigma(m)$, where $\chi(s)$ is a function of s alone.

3. *Summaries for* $c = 1, 2, 3$. Write

$$f \equiv x^2 + y^2 + p(z^2 + t^2), \tag{4}$$

$$g \equiv x^2 + 2y^2 + p(z^2 + 2t^2), \tag{5}$$

$$h \equiv x^2 + 3y^2 + p(z^2 + 3t^2), \tag{6}$$

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* Ibid., Vol. 6 (1890), p. 1.

for p prime, and let m, n, s be as in § 2.

Then, for the form f,

$$N[2^a m = f] = c\,N[m = f], \tag{4.1}$$

where $c = 2, 6$ according as $a = 1$ or $a > 1$;

$$p \equiv -1 \bmod 4, \quad N[p^a n = f] = N[n = f], \quad a \geqq 0; \tag{4.2}$$

$$p \equiv 1 \bmod 4, \quad N[p^a n = f] = \psi(a), \tag{4.3}$$

where $\psi(a)$ is as in (2), (3), with $k = 16\sigma(m)$ or $48\sigma(m)$ according as $s = 0$ or $s > 0$.

For the form g, with $p > 2$ (the case $p = 2$ is treated in § 5), we have

$$p \equiv 5, 7 \bmod 8, \quad N[p^a n = g] = N[n = g], \quad a \geqq 0; \tag{5.1}$$

$$p \equiv 1, 3 \bmod 8, \quad N[p^a n = g] = \psi(a), \tag{5.2}$$

where $\psi(a)$ is as before, with $k = 8\sigma(m)$, $16\sigma(m)$, $48\sigma(m)$ according as $s = 0$, $s = 1$, $s > 1$.

For the form h,

$$p \equiv 1 \bmod 6, \quad N[p^a n = h] = N[n = h], \quad a \geqq 0; \tag{6.1}$$

$$\begin{gathered} p \equiv -1 \bmod 6, \quad (p, 3) = (p, n) = (3, n) = 1, \\ N[p^a 3^b n = f] = N[p^a n = f], \quad b \geqq 0; \end{gathered} \tag{6.2}$$

$$\begin{gathered} p \equiv -1 \bmod 6, \quad (n, 3p) = (m, 2) = 1, \quad n = 2^s m, \\ N[p^a n = h] = \psi(a), \end{gathered} \tag{6.3}$$

with $k = 8\sigma(m)$, $8(2^{s+1} - 3)\sigma(m)$ according as $s = 0$, $s > 0$.

The enumerations (4.1) — (6.3) are evidently exhaustive for the respective forms; the only cases offering any difficulty are

$$(4.3),\ (5.2),\ (6.3), \tag{7}$$

to which we return in §§ 6-8.

4. *Connection of f, g, h with principal binary quadratic forms of determinant $-c$.* In order that n may be represented in the form $x^2 + cy^2$ of

negative determinant $-c$, it is necessary that $(-c \mid n) = 1$, where (as always henceforth) $(r \mid s)$ is the Jacobi-Legendre symbol. Hence, in order that n be not representable in $x^2 + cy^2$ it is sufficient that $(-c \mid n) = -1$.

Consider all the representations of $p^a n$, where p, a, n are as before, in the form

$$F \equiv x^2 + cy^2 + p(z^2 + ct^2).$$

If $(-c \mid p) = -1$, so that p is not representable in $x^2 + cy^2$, then $N[p^a n = F] \neq 0$, $a > 0$, only when $x \equiv y \equiv 0 \bmod p$. Put therefore $x = p\xi$, $y = p\eta$; then

$$N[p^a n = x^2 + cy^2 + p(z^2 + ct^2)]$$
$$= N[p^{a-1} n = z^2 + ct^2 + p(\xi^2 + c\eta^2)],$$

whence, repeating the reduction a sufficient number of times we have

$$N[p^a n = F] = N[n = F], \quad a \geqq 0.$$

Either this or a similar argument disposes of all cases in § 3 except (7), and the like can obviously be extended to any form $f = f_1 + p(f_2 + \cdots + f_s)$, where

$$f_j = \alpha x_j{}^2 + \beta x_j y_j + \gamma y_j{}^2 \quad (j = 1, \cdots, s),$$

and, with obvious modifications, to any forms f_j all of the same kind in different sets of indeterminates.

5. *The Algebraic Method.* To illustrate the simpler essentials of the algebraic method of finding theorems of the type § 3(7), we shall apply it to the form

$$\phi \equiv x^2 + y^2 + 2z^2 + 2t^2,$$

verifying first that

$$N[2^{a+1} m = \phi] = N[2^a m = x^2 + y^2 + z^2 + t^2],$$

where $(m, 2) = 1$ as before. Multiply within the second bracket by 2, in order to make the arguments $2^{a+1}m$, $2^a m$ agree, so that like powers of a parameter q in the generating functions of both N's may be compared:

$$N[2^{a+1} m = \phi] = N[2^{a+1} m = 2x^2 + 2y^2 + 2z^2 + 2t^2].$$

The respective generators are

$$\theta_3^4(q^2) = \sum q^{2n} N[2n = 2x^2 + 2y^2 + 2z^2 + 2t^2],$$

where $\sum$ refers to $n = 0, 1, 2, \cdots$, and the modification of

$$\theta_3^2(q)\theta_3^2(q^2) = \sum q^n N[n = x^2 + y^2 + 2z^2 + 2t^2],$$

from which are selected only those q^n such that $n \equiv 0 \bmod 2$. This is accomplished by changing q to $-q$ and adding;

$$\theta_3^2(q)\theta_3^2(q^2) + \theta_0^2(q)\theta_3^2(q^2) = 2\sum q^{2n} \; N[2n = \phi].$$

Hence it must be shown that

$$\theta_3^2(q)\theta_3^2(q^2) + \theta_0^2(q)\theta_3^2(q^2) = 2\theta_3^4(q^2),$$

or, what is the same,

$$\theta_3^2(q) + \theta_0^2(q) = 2\theta_3^2(q^2),$$

which is correct, by the transformation of the second order. The verification is therefore complete.

The particular point to be noted is that we select only those q^n in the second generator for which $n \equiv 0 \bmod p$, where p $(= 2$ in this case$)$ is the prime concerned in the given form ϕ. This generalizes to any p.

The foregoing however is a mere verification, and we are interested in finding such reduction formulas *ab initio*. For this the order of the steps is reversed. We start from any known theorem concerning numbers of representations in binary quadratic forms, translate it into its theta equivalent, multiply the result throughout by theta constants to obtain a new identity which, when read as a relation between N's, yields a theorem concerning numbers of representations. The appropriate theorems for binary forms will be found in Dirichlet's *Zahlentheorie*, or in the papers of Pepin and others abstracted in Volume 3 of Dickson's *History*.

6. *Form* $f \equiv x^2 + y^2 + p(z^2 + t^2)$. The appropriate binary theorem is

$$N[n = x^2 + y^2] = 4\xi(n),$$

where $\xi(n)$ = the excess of the number of $4t + 1$ divisors of n over the number of $4t + 3$ divisors; $\xi(r)\xi(s) = \xi(rs)$ if $(r, s) = 1$, and if $p \equiv 1 \bmod 4$

(p prime as always), $\xi(p^a) = a + 1$. Since we need attend only to $p \equiv 1$ mod 4 for f, we take only this case, and have, in the notation of § 2,

$$N[p^a n = x^2 + y^2] + N[p^{a-2} n = x^2 + y^2] = 2N[p^{a-1} n = x^2 + y^2].$$

as the simplest difference equation satisfied by ξ. The translation of this into an identity between generators is (coefficients of q^j, $j = p^a n$, are compared),

$$\Sigma\, q^{pn} N[pn = x^2 + y^2] + \Sigma\, q^{p^2 n} N[n = x^2 + y^2] = 2 \Sigma\, q^{pn} N[n = x^2 + y^2],$$

in which Σ refers to $n = 0, 1, 2, \cdots$. We now replace this by its θ-equivalent; the first Σ is the only one requiring any attention. To restate this Σ as a function of θ's we must select from

$$\theta_3{}^2(q) = \Sigma\, q^n N[n = x^2 + y^2]$$

only those terms for which the exponent of q is a multiple of p. Let r be a primitive (algebraic) pth root of unity. Then

$$\sum_{j=0}^{p-1} \theta_3{}^2(r^j q) = p \Sigma\, q^{pn} N[pn = x^2 + y^2],$$

and hence the required θ-equivalent is

$$\sum_{j=0}^{p-1} \theta_3{}^2(r^j q) + p\theta_3{}^2(q^{p^2}) = 2p\, \theta_3{}^2\, (q^p),$$

which is the analytic expression of our linear difference equation for N. Hence any consequence of this θ-identity will yield a true proposition. We wish here to derive from it a theorem concerning enumerations in quadratic forms in 4 indeterminates, and therefore we multiply the identity throughout by $\theta_3{}^2(q)$ or by $\theta_3{}^2(q^p)$, choosing the latter as it gives a simpler result. We have then

$$\theta_3{}^2(q^p) \sum_{j=0}^{p-1} \theta_3{}^2(r^j q) + p\theta_3{}^2(q^p)\, \theta_3{}^2(q^{p^2}) = 2p\theta_3{}^4(q^p).$$

From the way in which the Σ in this was obtained, only multiples of p will occur as exponents of q in the power series for the first product on the left. Equating coefficients of q we get

$$\begin{aligned} &N[pn = x^2 + y^2 + p(z^2 + t^2)] \\ &\quad + N[pn = (px)^2 + (py)^2 + p(z^2 + t^2)] \\ &\quad = 2N[pn = px^2 + py^2 + pz^2 + pt^2], \\ &\qquad 2N[n = x^2 + y^2 + z^2 + t^2], = 16[2 + (-1)^n]\sigma(m), \end{aligned}$$

the last by Jacobi's theorem on 4 squares, where as always $n = 2^s m, (m, 2) = 1$, $s \geqq 0$.

Hence, in the notation of § 2 we have

$$\psi(a+1) + \psi(a) = 16[2 + (-1)^n]\sigma(p^a m),$$

whose solution $\psi(a)$ is given by (3) with $k = 16[2 + (-1)^n]\sigma(m)$, and this is equivalent to (4.3).

In the foregoing analysis the use of pth roots of unity is not a necessity; we might have used throughout the full generator in its N form. But in more complicated situations of a similar kind it is advantageous to have all generators expressed in finite theta form on account of the transformations thus suggested.

7. *Form* $g \equiv x^2 + 2y^2 + p(z^2 + 2t^2)$. We start from the known N for the principal binary quadratic form of determinant -2,

$$N[2^s m = x^2 + 2y^2] = 2\sum (-2 \mid d) \equiv 2\omega(m),$$

where $(m, 2) = 1$, and Σ refers to all divisors d of m. From the properties of Jacobi's symbol it follows that if r, s are relatively prime integers, then $\omega(rs) = \omega(r)\omega(s)$. To obtain the initial difference equation for ω we need the explicit form of $\omega(p^\beta)$, $\beta > 0$ (p prime, as always). We have

$$\omega(p^\beta) = \sum_{j=0}^{\beta} (-2 \mid p^j) = \sum (-2 \mid p)^j,$$

and therefore

$$\omega(p^\beta) = \beta + 1, \quad \tfrac{1}{2}[1 + (-1)^\beta],$$

according as $(-2 \mid p) = 1$ or -1. Explicitly then

$$\begin{aligned} \omega(p^\beta) = \tfrac{1}{2}[1 + (-2 \mid p)](\beta + 1) \\ + \tfrac{1}{2}[1 - (-2 \mid p)] \cdot \tfrac{1}{2}[1 + (-1)^\beta], \end{aligned}$$

so that

$$4\omega(p^{\beta}) = 2\beta + 3 + (-1)^{\beta} + (-2 \mid p)\,[2\beta + 1 - (-1)^{\beta}].$$

Hence, for $a > 0$,

$$\begin{aligned} 4\omega(p^{a+1}) &= 2a + 5 - (-1)^{a} + (-2 \mid p)\,[2a + 3 + (-1)^{a}], \\ 4\omega(p^{a}) &= 2a + 3 + (-1)^{a} + (-2 \mid p)\,[2a + 1 - (-1)^{a}], \\ 4\omega(p^{a-1}) &= 2a + 1 - (-1)^{a} + (-2 \mid p)\,[2a - 1 + (-1)^{a}]; \end{aligned}$$

whence we have the required difference equation

$$\omega(p^{a+1}) + \omega(p^{a-1}) = 2\omega(p^{a}) + (-1)^{a}\,[(-2 \mid p) - 1].$$

Referring to § 3 (7) we see that it is necessary for our present purpose to attend only to this equation in the case $(-2 \mid p) = 1$, so that

$$(-2 \mid p) = 1, \quad \omega(p^{a+1}) + \omega(p^{a-1}) = 2\omega(p^{a}), \quad a > 0.$$

Proceeding now exactly as in § 6, we write down from this

$$\begin{aligned} \sum q^{np} N[np = x^2 + 2y^2] + \sum q^{np^2} N[n = x^2 + 2y^2] \\ = 2\sum q^{np}\, N[n = x^2 + 2y^2], \end{aligned}$$

where Σ refers to $n = 0, 1, 2, \cdots$, and (as in all that follows) $(-2 \mid p) = 1$. Translating as before to theta functions, with $r =$ a primitive pth root of unity, we get

$$\sum_{j=0}^{p-1} \theta_3(r^{j}q)\theta_3(r^{2j}q^2) + p\theta_3(q^{p^2})\theta_3(q^{2p^2}) = 2p\theta_3(q^{p})\theta_3(q^{2p}),$$

and therefore

$$\begin{aligned} \theta_3(q^{p})\theta_3(q^{2p}) \sum_{j=0}^{p-1} \theta_3(r^{j}q)\theta_3(r^{2j}q) \\ + p\theta_3(q^{p})\theta_3(q^{2p})\theta_3(q^{p^2})\theta_3(q^{2p^2}) = 2p\theta_3^{2}(q^{p})\theta_3^{2}(q^{2p}), \end{aligned}$$

which steps up from binary to quaternary forms upon equating coefficients of like powers of q. With $(n, p) = 1$ we get

$$N[p^{a+1}n = g] + N[p^{a}n = g] = 2N[p^{a}n = x^2 + 2y^2 + z^2 + 2t^2].$$

The last N being a known function we can proceed. For $(2, m) = 1$,

$$N[2^s m = x^2 + y^2 + 2(z^2 + t^2)] = b\sigma(m),$$

where $b = 4, 8, 24$ according as $s = 0, 1, > 1$, which follows at once from § 5 and Jacobi's theorem on 4 squares. Hence for $n = 2^s m$, $s \geqq 0$, we have

$$b = 2\{1 - (-1)^n\} + 4\{2 + (-1)^{n/2}\}\{1 + (-1)^n\},$$

and therefore in the notation of § 2 (1)

$$\psi(a+1) + \psi(a) = k\sigma(p^a),$$

with here

$$k = 2b\sigma(m),$$

whose solution is given by § 2 (3) with this value of k. This is equivalent to (5.2).

8. *Form* $h \equiv x^2 + 3y^2 + p(z^2 + 3t^2)$. An even integer n is represented in $x^2 + 3y^2$ only if $n \equiv 0$ mod 4. From the known theory of binary forms we have

$$N[m = x^2 + 3y^2] = 2\sum (-3 \mid d) \equiv 2\delta(m), \quad (m, 2) = 1,$$
$$N[2^{2s} m = x^2 + 3y^2] = 6\delta(m), \quad s > 0,$$

the Σ referring to all divisors d of m. When $m \equiv 0$ mod 3 the Jacobi symbol $(-3 \mid 3d) = 0$. Proceeding as in § 7 we find

$$4\delta(p^\beta) = 2\beta + 3 + (-1)^\beta + (-3 \mid p)\, [2\beta + 1 - (-1)^\beta],$$

which gives the difference equation

$$\delta(p^{a+1}) + \delta(p^{a-1}) = 2\delta(p^a) + (-1)^a[(-3 \mid p) - 1],$$

for which, by § 2 (7), we need discuss only the case $p \equiv -1$ mod 6; whence

$$\delta(p^{a+1}) + \delta(p^{a-1}) = 2\delta(p^a).$$

As before, with $(n, p) = 1$, we find from the identity between products of 4 theta functions derived from this,

$$N[p^{a+1}n = h] + N[p^a n = h] = 2N[p^a n = x^2 + 3y^2 + z^2 + 3t^2].$$

The last N is well known; it may be obtained if desired analytically from the transformation of the third order of the theta functions. It can be written

$$N[n = x^2 + 3y^2 + z^2 + 3t^2] = 4[\sigma(n) - \{1 - (-1)^n\} \; \sigma(m)],$$

where $n = 2^s m$, $s \geqq 0$, $(m, 2) = 1$. Hence in this case

$$\psi(a+1) + \psi(a) = k\sigma(p^a)$$

where

$$k = 4[1 - (-1)^n + \{1 + (-1)^n\}(2^{s+1} - 3)] \; \sigma(m),$$

and for this k we get (6.3) from (3).

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Generalization of Certain Theorems of Bohl, [Second Paper].

By F. H. Murray.

In a preceding communication with the same title [*] it was shown that a number of existence theorems of Bohl [†] can be generalized in such a manner as to apply to the solutions of a system of differential equations in the canonical form

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}; \qquad i=1,\cdots,n,$$

near a point solution $x_i=a_i$ $(i=1,\cdots,n)$ when certain conditions are satisfied. In the present paper these conditions are replaced by less stringent ones; the methods of proof of certain existence theorems are very similar to those employed in the first paper, and these proofs are given here in an abbreviated form. In addition, the asymptotic properties of certain trajectories are discussed by an extension of the methods of Bohl.

On account of the more complicated form of certain quadratic forms which occur here, it has been convenient to leave undetermined certain constants which are determined explicitly in the first paper; this procedure, together with the algebraic method of transforming the canonical equations, reduces to a small amount the results common to both papers.

1. *Transformation of the canonical equations.*

Assume the function $F(x_1,\cdots,x_n;\,y_1,\cdots,y_n)$ to be continuous, together with its partial derivatives of the first and second orders, within a region

$$\text{(D)} \qquad |x_i|\leqq D, \qquad |y_i|\leqq D, \qquad i=1,\cdots,n,$$

and suppose $\dfrac{\partial F}{\partial x_i}=\dfrac{\partial F}{\partial y_i}=0$, $(i=1,\cdots,n)$ if $x_k=y_k=0$. The canonical equations

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[*] *Amer. Journal of Math.*, vol. XLVII (1925), pp. 25-44.

[†] P. Bohl, "Ueber die Bewegungen eines mechanischen Systems, etc.", *Crelle's Journal*, Band 127 (1904), pp. 179-276.

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i},\qquad i=1,\cdots,n, \tag{1.01}$$

become

$$\begin{aligned}\frac{dx_i}{dt}&=\sum_{r=1}^{n}\frac{\partial^2 F}{\partial y_i\partial x_r}x_r+\sum_{r=1}^{n}\frac{\partial^2 F}{\partial y_i\partial y_r}y_r+F_i',\\ \frac{dy_i}{dt}&=-\sum_{r=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_r}x_r-\sum_{r=1}^{n}\frac{\partial^2 F}{\partial x_i\partial y_r}y_r+F_i'',\end{aligned}\qquad i=1,\cdots,n, \tag{1.02}$$

and the equations of variation

$$\begin{aligned}\frac{d\bar{\xi}_i}{dt}&=\sum_{r=1}^{n}\frac{\partial^2 F}{\partial y_i\partial x_r}\bar{\xi}_r+\sum_{r=1}^{n}\frac{\partial^2 F}{\partial y_i\partial y_r}\bar{\eta}_r\\ \frac{d\bar{\eta}_i}{dt}&=-\sum_{r=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_r}\bar{\xi}_r-\sum_{r=1}^{n}\frac{\partial^2 F}{\partial x_i\partial y_r}\bar{\eta}_r\end{aligned}\qquad i=1,\cdots,n, \tag{1.03}$$

possess characteristic exponents which satisfy an equation with real coefficients

$$\Delta(a)=0. \tag{1.04}$$

By hypothesis the functions F_k', F_k'' possess continuous derivatives of the first order which vanish with the variables x_i, y_i $(i, k=1,\cdots,n)$.

From the classical theory of differential systems of the form (1.03) it is known that if a is a root of equation (1.04), the solutions can be associated in certain groups of the form

$$\begin{aligned}&\bar{\xi}_i'=e^{at}\xi_i', && \bar{\eta}_i'=e^{at}\eta_i',\\ &\bar{\xi}_i''=e^{at}(\xi_i''+t\xi_i'), && \bar{\eta}_i''=e^{at}(\eta_i''+t\eta_i'),\\ &\cdots\cdots\cdots\cdots\\ &\bar{\xi}_i^{(\nu)}=e^{at}\left(\xi_i^{(\nu)}+t\xi_i^{(\nu-1)}+\cdots+\frac{t^{\nu-1}}{\underline{|\nu-1}}\xi_i'\right),\\ &\bar{\eta}_i^{(\nu)}=e^{at}\left(\eta_i^{(\nu)}+t\eta_i^{(\nu-1)}+\cdots+\frac{t^{\nu-1}}{\underline{|\nu-1}}\eta_i'\right).\end{aligned} \tag{1.05}$$

Suppose another group of solutions represented by

$$\begin{aligned}&\bar{x}_i'=e^{a't}x_i', && \bar{y}_i'=e^{a't}y_i',\\ &\cdots\cdots\cdots\cdots\\ &\bar{x}_i^{(\mu)}=e^{a't}\left(x_i^{(\mu)}+\cdots+\frac{t^{\mu-1}}{\underline{|\mu-1}}x_i'\right),\\ &\bar{y}_i^{(\mu)}=e^{a't}\left(y_i^{(\mu)}+\cdots+\frac{t^{\mu-1}}{\underline{|\mu-1}}y_i'\right).\end{aligned} \tag{1.051}$$

Then from the relation which is satisfied by any pair of solutions of (1.03),

$$(1.06)\qquad \sum_{i=1}^{n}(\xi_i{}^{(r)}\bar{y}_i{}^{(s)}-\bar{\eta}_i{}^{(r)}\bar{x}_i{}^{(s)})=C,$$

we obtain, if $a+a'\neq 0$,

$$\sum_{i=1}^{n}(\xi_i'y_i'-\eta_i'x_i')=0,$$

and in general

$$\sum_{i=1}^{n}(\xi_i{}^{(r)}y_i{}^{(s)}-\eta_i{}^{(r)}x_i{}^{(s)})+t\,\Big[\sum_{i=1}^{n}(\xi_i{}^{(r-1)}y_i{}^{(s)}-\eta_i{}^{(r-1)}x_i{}^{(s)})$$
$$+\sum_{i=1}^{n}(\xi_i{}^{(r)}y_i{}^{(s-1)}-\eta_i{}^{(r)}x_i{}^{(s-1)})\Big]+\cdots=0,$$

whence

$$(1.07)\qquad \sum_{i=1}^{n}(\xi_i{}^{(r)}y_i{}^{(s)}-\eta_i{}^{(r)}x_i{}^{(s)})=0,$$
$$r=1,2,\cdots,\nu,\quad s=1,2,\cdots,\mu,\quad (r+s>1).$$

If $a+a'=0$, we obtain in the same manner

$$\sum_{i=1}^{n}(\bar{\xi}_i{}^{(r)}\bar{y}_i{}^{(s)}-\bar{\eta}_i{}^{(r)}\bar{x}_i{}^{(s)})=\sum_{i=1}^{n}(\xi_i{}^{(r)}y_i{}^{(s)}-\eta_i{}^{(r)}x_i{}^{(s)})+t$$
$$\Big[\sum_{i=1}^{n}(\xi_i{}^{(r)}y_i{}^{(s-1)}-\eta_i{}^{(r)}x_i{}^{(s-1)})+\sum_{i=1}^{n}(\xi_i{}^{(r-1)}y_i{}^{(s)}-\eta_i{}^{(r-1)}x_i{}^{(s)})\Big]$$
$$+\cdots=C',$$

from which further relations can be derived; but these will not be used in what follows.

Suppose that no exponent is zero, and that the function F is real for real values of its arguments. Then it is known that to each group of ν solutions of (1.03) corresponding to the exponent a there is a group of ν solutions with the exponent $-a$.

If a is one exponent, the conjugate imaginary $\bar{a}$ is another, and a fundamental set of solutions of (1.03) can be so chosen that the constants which appear in any group corresponding to $\bar{a}$ are the conjugate imaginaries of the corresponding constants appearing in the group with the exponent a.

It will be assumed that there is only one solution in each group corresponding to a pure imaginary exponent; the solutions corresponding to a pure imaginary exponent a_r, $\left(\dfrac{a_r}{\sqrt{-1}}>0\right)$, can be normalized * in such a manner

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* *Amer. Journal.*, *l. c.*, pp. 28-30.

that if (ξ_i, η_i), corresponding to a_r, and (x_i, y_i) corresponding to $-a_r$ are conjugate imaginary sets, then

$$\sum_{i=1}^{n} (\xi_i y_i - \eta_i x_i) = \epsilon_r = \pm (1/a_r), \tag{1.08}$$

while if (x'_i, y'_i) is any other solution corresponding to $-a_r$,

$$\sum_{i=1}^{n} (\xi_i y_i' - \eta_i x_i') = 0. \tag{1.09}$$

Suppose

$$2Q(\xi, \eta) = \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial x_s} \xi_r \xi_s + 2 \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial y_s} \xi_r \eta_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial y_s} \eta_r \eta_s \tag{1.1}$$

and define

$$\begin{aligned} Q(\xi\eta, \xi'\eta') &= \sum_{r=1}^{n} \xi'_r \frac{\partial Q}{\partial \xi_r} + \sum_{r=1}^{n} \eta'_r \frac{\partial Q}{\partial \eta_r} = Q(\xi'\eta', \xi\eta) \\ &= \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial x_s} \xi'_r \xi_s + \sum_{r,s} \frac{\partial^2 F}{\partial x_r \partial y_s} \xi'_r \eta_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial x_s} \eta'_r \xi_s + \sum_{r,s} \frac{\partial^2 F}{\partial y_r \partial y_s} \eta'_r \eta_s \end{aligned} \tag{1.11}$$

The functions of (1.05) satisfy the equations

$$\begin{aligned} & d\bar{\xi}_i'/dt = a\bar{\xi}_i', \quad d\bar{\xi}_i''/dt = \bar{\xi}_i' + a\bar{\xi}_i'', \quad \cdots, \quad d\bar{\xi}_i^{(\nu)}/dt = \xi_i^{(\nu-1)} + a\bar{\xi}_i^{(\nu)}, \\ & d\bar{\eta}_i'/dt = a\bar{\eta}_i', \quad d\bar{\eta}_i''/dt = \bar{\eta}_i' + a\bar{\eta}_i'', \quad \cdots, \quad d\bar{\eta}_i^{(\nu)}/dt = \bar{\eta}_i^{(\nu-1)} + a\bar{\eta}_i^{(\nu)}, \end{aligned} \tag{1.12}$$

consequently

$$\begin{aligned} & a\xi_i' = Q_{\eta_i}(\xi', \eta'), \quad \xi_i^{(r-1)} + a\xi_i^{(r)} = Q_{\eta_i}(\xi^{(r)}, \eta^{(r)}), \\ & a\eta_i' = - Q_{\xi_i}(\xi', \eta'), \quad \eta_i^{(r-1)} + a\eta_i^{(r)} = - Q_{\xi_i}(\xi^{(r)}, \eta^{(r)}), \\ & \qquad\qquad\qquad\qquad r = 2 \cdots \nu; \end{aligned} \tag{1.2}$$

and if $a + a' \neq 0$,

$$\begin{aligned} Q(\xi^{(r)}\eta^{(r)}, x^{(s)}y^{(s)}) &= \sum_{i=1}^{n} [x_i^{(s)} Q_{\xi_i}(\xi^{(r)}, \eta^{(r)}) + y_i^{(s)} Q_{\eta_i}(\xi^{(r)}, \eta^{(s)})] \\ &= a \sum_{i=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) = 0, \quad r = 1, \\ &= \sum_{i=1}^{n} (\xi_i^{(r-1)} y_i^{(s)} - \eta_i^{(r-1)} x_i^{(s)}) \\ &\qquad + a \sum_{r=1}^{n} (\xi_i^{(r)} y_i^{(s)} - \eta_i^{(r)} x_i^{(s)}) = 0, \\ &\qquad r = 2 \cdots \nu, \quad s = 1 \cdots \mu. \end{aligned} \tag{1.3}$$

Suppose $a_1, a_2, \cdots, a_M$ the pure imaginary exponents such that $a_r/\sqrt{-1}$

> 0, $(r = 1, \cdots, M)$, while $a_{M+1}, \cdots, a_n$ have their real parts positive; suppose also $a_{n+r} = -a_r$, and let the fundamental set of solutions be so numbered that if the solutions $(\xi_{ki}{}^{(r)}, \overline{\eta}_{ki}{}^{(r)})$ $(r = m, m+1, \cdots, m + \nu_k - 1)$ form a group corresponding to a_k, the solutions $(\xi^{(r)}{}_{n+k,i}, \overline{\eta}^{(r)}{}_{n+k,i})$ $(r = m, m+1, \cdots, m + \nu_k - 1)$ form a group corresponding to a_{n+k}. Under these conditions make the transformation

$$
\begin{aligned}
x_i &= \sum_k \left(\xi_{ki}{}^{(\nu_k)} u_1{}^{(k)} + \xi_{ki}{}^{(\nu_k-1)} u_2{}^{(k)} + \cdots + \xi'_{ki} u_{\nu_k}{}^{(k)}\right) \\
&\qquad + \sum_k \left(\xi^{(\nu_k)}{}_{n+k,i} v_1{}^{(k)} + \cdots + \xi'_{n+k,i} v_{\nu_k}{}^{(k)}\right), \\
y_i &= \sum_k \left(\eta_{ki}{}^{(\nu_k)} u_1{}^{(k)} + \eta_{ki}{}^{(\nu_k-1)} u_2{}^{(k)} + \cdots + \eta'_{ki} u_{\nu_k}{}^{(k)}\right) \\
&\qquad + \sum_k \left(\eta^{(\nu_k)}{}_{n+k,i} v_1{}^{(k)} + \cdots + \eta'_{n+k,i} v_{\nu_k}{}^{(k)}\right),
\end{aligned}
\tag{1.4}
$$

the first summation extending over all the groups with the exponents $a_1, \cdots, a_n$, the second over the groups with the exponents $a_{n+1}, \cdots, a_{2n}$.

If the transformation (1.4) is made in the right-hand members of (1.02), these become

$$
\begin{aligned}
dx_i/dt &= \sum_k \left[Q_{\eta_i}(\xi_k{}^{(\nu_k)}, \eta_k{}^{(\nu_k)}) u_1{}^{(k)} + \cdots + Q_{\eta_i}(\xi_k', \eta_k') u_{\nu_k}{}^{(k)}\right] \\
&+ \sum_k \left[Q_{\eta_i}(\xi_{n+k}{}^{(\nu_k)}, \eta_{n+k}{}^{(\nu_k)}) v_1{}^{(k)} + \cdots + Q_{\eta_i}(\xi'_{n+k}, \eta'_{n+k}) v_{\nu_k}{}^{(k)}\right] + F'_i, \\
dy_i/dt &= -\sum_k \left[Q_{\xi_i}(\xi_k{}^{(\nu_k)}, \eta_k{}^{(\nu_k)}) u_1{}^{(k)} + \cdots + Q_{\xi_i}(\xi'_k, \eta'_k) u_{\nu_k}{}^{(k)}\right] \\
&- \sum_k \left[Q_{\xi_i}(\xi_{n+k}{}^{(\nu_k)}, \eta_{n+k}{}^{(\nu_k)}) v_1{}^{(k)} + \cdots + Q_{\xi_i}(\xi'_{n+k}, \eta'_{n+k}) v_{\nu_k}{}^{(k)}\right] + F_i'', \\
&\qquad i = 1 \cdots n.
\end{aligned}
\tag{1.5}
$$

If for the functions Q_{ξ_i}, Q_{η_i} we substitute their equals from (1.2), and rearrange the linear terms, then for the derivatives $\dfrac{dx_i}{dt} \cdot \dfrac{dy_i}{dt}$ substitute their equals from (1.4), linear expressions in the variables $u_s{}^{(r)}$, $v_s{}^{(r)}$ and their derivatives with the same coefficients are obtained on both sides of the equations, their determinant being the determinant of the fundamental set of solutions and hence different from zero. Solving, we obtain

$$
\begin{aligned}
du_1{}^{(k)}/dt &= a_k u_1{}^{(k)} + \phi_1{}^{(k)}, & dv_1{}^{(k)}/dt &= a_{n+k} v_1{}^{(k)} + \psi_1{}^{(k)}, \\
du_2{}^{(k)}/dt &= u_1{}^{(k)} + a_k u_2{}^{(k)} + \phi_2{}^{(k)}, & dv_2{}^{(k)}/dt &= v_1{}^{(k)} + a_{n+k} v_2{}^{(k)} + \psi_2{}^{(k)}, \\
du_{\nu_k}{}^{(k)}/dt &= u_{\nu_k-1}{}^{(k)} + a_k u_{\nu_k}{}^{(k)} + \phi_{\nu_k}{}^{(k)}, & dv_{\nu_k}{}^{(k)}/dt &= v_{\nu_k-1}{}^{(k)} + a_{n+k} v_{\nu_k}{}^{(k)} + \psi_{\nu_k}{}^{(k)},
\end{aligned}
\tag{1.6}
$$

the functions $\phi_k{}^{(r)}$, $\psi_k{}^{(r)}$ being linear combinations of the functions F_i', F_i''.

The function $Q(xy)$, representing the terms of the second order in the

Taylor expansion of F, can easily be calculated from (1.1), (1.11), (1.3), (1.4); we obtain

$$
\begin{aligned}
Q(x,y) &= \tfrac{1}{2}\sum_i [x_i Q_{x_i}(xy) + y_i Q_{y_i}(xy)] \\
&= \sum_{klrs} Q(\xi_{n+k}{}^{\nu_k-r+1}\eta_{n+k}{}^{\nu_k-r+1}, \xi_l{}^{\nu_l-s+1}\,\eta_l{}^{\nu_l-s+1})\, u_s{}^{(l)} v_r{}^{(k)} \qquad (1.7) \\
&= \sum_{i=1}^{M} a_i \epsilon_i u_1{}^{(i)} v_1{}^{(i)} + \sum_{klrs}{}' Q(\xi_{n+k}{}^{\nu_k-r+1}\eta_{n+k}{}^{\nu_k-r+1}, \xi_l{}^{\nu_l-s+1}\,\eta_l{}^{\nu_l-s+1})\, u_s{}^{(l)} v_r{}^{(k}
\end{aligned}
$$

where Σ' denotes a summation over the groups for which the real part of the exponent is different from zero; the coefficient $Q(\xi_{n+k}{}^{(r)}\eta_{n+k}{}^{(r)}\xi_l{}^{(s)}\eta_l{}^{(s)})$ is different from zero only if $a_l + a_{n+k} = 0$.

Since the function F is defined only for real values of its arguments, in general, it will be necessary to give $u_s{}^{(r)}$, $v_s{}^{(r)}$ such values that x_i, y_i, will be real; this result will be obtained if the following conditions are satisfied:

(a) when a_r is a pure imaginary, $a_r = p_r\sqrt{-1}$, $p_r > 0$, $\bar{x}_r$ and $\bar{y}_r$ are real, and
$$u_1{}^{(r)} = \bar{x}_r + i\bar{y}_r, \qquad v_1{}^{(r)} = \bar{x}_r - i\bar{y}_r;$$

(b) when a_s is real, $a_s = q_s > 0$, $u_k{}^{(s)} = u_{sk}$, $v_k{}^{(s)} = v_{sk}$, where u_{sk} and v_{sk} are real; and (c) when a_r is neither real nor pure imaginary,

$$
\begin{aligned}
&a_r = q_r + iq_r', \quad a_s = q_r - iq_r', \quad q_r > 0, \\
&u_k{}^{(r)} = u_{rk} + iu_{sk} \qquad v_k{}^{(r)} = v_{rk} + iv_{sk} \\
&u_k{}^{(s)} = u_{rk} - iu_{sk} \qquad v_k{}^{(s)} = v_{rk} - iv_{sk}
\end{aligned}
$$

where u_{rk}, v_{rk}, u_{sk}, v_{sk} are real variables.

Of the exponents $a_1, \cdots, a_n$ suppose there are M pure imaginaries, K real, and L pairs of conjugates, each exponent counted just once for each group (1.6). Then these equations take one of the following forms, a_r and a_{L+r} being conjugates:

$$
\begin{aligned}
&\text{(a)} \quad d\bar{x}_i/dt = -p_i\bar{y}_i + X_i, \qquad d\bar{y}_i/dt = p_i\bar{x}_i + Y_i, \quad i = 1 \cdots M. \\
&(1.8) \quad du_{r1}/dt = q_r u_{r1} + U_{r1}, \qquad dv_{r1}/dt = -q_r v_{r1} + V_{r1}, \\
&\text{(b)} \quad du_{ri}/dt = u_{r,i-1} + q_r u_{ri} + U_{ri}, \qquad dv_{ri}/dt = v_{r,i-1} - q_r v_{ri} + V_{ri}, \\
&\qquad\qquad \begin{pmatrix} i = 2 \cdots \nu_r, \\ r = 1 \cdots k \end{pmatrix} \\
&du_{r1}/dt = q_r u_{r1} - q'_r u_{L+r,1} + U_{r1}, \\
&\qquad dv_{r1}/dt = \qquad -q_r v_{r1} + q'_r v_{L+r,1} + V_{r1},
\end{aligned}
$$

$$(1.8)\quad du_{ri}/dt = u_{r,i-1} + q_r u_{ri} - q'_r u_{L+r,i} + U_{ri},$$
$$dv_{ri}/dt = v_{r,i-1} - q_r v_{ri} + q'_r v_{L+r,i} + V_{ri},$$
$$i = 2, \cdots \nu_r;$$

$$(c)\quad du_{L+r,1}/dt = q'_r u_{r1} + q_r u_{L+r,1} + U_{L+r,1},$$
$$dv_{L+r,1}/dt = -q'_r v_{r1} - q_r v_{L+r,1} + V_{L+r,1},$$
$$du_{Lr+,i}/dt = u_{L+r,i-1} + q'_r u_{ri} + q_r u_{L+r,i} + U_{L+r,i},$$
$$dv_{L+r,i}/dt = v_{L+r,i-1} - q_r' v_{r,i} - q_r v_{L+r,i} + V_{L+r,i}.$$
$$i = 2, \cdots \nu_r; \quad r = K + 1 \cdots K + L.$$

From the equations (a), (b), (c) can be obtained certain fundamental *differential inequalities* similar to those employed by Bohl;
if for each system of type (b),

$$g_r(u_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm}^2, \quad g_r(v_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm}^2, \quad r = 1 \cdots K,$$

we obtain

$$dg_r(u_r)/dt = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^m u_{rm} + q_r^{m-1} u_{r,m-1})^2 + 2 \sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm} U_{rm};$$
$$(1.9)$$
$$dg_r(v_r)/dt = -q_r^2 v_{r1}^2 - q_r^{2\nu_r} v^2_{r,\nu_r} - \sum_{m=2}^{\nu_r} (q_r^m v_{rm} - q_r^{m-1} v_{r,m-1})^2 + 2 \sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm} V_{rm}.$$

For each system of type (c), if

$$g_r(u_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} u_{rm}^2, \quad g_r(v_r) = \sum_{m=1}^{\nu_r} q_r^{2m-1} v_{rm}^2,$$

then

$$d/dt[g_r(u_r) + g_r(u_{L+r})] = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^{m-1} u_{r,m-1} + q_r^m u_{rm})^2$$
$$+ q_r^2 u^2_{L+r,1} + q_r^{2\nu_r} u^2_{L+r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^{m-1} u_{L+r,m-1} + q_r^m u_{L+r,m})^2$$
$$(1.91)\quad + 2 \sum_{m=1}^{\nu_r} q_r^{2m-1} (u_{rm} U_{rm} + u_{L+r,m} U_{L+r,m})$$
$$d/dt[g_r(v_r) + g_r(v_{L+r})] = -q_r^2 v_{r1}^2 - q_r^{2\nu_r} v^2_{r,\nu_r}$$
$$- \sum_{m=2}^{\nu_r} (q_r^{m-1} v_{r,m-1} - q_r^m v_{r,m})^2 - q_r^2 v^2_{L+r,1} - q^{2\nu_r} v^2_{L+r,\nu_r}$$
$$- \sum_{m=2}^{\nu_r} (q_r^{m-1} v_{L+r;m-1} - q_r^m v_{L+r,m})^2 + 2 \sum_{m=1}^{\nu_r} q_r^{2m-1} \left(\begin{matrix} v_{rm} \; V_{rm} \\ + v_{L+r,m} \; V_{L+r,m} \end{matrix} \right)$$

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[*] P. Bohl, "Sur certaines équations différentielles d'un type général utilisables en mécanique," *Bull. Soc. Math. de France*, tome 38 (1910), p. 18.

From the manner in which equations (1.6) were derived, it is seen that if a_k, a_l are conjugate imaginaries, the functions $\phi_i^{(k)}$, $\phi_i^{(l)}$ are conjugates, as are $\psi_i^{(k)}$, $\psi_i^{(l)}$; and the functions X_i, Y_i, U_{rs}, V_{rs} are real functions of the present variables. From (1.08), (1.7), we obtain

$$(1.92) \qquad F = F_0 + \sum_{i=1}^{M} \pm (\bar{x}_i^2 + \bar{y}_i^2) + \sum_{klrs}{}' A_{r,s}^{k,l} u_{kr} v_{ls} + F_3,$$

in which the constants A_{rs}^{kl} are real.

Suppose (a): $| \bar{x}_i | \leqq a$, $| \bar{y}_i | \leqq a$, $| u_{rs} | \leqq a$, $| v_{rs} | \leqq a$, a region such that the original variables x_i, y_i lie within (D) when the new variables lie within (a). The functions X_i, Y_i, U_{rs}, V_{rs} possess continuous partial derivatives of the first order with respect to the new variables, which vanish with their arguments. If $g(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs})$ is the largest of the absolute values of these derivatives at the point $\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs}$), then

$$| X_i | \leqq g(\tau\bar{x}_i, \tau\bar{y}_i, \tau u_{rs}, \tau y_{rs}) \left[\sum_i (| \bar{x}_i | + | \bar{y}_i |) + \sum_{r,s} (| u_{rs} | + | v_{rs} |)\right], \qquad 0 < \tau < 1,$$

and similar inequalities hold, for other values of τ, for the other functions.

Instead of a single trajectory consider two, represented by the variables

$$\bar{x}_i, \ \bar{y}_i, \ u_{rs}, \ v_{rs}, \quad \bar{x}_i + x'_i, \ \bar{y}_i + y'_i, \ u_{rs} + u'_{rs}, \ v_{rs} + v'_{rs};$$

since equations (1.8) are valid for both, we obtain a similar set of equations

$$(1.81) \qquad dx_i'/dt = -p_i y_i' + X_i', \quad dy_i'/dt = p_i x_i' + Y_i', \quad \text{etc.},$$

such that if X'_i, Y'_i, U'_{rs}, V'_{rs} represent the differences between the unprimed functions at the two points considered, the new equations (1.81) contain the primed variables and functions where the equations (1.8) contain the unprimed variables and functions. Equations similar to (1.9), (1.91) will also be obtained. We shall have

$$| X_i' | \leqq g(\bar{x}_i + \tau x_i', \bar{y}_i + \tau y_i', u_{rs} + \tau u_{rs}', v_{rs} + \tau v_{rs}') \left[\sum_i (| x_i' | + | y_i' |) + \sum_{r,s} (| u_{rs}' | + | v_{rs}' |)\right], \qquad 0 < \tau < 1, \quad \text{etc.}$$

2. *Certain fundamental inequalities.*

Suppose $l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_{K+L}$ any set of positive numbers, and $s_1, s_2, \cdots, s_M, t_1, t_2, \cdots, t_M, z_{rs}$ ($s = 1, \cdots, v_r$, $r = 1, \cdots, K + L$) a set of real variables. Then from the inequality

$$\sqrt{m} \sqrt{\sum_{i=1}^{m} a_i^2} \geqq \sum_{r=1}^{m} | a_i |$$

and the introduction of the constants

$$\bar{k} \leqq l_i, \quad \bar{k} \leqq m_r q_r^{s-\frac{1}{2}}, \quad \bar{g} \geqq l_i, \quad \bar{g} \geqq m_r q_r^{s-\frac{1}{2}}, \quad N = n - M, \tag{2.1}$$

$$\Gamma = (1/\bar{k}) \sqrt{2M + N}, \qquad \Delta = \bar{g} \sqrt{2M + N}$$

we find

$$\Gamma \sqrt{\sum_{i=1}^{M} l_i^2 (s_i^2 + t_i^2) + \sum_{r=1}^{K} m_r^2 g_r(z_r) + \sum_{r=K+1}^{K+L} m_r^2 [g_r(z_r) + g_r(z_{L+r})]}$$

$$\geqq \sum_{r=1}^{M} (|\, s_i \,| + |\, t_i \,|) + \sum_{r,s} |\, z_{rs} \,| \tag{2.2}$$

$$\Delta \sqrt{\sum_{i=1}^{M} l_i^2 (s_i^2 + t_i^2) + \sum_{r=1}^{K} m_r^2 g_r(z_r) + \sum_{r=K+1}^{K+L} m_r^2 [g_r(z_r) + g_r(z_{L+r})]} \geqq$$

$$\geqq \sum_{i=1}^{M} l_i^2 (|\, s_i \,| + |\, t_i \,|) + \sum_{r=1}^{K} \sum_{s=1}^{\nu_r} m_r^2 q_r^{2s-1} |\, z_{rs} \,|$$

$$+ \sum_{r=K+1}^{K+L} \sum_{s=1}^{\nu_r} m_r^2 q_r^{2s-1} (|\, z_{rs} \,| + z_{L+r,s} \,|)$$

Suppose $q_{K+L+r} = q_{K+r}$; let

$$G(u) = \sum_{r=1}^{K-2L} g_r(u_r)$$

and

$$h_r(u_r) = q_r^2 u_{r1}^2 + q_r^{2\nu_r} u^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^m u_{rm} + q_r^{m-1} u_{r,m-1})^2,$$

$$r = 1, 2 \cdots K + 2L \tag{2.3}$$

$$\bar{h}_r(v_r) = q_r^2 v_{r1}^2 + q_r^{2\nu_r} v^2_{r,\nu_r} + \sum_{m=2}^{\nu_r} (q_r^m v_{rm} - q_r^{m-1} v_{r,m-1})^2,$$

$$H(u) = \sum_{r=1}^{K+2L} h_r(u_r), \qquad \bar{H}(v) = \sum_{r=1}^{K+2L} \bar{h}_r(v_r).$$

Since the quadratic forms $h_r(u_r)$, $\bar{h}_r(v_r)$ are positive definite, there exist positive constants λ_r, $\bar{\lambda}_r$ such that these forms satisfy the inequalities

$$h_r(u_r) - \lambda_r g_r(u_r) \geqq 0, \quad \bar{h}_r(v_r) - \bar{\lambda}_r g_r(v_r) \geqq 0,$$

and it will be assumed that these constants have their greatest values.*

---

* It follows that the quantities λ_r, $\bar{\lambda}_r$ are roots of the characteristic equations of the pencils of forms

$$h_r(u_r) - \lambda_r g_r(u_r), \quad h_r(v_r) - \bar{\lambda} g_r(v_r),$$

respectively. Since the transformation

$$v_{r1} = u_{r1}, \quad v_{r2} = -u_{r2}, \quad v_{r,2m-1} = u_{r,2m-1}, \quad v_{r,2m} = -u_{r,2m}, \; \cdots$$

carries the second form into the first, the characteristic equations have the same roots, the smallest of these roots being the values of λ_r, $\bar{\lambda}_r$.

Suppose P, Q constants such that $Q \geqq q_r^{2s-1} \geqq P$, $(s = 1, \cdots \nu_r, r = 1, \cdots K + L)$; then

$$\sum_{r,s} q_r^{2s-1} \mid z_{rs} \mid \leqq \sqrt{NQ} \ \sqrt{G(z)}, \quad \sum_{r,s} \mid z_{rs} \mid \leqq \sqrt{N/P} \cdot \sqrt{G(z)}. \tag{2.4}$$

If

$$r' = \sum_{i=1}^{M} l_i^2(x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(v_r') + \sum_{r=K+1}^{K+L} m_r^2 [g_r(v_r') + g_{L+r}(\dot{v}'_{L+r})] - G(u'),$$

then from (1.9), (1.91), if q is the smallest of the constants $\bar{\lambda}_1 \cdots \bar{\lambda}_{K+L}$, $\lambda_1 \cdots \lambda_{K+L}$, and if $r' \leqq 0$, $t = t_0$, we obtain easily

$$dr'/dt \leqq [-q + 2g(\Gamma + \sqrt{N/P})(\Delta + \sqrt{NQ})] G(u'), \quad t = t_0.$$

If in (a),

$$g < \frac{q}{2(\Gamma + \sqrt{N/P})(\Delta + \sqrt{NQ})} \tag{2.5}$$

then $dr'/dt \leqq 0$.

Suppose this inequality for g satisfied within (a). If the two trajectories lie in (a) for $t_0 \leqq t \leqq t_1$, and $r' \leqq 0$ for $t = t_0$, while $G(u') > 0$, then $dr'/dt < 0$, $t = t_0$. Suppose $r' \geqq 0$ for $t = t_1$; at some time $t' \leqq t_1$, $r' = 0$, and $r' < 0$, $t_0 < t < t'$. Hence there exists also a time $t'' < t'$ for which $dr'/dt = 0$; but at this time $r' < 0$, hence $dr'/dt < 0$. From this contradiction we conclude that $r' < 0$ when $t = t_1$, and $r' < 0$, $t_0 < t \leqq t_1$. Hence from the definition of r', $G(u') > 0$, $t_0 < t \leqq t_1$, and if $G(u') \neq 0$, $t = t_0$, then $G(u') > 0$, $t_0 \leqq t \leqq t_1$.

Under the assumption $r' \leqq 0$, and the preceding inequalities,

$$\begin{aligned} d/dt \ G(u') &= H(u') + 2 \Big[\sum_{r=1}^{K} \sum_{s} q_r^{2s-1} u_{rs}' U_{rs}' + \sum_{r=K+1}^{K+L} \sum_{s} q_r^{2s-1} (u_{rs}' U_{rs}' + u'_{L+r,s} U'_{L+r,s}) \Big] \\ &\geqq H(u') - 2g \Big[\sum_{r=1}^{K} \sum_{s} q_r^{2s-1} \mid u_{rs}' \mid + \sum_{r=K+1}^{K+L} \sum_{s} q_r^{2s-1} (\mid u_{rs}' \mid + \mid u'_{L+r,s} \mid) \Big] \\ &\qquad [\sum_{i} (\mid x_i' \mid + \mid y_i' \mid + \sum_{r,s} \mid u_{rs}' \mid + \sum_{r,s} \mid v'_{rs} \mid] \\ &\geqq H(u') - 2g \sqrt{NQ} \sqrt{G(u')} (\Gamma + \sqrt{N/P}) \sqrt{G(u')} \\ &\geqq [q - 2g \ (\Gamma + \sqrt{N/P}) \sqrt{NQ}] \ G(u'). \end{aligned}$$

If the constant g satisfies (2.5), there can be found a constant σ such that

$$q > \sigma > 2g \sqrt{NQ} (\Gamma + \sqrt{N/P});$$

and

$$dG(u')/dt > (q-\sigma)G(u'),$$

whence

$$[G(u')]_{t=t_1} > e^{(q-\sigma)(t_1-t_0)}[G(u')]_{t=t_0}.$$

From the preceding results we have the theorem: *If* $\sigma < q$, and l_1, l_2, $\cdots$, l_M, m_1, m_2, $\cdots$, m_{K+L} *are any positive numbers, then if a neighborhood* (a) *of the origin is so chosen that within it the partial derivatives of* X_i, Y_i, U_{rs}, V_{rs} *are less in absolute value than either of the quantities*

$$\frac{q}{2(\Gamma+\sqrt{N/P})(\Delta+\sqrt{NQ})}, \quad \frac{\sigma}{2\sqrt{NQ}\,(\Gamma+\sqrt{N/P})}$$

where Δ, Γ, P, Q, *are the constants previously defined, this neighborhood has the following properties:* 1. *if the coordinates of two trajectories lie in* (a) *for* $t_0 \leqq t \leqq t_1$ *and if when* $t=t_0$, $G(u') \neq 0$,

$$r' = \sum_{i=1}^{M} l_i^2\,(x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(v_r') + \sum_{r=K+1}^{K+L} m_r^2[g_r(v_r') + g_r(v'_{L+r})] - G(u') \leqq 0,$$

then $r' < 0$ *for* $t_0 < t \leqq t_1$, *and*

$$[G(u')]_{t=t_1} > e^{(q-\sigma)(t_1-t_0)}[G(u')]_{t=t_0}.$$

2. *If the coordinates defining two trajectroies lie in* (a) *for* $t_0 \geqq t \geqq t_1$ $(t_0 > t_1)$, *and if for* $t=t_0$, $G(v') \neq 0$, and

$$s' = \sum_{i=1}^{M} l_i^2(x_i'^2 + y_i'^2) + \sum_{r=1}^{K} m_r^2 g_r(u'_r) + \sum_{r=K+1}^{K+L} m_r^2[g_r(u_r') + g_r(u'_{L+r})] - G(v') \leqq 0,$$

then $s' < 0$ *for* $t_0 > t \geqq t_1$, *and*

$$[G(v')]_{t=t_1} > e^{(q-\sigma)(t_0-t_1)}[G(v')]_{t=t_0}.$$

The last part of the theorem can be demonstrated in the same manner.

3. *An application of the energy integral.*

In what follows it will be assumed that the form $\sum_{i=1}^{M} \pm (\bar{x}_i^2 + \bar{y}_i^2)$ is definite; suppose the constant A so large that for all values of the variables concerned,

$$-A[G(u)+G(v)] \leqq \sum_{klrs}' A_{rs}^{k,l} u_{kr} v_{ls} \leqq A[G(u)+G(v)].$$

Assume μ, ν, ρ positive constants which satisfy the inequality

$$\rho + \mu + \rho\mu + 4\nu(A + \mu/P) < 1.$$

Since the first and second partial derivatives of F_3 vanish with its arguments, we can assume that in (E), a sub-region of (a),

$$(3.1) \qquad | F_3 | \leqq \mu \left[\sum_i (\bar{x}_i^2 + \bar{y}_i^2) + \sum_{r,s} (u_{rs}^2 + v_{rs}^2) \right].$$

Then suppose $\delta > 0$ such that the region defined by

$$(\mathrm{R}') \qquad \sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \delta^2,$$

$$(\mathrm{S}) \qquad G(u) \leqq \nu\delta^2, \qquad G(v) \leqq \nu\delta^2$$

lies within (E), and let (R″) be the region

$$(\mathrm{R}'') \qquad \sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho\delta^2.$$

From the energy integral, if a trajectory lies in (R′), (S) at $t = t_1$, in (R″), (S) at $t = t_2$,

$$\begin{aligned} \left[\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2)\right]_{t=t_1} &= \pm \left[\pm\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) + \sum_{klrs}{}' A_{rs}^{k,l} u_{kr} v_{ls} + F_3 \right]_{t=t_2} \\ &\quad \mp \left[\sum_{klrs}{}' A_{rs}^{kl} u_{kr} v_{ls} + F_3 \right]_{t=t_1} \\ &\leqq \rho\delta^2 + 4A\nu\delta^2 + \mu[\delta^2 + \rho\delta^2 + 1/P(2\nu\delta^2 + 2\nu\delta^2)] \\ (3.2) \qquad &\leqq [\rho + \mu + \rho\mu + 4\nu(A + \mu/P)]\delta^2 < \delta^2. \end{aligned}$$

Let (H) be a region lying within (E), and such that in (H) the partial derivatives of X_i, Y_i, U_{rs}, V_{rs} are less in absolute value than θ where

$$\theta < \frac{q\nu}{2\sqrt{NQ\nu}\,(\sqrt{2M} + 2\sqrt{N\nu/P})}.$$

Suppose d so small that the region

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq d^2, \quad G(u) \leqq \nu d^2, \quad G(v) \leqq \nu d^2$$

lies within (H), and construct the regions

$$(\mathrm{A}) \qquad \sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq d^2,$$

$$(a) \qquad \sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2,$$

$$(\mathrm{B}) \qquad G(u) \leqq \nu d^2, \quad G(v) \leqq \nu d^2.$$

It results that *if* $(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs})$ *lies in* (a, B) *for* $t = t_2$, *in* (A, B) *for* $t = t_1$, *and if* $G(u) = \nu d^2$ *for* $t = t_1$, *then* $dG(u)/dt > 0$, $t = t_1$.

For when $t = t_1$, from (3.2)

$$dG(u)/dt \geqq H(u) - 2\theta\sqrt{NQG(u)}$$
$$[\sqrt{2M}\sqrt{\sum_i (\bar{x}_i^2 + \bar{y}_i^2)} + \sqrt{2N/P}\sqrt{G(u) + G(v)}]$$
$$\geqq [q\nu - 2\theta\sqrt{NQ\nu}\,(\sqrt{2M} + 2\sqrt{N\nu/P})]d^2 > 0.$$

4. *Existence theorems.*

With the definitions of the domains (A), (a), (B) of the preceding paragraph, we obtain at once, by the reasoning of the first paper (§ 4, pp. 40-42) the following theorems: 1. *To every set of values* $(\bar{x}_i, \bar{y}_i)$, (v_{rs}) *of the region*

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2, \quad G(v) \leqq \nu d^2,$$

can be adjoined a set of values (u_{rs}) *of the region* $G(u) \leqq \nu d^2$ *such that the trajectory having these initial values at* $t = t_0$ *remains in the region* (A, B) *for* $t > t_0$.

2. *To every set of values* $(\bar{x}_i, \bar{y}_i)$, (u_{rs}) *of the region*

$$\sum_{i=1}^{M} (\bar{x}_i^2 + \bar{y}_i^2) \leqq \rho d^2, \quad G(u) \leqq \nu d^2,$$

can be adjoined a set of values (v_{rs}) *of the region* $G(v) \leqq \nu d^2$ *such that the trajectory having these initial values at* $t = t_0$ *lies in the region* (A, B) *for* $t < t_0$.

Let (H′) be the region (H) for the particular set of values $l_1 = l_2 = \cdots = l_M = 1$, $m_1 = m_2 = \cdots = m_{K+L} = m < 1$; from § 2 the quantity r' formed for two trajectories which lie in (H′) for $t \geqq t_0$,

$$r' = \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(v') - G(u')$$

must be positive or zero, otherwise $G(u')$ would increase indefinitely with t. Consequently

$$G(u') \leqq \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(v'), \quad t \geqq t_0; \tag{4.1}$$

and *if two trajectories lie in* (H') *for* $t \geqq t_0$, *and if at any instant* $t' > t_0$, $x_i' = y_i' = v_{rs}' = 0$, *then* $u_{rs}' = 0$, *and the two trajectories coincide.*

In the same manner we obtain the result that *if two trajectories lie in* (H') *for* $t \leqq t_0$, *and if at any instant* $t' < t_0$, $x_i' = y_i' = u_{rs}' = 0$, *then* $v_{rs}' = 0$, *and the two trajectories coincide.*

From the last two theorems it follows that the set of values (u_{rs}) corresponding to the given set of values $(\bar{x}_i, \bar{y}_i, v_{rs})$ in the first existence theorem above is uniquely determined; these values, functions of the remaining variables, will be denoted by $[u_{rs}]$: similarly, the values (v_{rs}) whose existence is asserted in the second existence theorem, will be denoted by $[v_{rs}]$. The continuity of these functions follows at once from (4.1) and the corresponding inequality

$$G(v') \leqq \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(u'), \quad t \leqq t_0. \tag{4.2}$$

If a set of values $(\bar{x}_i, \bar{y}_i)$ satisfying (α), § 3, is fixed, the functions

$$u_{rs} - [u_{rs}], \quad v_{rs} - [v_{rs}], \quad s = 1 \cdots \nu_r, \quad r = 1 \cdots K + 2L$$

are continuous in u_{rs}, v_{rs} within and on the boundary of

$$G(u) \leqq \nu d^2, \qquad G(v) \leqq \nu d^2. \tag{B}$$

If no set of values of u_{rs}, v_{rs} exists for which these functions vanish simultaneously, from a theorem of Bohl a set of values must exist on the boundary for which one of the systems of equations must hold:

$$u_{rs} - [u_{rs}] = -\lambda' u_{rs}, \quad v_{rs} - [v_{rs}] = -\lambda' v_{rs}, \quad \lambda' > 0,$$

$$(1 + \lambda') u_{rs} = [u_{rs}], \quad (1 + \lambda') v_{rs} = [v_{rs}], \quad s = 1 \cdots \nu_r, \quad r = 1 \cdots K + 2l$$

Consequently,

$$(1 + \lambda')^2 G(u) = G([u]) \text{ or } (1 + \lambda')^2 G(v) = G([v])$$

and therefore either

$$(1 + \lambda')^2 \nu d^2 = G([u]) \text{ or } (1 + \lambda')^2 \nu d^2 = G([v]).$$

But this is impossible, since the values $[u_{rs}]$, $[v_{rs}]$ satisfy the inequalities (B). Consequently values must exist for which

$$u_{rs} = [u_{rs}], \quad v_{rs} = [v_{rs}]$$

from which we obtain the theorem: *to every set of values of* (α) *can be adjoined a set of values of* (B) *such that the trajectory possessing these initial values at any instant remains in the region* (A, B) *for all values of* t.

If two trajectories remain in (A, B) for all values of t, we shall have

$$\sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(v') \geqq G(u'),$$

$$\sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(u') \geqq G(v'),$$

from which

$$\sum_{i=1}^{M} (x_i'^2 + y_i'^2) \geqq \frac{1 - m^2}{2} \, [G(u') + G(v')].$$

Hence if at any instant $x_i' = y_i' = 0$, then $u_{rs}' = v_{rs}' = 0$, and the two trajectories coincide.

5. *An inequality.*

Let l, m, h be positive, and let a, β satisfy the inequalities $a \leqq l$, $mq_r^{s-\frac{1}{2}}$, $q_r^{s-\frac{1}{2}} \leqq \beta$. Assume k a number satisfying

$$k < \frac{qa}{8\beta(M+N)}, \quad k < \frac{ha}{2\beta(M+N)}.$$

If

$$\omega = e^{2ht}[l^2 \sum_{i=1}^{M} (x_i'^2 + y_i'^2) + m^2 G(u') - G(v')],$$

where x_i', y_i' u_{rs}', v_{rs}' are the differences for two trajectories in the region (C) where the partial derivatives of X_i', Y_i', U_{rs}', V_{rs}' are less in absolute value than k, then from (1.81),

$$\begin{aligned} d\omega/dt = 2h\omega + e^{2ht}[m^2 H(u') + 2l^2 \sum_{i=1}^{M} (x_i' X_i' + y_i' Y_i') \\ + \sum_{r,s} 2m^2 q_r^{2s-1} u_{rs}' U_{rs}'] \\ - e^{2ht}[-H(v') + 2 \sum_{r,s} q_r^{2s-1} v_{rs}' V_{rs}'] \end{aligned}$$

Now

$$\begin{aligned} |X_i'|, |Y_i'|, |U_{rs}'|, |V_{rs}'| &\leqq k[\sum_i (|x_i'| + |y_i'|) + \sum_{r,s}(|u_{rs}'| + |v_{rs}'|)] \\ &\leqq k/a \sqrt{2(M+N)} \sqrt{\sum_i l^2(x_i'^2 + y_i'^2) + m^2 G(u') + G(v')} \end{aligned}$$

Also,

$$\begin{aligned} l^2 \sum_i (|x_i'| + |y_i'|) + m^2 \sum_{r,s} q_r^{2s-1} |u_{rs}'| + \sum_{r,s} q_r^{2s-1} |v_{rs}'| \\ \leqq \beta\, [l \sum_i (|x_i'| + |y_i'|) + m \sum_{r,s} q_r^{s-\frac{1}{2}} |u_{rs}'| + \sum_{r,s} q_r^{s-\frac{1}{2}} |v_{rs}'|] \\ \leqq \beta \sqrt{2(M+N)} \sqrt{l^2 \sum_i (x_i'^2 + y_i'^2) + m^2 \, G(u') + G(v')} \end{aligned}$$

and consequently

$$d\omega/dt \geqq 2h\omega + e^{2ht}[qm^2G(u') + qG(v') - 2\,\frac{k\beta}{a}\,(2M + 2N)$$

$$\{\sum_{i=1}^{M} l^2(x_i'^2 + y_i'^2) + m^2G(u') + G(v')\}]$$

$$\geqq 2[h - \frac{k\beta}{a}\,(2M + 2N)]\omega$$

$$+ e^{2ht}[q - \frac{8\beta k}{a}\,(M + N)]G(v') + qm^2G(u')e^{2ht}$$

$$\geqq 2\rho\omega + \theta e^{2ht}G(v'), \quad (\rho = h - \frac{2k\beta}{a}\,(M + N), \quad \theta = q - \frac{8\beta k}{a}\,(M + N)).$$

From which

$$d/dt\,[e^{-2\rho t}\omega] \geqq 0.$$

If ω is positive for any value of $t > t_0$, ω will therefore increase until one of the trajectories leaves the region (C); if $\omega = 0$ for $t = t_0$ but the two trajectories do not coincide, $G(v') > 0$, hence $d\omega/dt > 0$ and ω takes on positive values. Hence if both trajectories remain in (C) for $t > t_0$, and ω is bounded, then $\omega < 0$ unless the two trajectories coincide.

6. *Asymptotic character of certain trajectories.*

Suppose h a positive constant less than q, q_r, and suppose A, B so small that the region

$$\text{(J)} \qquad |\bar{x}_i|,\ |\bar{y}_i| \leqq A, \qquad |u_{rs}|,\ |v_{rs}| \leqq B$$

lies within (a) and in (J) the partial derivatives of X_i, Y_i, U_{rs}, V_{rs} are less in absolute value than η, a constant to be fixed later. Then we have the theorem: *The constants* η, $\bar{\bar{H}}$, H *can be so chosen that if* $H < A$, $H < B$, *and* (K) *is the region*

$$\text{(K)} \qquad |\bar{x}_i|,\ |\bar{y}_i| \leqq A - H, \qquad |u_{rs}|,\ |v_{rs}| \leqq B - H,$$

then if a trajectory σ *lies in* (K) *for* $t \geqq t_0$, *suppose* (v_{rs}^0) *its set of v-values at* $t = t_0$: *if an arbitrary set of v-values is chosen such that*

$$\sum_{r,s} (v_{rs} - v_{rs}^0)^2 (q_r - h)^{2s-1} \leqq \bar{H}^2,$$

then there is one and only one trajectory whose v-values coincide with these for $t = t_0$, *which remains in* (J) *for* $t > t_0$, *and approaches* σ *asymptotically as* $t \rightarrow \infty$, *in such a manner that the differences of the coordinates, multiplied by* e^{ht}, *remain bounded for* $t > t_0$.

Let the coordinates of σ be represented by the variables $\bar{\bar{x}}_i$, $\bar{\bar{y}}_i$, $\bar{u}_{rs}$, $\bar{v}_{rs}$, and suppose ϵ so small that the region

$$|\bar{x}_i|,\ |\bar{y}_i| \leqq A + 2(e^{h\epsilon}-1)H, \quad |u_{rs}|,\ |v_{rs}| \leqq B + 2(e^{h\epsilon}-1)H,$$

lies in (a). Choose a constant $\theta < \epsilon$ so small that for $t_0 - \theta \leqq t \leqq t_0$,

$$|\bar{\bar{x}}_i|,\ |\bar{\bar{y}}| \leqq A - H + (e^{h\epsilon}-1)H, \quad |\bar{u}_{rs}|,\ |\bar{v}_{rs}| \leqq B - H + (e^{h\epsilon}-1)H.$$

Then if

$$(6.01) \qquad |\xi_i|,\ |\eta_i| \leqq e^{h(\epsilon-\theta)}H, \quad |z_{rs}|,\ |w_{rs}| \leqq e^{h(\epsilon-\theta)}H,$$

we shall have

$$|\bar{\bar{x}}_i + e^{-h\tau}\xi_i|; \quad |\bar{\bar{y}}_i + e^{-h\tau}\eta_i| \leqq A + 2(e^{h\epsilon}-1)H,$$
$$|\bar{u}_{rs} + e^{-h\tau}z_{rs},| \quad |\bar{v}_{rs} + e^{-h\tau}w_{rs}| \leqq B + 2(e^{h\epsilon}-1)H.$$
$$\tau = t - t_0 > -\theta,$$

Making the substitution

$$(6.1) \qquad \bar{x}_i = \bar{\bar{x}}_i + e^{-h\tau}\xi_i, \quad \bar{y}_i = \bar{\bar{y}}_i + e^{-h\tau}\eta_i,$$
$$u_{rs} = \bar{u}_{rs} + e^{-h\tau}z_{rs}, \quad v_{rs} = \bar{v}_{rs} + e^{-h\tau}w_{rs}$$

we obtain from (1.8) the equations for the new variables ξ_i, η_i, z_{rs}, w_{rs}. These become

$$\text{(a)} \quad d\xi_i/dt = h\xi_i - p_i\eta_i + \bar{X}_i', \quad d\eta_i/dt = p_i\xi_i + h\eta_i + \bar{Y}_i', \quad i = \cdots M.$$

$$\text{(b)} \quad dz_{r1}/dt = (h+q_r)z_{r1} + \bar{U}_{r1}', \quad dw_{r1}/dt = (h-q_r)w_{r1} + \bar{V}_{r1}'$$
$$i = 2 \cdots v_r,$$
$$dz_{ri}/dt = z_{r,i-1} + (h+q_r)z_{ri} + \bar{U}_{ri}',$$
$$dw_{ri}/dt = w_{r,i-1} + (h-q_r)w_{ri} + \bar{V}_{ri}', \qquad \mathrm{r} = 1 \cdots K.$$

$$(6.2) \quad dz_{r1}/dt = (h+q_r)z_{r1} + \bar{U}_{r1}', \quad dw_{r1}/dt = (h-q_r)w_{r1} + \bar{V}_{r1}',$$
$$dz_{r,i}/dt = z_{r,i-1} + (h+q_r)z_{ri} - q_r'z_{L+r,i} + \bar{U}_{ri}',$$
$$dw_{ri}/dt = w_{r,i-1} + (h-q_r)w_{ri} + q_r'w_{L+r,i} + \bar{V}_{ri}',$$
$$\text{(c)} \quad dz_{L+r,1}/dt = q_r'z_{r1} + (h+q_r)z_{L+r,1} + \bar{U}'_{L+r,1},$$
$$dw_{L+r,1}/dt = -q_r'w_{r1} + (h-q_r)w_{L+r,1} + \bar{V}'_{L+r,1},$$
$$dz_{L+r,i}/dt = z_{L+r,i-1} + q_r'z_{ri} + (h+q_r)z_{L+r,i} + \bar{U}'_{L+r,i},$$
$$dw_{L+r,i}/dt = w_{L+r,i-1} - q_r'w_{ri} + (h-q_r)w_{L+r,i} + \bar{V}'_{L+r,i},$$
$$i = 2 \cdots v_r, \quad r = K+1 \cdots K+L.$$

where

$$\bar{X}_i' = e^{h\tau}[X_i(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs}) - X_i(\bar{\bar{x}}_i, \bar{\bar{y}}_i, \bar{u}_{rs}, \bar{v}_{rs})],$$
$$\bar{U}_{rs}' = e^{h\tau}[U_{rs}(\bar{x}_i, \bar{y}_i, u_{rs}, v_{rs}) - U_{rs}(\bar{\bar{x}}_i, \bar{\bar{y}}_i, \bar{u}_{rs}, \bar{v}_{rs})], \text{ etc.}$$

the variables $\bar{x}_i$, $\bar{y}_i$, u_{rs}, v_{rs} being given by the substitution (6.1).

The right-hand members of (6.2) are bounded and satisfy a Lipschitz condition for $t \geqq t_0 - \theta$, if the dependent variables are in the region specified above, hence the existence of solutions in a certain neighborhood of $t = t_0$ is assured.

Suppose the forms $G_h(z)$, $\bar{G}_h(w)$ defined by

$$G_h(z) = \sum_{r,s} (q_r + h)^{2s-1} z_{rs}^2, \quad \bar{G}_h(w) = \sum_{r,s} (q_r - h)^{2s-1} w_{rs}^2.$$

If $t \geqq t_0$, and the inequalities

$$(6.3) \qquad \sum_i (\xi_i^2 + \eta_i^2) + G_h(z) \leqq \bar{H}^2, \quad \bar{G}_h(w) \leqq \bar{H}^2,$$

are satisfied, then if $H_h(z)$ is obtained from $H(z)$ by replacing q_r by $q_r + h$, while $\bar{H}_h(w)$ is obtained from $\bar{H}(w)$ by replacing q_r by $q_r - h$, we obtain as in § 2,

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{i=1}^{M}(\xi_i^2 + \eta_i^2) + G_h(z)\Big] &\geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + H_h(z) \\ -2\Big[\sum_{i=1}^{M}(|\xi_i| \cdot |\bar{X}_i'| + |\eta_i| \cdot |\bar{Y}_i'|) &+ \sum_{r,s} (q_r + h)^{2s-1} |z_{rs}| \cdot |\bar{U}_{rs}'|\Big] \\ &\geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + H_h(z) \\ &\quad - 2\eta \Big[\sum_{i=1}^{M}(|\xi_i| + |\eta_i|) + \sum_{r,s} (q_r + h)^{2s-1} |z_{rs}|\Big] \\ &\quad \Big[\sum_{i=1}^{M}(|\xi_i| + |\eta_i|) + \sum_{r,s}(|z_{rs}| + |w_{rs}|)\Big]\end{aligned}$$

If $\bar{\mu}$, $\bar{\nu}$, P_h, Q_h, $\bar{P}_h$, $\bar{Q}_h$ are constants such that

$$H_h(z) \geqq \bar{\mu}\, G_h(z), \quad \bar{H}_h(w) \geqq \bar{\nu}\, \bar{G}_h(w),$$

$$Q_h \geqq (q_r + h)^{2s-1} \geqq P_h, \quad \bar{Q}_h \geqq (q_r - h)^{2s-1} \geqq \bar{P}_h,$$

then

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{i=1}^{M}(\xi_i^2 + \eta_i^2) + G_h(x)\Big] &\geqq 2h \sum_{i=1}^{M} (\xi_i^2 + \eta_i^2) + \bar{\mu} G_h(z) \\ &\quad - 2\eta \,[\sqrt{2M \textstyle\sum (\xi_i^2 + \eta_i^2)} + \sqrt{NQ_h}\,\sqrt{G_h(z)}\,] \times \\ &\quad \times \Big[\sqrt{2M \sum_i (\xi_i^2 + \eta_i^2)} + \sqrt{\frac{N}{P_h}\, G_h(z)} + \sqrt{\frac{N}{\bar{P}_h}\, \bar{G}_h(w)}\,\Big]\end{aligned}$$

If $2\bar{\rho}$ is the smaller of $2h$, $\bar{\mu}$ and if $\bar{H} < H/\sqrt{P_h}$, $\bar{H} < H/\sqrt{\bar{P}_h}$, $\bar{H} < H$ while

$$\sum_i (\xi_i^2 + \eta_i^2) + G_h(z) = \bar{H}^2, \quad \bar{G}_h(w) \leqq H^2$$

then

$$(6.4)\quad d/dt\,[\sum_i(\xi_i^2+\eta_i^2)+G_h(z)]\geqq 2\{\bar{\rho}-\eta[\sqrt{2M}+\sqrt{N/P_h}+\sqrt{N/\bar{P}_h}]\\ (\sqrt{2M}+\sqrt{NQ_h})\}\bar{H}^2.$$

We shall assume η so small that the expression on the right multiplying $\bar{H}^2$ is positive; then the left-hand member is also positive.

Also,

$$\frac{d}{dt}\bar{G}_h(w)\leqq-\bar{H}(w)+2\eta\sum_{r,s}(q_r-h)^{2s-1}|w_{rs}|\\ [\sum(|\xi_i|+|\eta_i|)+\sum_{r,s}(|z_{rs}|+|w_{rs}|)]$$

$$\leqq-\bar{H}(w)+2\eta\sqrt{Q_hN}\Big[\sqrt{2M\sum_i(\xi_i^2+\eta_i^2)}+\sqrt{\frac{N}{P_h}\,G_h(z)}\\ +\sqrt{\frac{N}{\bar{P}_h}\,\bar{G}_h(w)}\,\Big]\times\sqrt{\bar{G}_h(w)}$$

Now suppose $\bar{G}_h(w)=\bar{H}^2$; then if $\sum_i(\xi_i^2+\eta_i^2)+G_h(z)\leqq\bar{H}^2$,

$$(6.4)\quad\frac{d}{dt}\bar{G}_h(w)\leqq[-\bar{\nu}+2\eta\sqrt{Q_hN}(\sqrt{2M}+\sqrt{N/P_h}+\sqrt{N/\bar{P}_h})]\bar{H}^2.$$

it will be assumed that η is so small that the term on the right multiplying H^2 is negative. From the choice of $\bar{H}$ it follows that when the inequalities (6.3) are satisfied, the first inequalities for the present variables are satisfied also. It will be assumed that A, B are so small that (J) belongs to the region (C) of § 5.

On the basis of these inequalities, we can show* that to every set of w-values satisfying $\bar{G}_h(w)\leqq\bar{H}^2$ at least one set of values $(\xi_i,\eta_i,z_{rs})_0$ can be chosen satisfying

$$(6.5)\qquad\sum_i(\xi_i^2+\eta_i^2)+G_s(z)\leqq\bar{H}^2$$

such that $(w_{rs})_0$, $(\xi_i,\eta_i,z_{rs})_0$ are the values for $t=t_0$ of a solution remaining in the region (6.5) for $t>t_0$.

Now suppose a set of v-values (V_{rs}) chosen, such that $\bar{G}_h(V-v^0)\leqq\bar{H}^2$, and let a solution of equations (6.2) be determined in accordance with the existence theorem above, such that at $t=t_0$, $w_{rs}=-v_{rs}^0+V_{rs}$, $(s=1$

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* Bohl, *l. c.*, p. 230. The proof of the proposition stated above is so similar that it is unnecessary to repeat it here.

$\cdots v_r$, $r = 1 \cdots K + 2L$), while the solution remains in the region designated for $t > t_0$; for $t > t_0$ the variables ξ_i, η_i, z_{rs}, w_{rs} satisfy

$$|\bar{\bar{x}}_i + e^{-h\tau}\xi_i|, \quad |\bar{\bar{y}}_i + e^{-h\tau}\eta_i| \leqq A,$$
$$|\bar{u}_{rs} + e^{-h\tau}z_{rs}|, \quad |\bar{v}_{rs} + e^{-h\tau}w_{rs}| \leqq B$$

while for $t = t_0$, $v_{rs}^0 + w_{rs} = V_{rs}$. The trajectory defined by

$$\bar{x}_i = \bar{\bar{x}}_i + e^{-h\tau}\xi_i, \quad \bar{y}_i = \bar{\bar{y}}_i + e^{-h\tau}\eta_i, \quad i = 1 \cdots M,$$
$$u_{rs} = \bar{u}_{rs} + e^{-h\tau}\xi_{rs}, \quad v_{rs} = \bar{v}_{rs} + e^{-h\tau}w_{rs},$$
$$s = 1 \cdots v_r, \quad r = 1 \cdots K + 2L,$$

satisfies the first part of the theorem; it is evident that the differences $(e^{-h\tau}\xi_i, e^{-h\tau}\eta_i, e^{-h\tau}z_{rs}, e^{-h\tau}w_{rs})$ multiplied by e^{ht}, remain bounded.

The second part of the theorem follows easily from § 5. In these inequalities suppose $l = m = 1$, and suppose the quantity ω constructed for two solutions remaining in (J) for $t \geqq t_0$, in such a manner that the coordinate differences, multiplied by e^{ht}, remain bounded for $t > t_0$, the constant h being such as to satisfy the inequalities of the present paragraph. Then

$$\omega = e^{2ht}\left[\sum_{i=1}^{M}(x_i'^2 + y_i'^2) + G(u') - G(v')\right]$$

is bounded, and consequently $\omega \leqq 0$ for $t > t_0$. If the differences v_{rs}' vanish at $t = t_0$, it follows that the remaining differences x_i', y_i', u_{rs}', vanish also, and the two trajectories coincide. Hence when the v-values are fixed, only one trajectory having the properties stated can exist.

Applications of the Determinant and Permanent Tensors to Determinants of General class and Allied Tensor Functions.

BY CLYDE M. CRAMLET.

1. *Introduction.* The geometric scheme of representing the elements of a detetrminant as an arrangement of rows and columns or of layers, sections and sheets * becomes cumbersome in the general study of determinants of elements with several indices. The natural analytical method in these cases is to confine attention to the general term of the determinant. Various methods are used to determine the signs of this term and to exclude the terms that should not appear. The method most like that used here is the scheme of Cayley † of using a square determinant in this capacity. Recently ‡¶ papers have appeared dealing with square determinants by a new method. It is along these lines that the work of this paper will proceed. Tensor notation will be used throughout but that is merely for considerations of symmetry. The results are valid for any tensor character that might be assigned. It is important to note, however, that with the transformation character that we shall assign our elements, the laws of transformation of the functions of the elements can be read from the indices. The modifications for different transformations on the elements will be at once apparent. To illustrate, suppose g_{rst} is one of n^3 elements that appears in our work, and that it is a coefficient in the trilinear form $g_{\rho\sigma\tau}x^\rho y^\sigma z^\tau$, then since the ground form is invariant the indices in g_{rst} are all covariant. The transformation character of the determinant or whatever function of these elements that is discussed in our work can be read off at once. On the other hand suppose the g_{rst} are the n^3 coefficients in a system of n quadratics $g_{r\sigma\tau}x^\sigma x^\tau$, $r = 1 \cdots n$, then the first index on the g is not of tensor (or transformation) character. It might be well in this case to indicate it as $g_{\bar{r}st}$. The modifications for this situation can be made in any equations in which the first index is not multiplied by inner multiplication by a tensor.

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* Hedrick, *Annals of Mathematics*, Vol. 1 (1899), page 49.

† *Trans. of Cambridge Phil. Soc.*, London, Vol. 8, Part 1, 1844 (2-43), S. 11, p. 85-8. (Reference from Lecat's *Leçons sur la Théorie des Déterminants à n dimensions.*)

‡ Murnaghan, *Math. Monthly*, XXXII, 1925, p. 233.

¶ The writer: *Annals of Mathematics*, Vol. 27, No. 4, September, 1926. This will be referred to hereafter as paper I.

2. *Definitions and Conventions.* We define a tensor* by the equation,

$$\gamma^{a_1 \cdots a_n}_{b_1 \cdots b_n} = \sum_p A^{(p)} \delta^{a_1}_{b_1} \cdots \delta^{a_n}_{b_n} \tag{2.1}$$

where the $n!$ terms formed by permuting the a's and multiplying each by an arbitrary scalar, are added. The $n!$ scalors are distinguished by the $n!$ permutations p. When the scalors are ± 1 according as the permutation p is even or odd we have

$$\delta^{a_1 \cdots a_n}_{b_1 \cdots b_n} \tag{2.2}$$

This tensor vanishes unless the a's and b's are the same set of distinct integers when it equals ± 1 according as the substitution to bring them to the same order is even or odd. This will be called the *determinant tensor*.† When the scalars are all equal to 1 we have

$$\pi^{a_1 \cdots a_n}_{b_1 \cdots b_n} \tag{2.3}$$

which will be called the *permanent tensor*. It vanishes unless the a's and b's are identical sets. For the non-vanishing cases the values are 1 if the sets are composed of distinct integers. If there are q groups with $p_1 \cdots p_q$ repeated or equal indices $p_1 + \cdots p_q \ngtr n$ the value of the tensors is $p_1! + \cdots p_q!$.

An inner product of γ and a set of elements such as

$$\gamma^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} \quad g_{r_1\rho_1} \cdots g_{r_n\rho_n} \tag{2.4}$$

will be called a *tensor function* of the elements. The γ's are of the tensor character indicated yet they are absolute invariants. If a δ is used in place of the γ we have a *determinant function,* and likewise, using a π, a *permanent function.* For functions involving elements with more than two indices we may use combinations of these three. The *degree* of a tensor function will be equal to the number of elements occurring in a term. This term will replace the less significant word, *order.* The set of n indices in the same position on the elements of a term will be called a *range.* Greek letters, only, when repeated will indicate a summation. Repeated letters are called umbral. For the non-repeated we venture the name *virtual.* The virtual ranges on the elements will be called fixed as is customary. The *class* is one greater than the number of umbral ranges, i. e. one greater than the number of tensor multipliers when each multiplier has one virtual range. If there is but one fixed range the class is equal to the number of ranges.

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* Properties of this tensor were given in a paper read before the Society April 3, 1926.

† Murnaghan has suggested the name "generalized Kronecker symbol." Either (2.2) or (2.3) for $n = 1$ becomes the Kronecker symbol.

These definitions and certain general properties will be illustrated by the function,

$$g_{[r_n](s_n)(t_n)} = \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \ \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \ g_{r_1\sigma_1\tau_1} \cdots g_{r_n\sigma_n\tau_n} \qquad (2.5)$$

This is a class 3 (or 3 range) determinant function of degree n. This expression has n^{3n} components corresponding to the integers assigned the virtual ranges. The value of the component of such an expression when the virtual ranges take the sequence $1 \cdots n$ is ordinarily called a *determinant* or in case π's replace δ's, a *permanent* with appropriate names for combinations. We shall not be careful, however, to make this distinction and will speak of 2.5 for example, thinking of any component, as a determinant.

3. *Elementary properties of the fundamental tensors.* The elementary properties of a tensor function are apparent in the notation.

(1) The virtual range on the multiplying tensor π or δ is permanent or alternant.

(2) Any substitution on an umbral range of elements only is equivalent to the same substitution on the virtual indices of the multiplying tensors carrying the same umbral indices.

(3) If the tensor function has a γ for a multiplying tensor the fixed range is in general neither permanent nor alternant, otherwise the fixed range is permanent or alternant according as the number of δ's having one range multiplied by inner multiplication (i. e. summed) is even or odd. The proof of the property is as follows: Consider the right member of (2.5) (to illustrate) when the r's are rearranged by a substitution S. They may be arranged in the original order by a permutation S^{-1} on the g's. But this rearranges the remaining umbral ranges by a substitution S^{-1}. The order of these, however, may be restored by renaming the umbral ranges. This will leave the umbral indices on the multiplying tensors only, permuted by substitution S. If the number of δ's is even these indices may be arranged in the original order and the determinant has been brought from one form to another with a different arrangement of indices in the fixed range without a change in value. If the number of δ's is odd there will be a change in sign if S is odd and the range is alternating. An illustration of this is found in the two range determinant

$$g_{(r_n)(s_n)} = \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \ g_{r_1\sigma_1} \cdots g_{r_n\sigma_n} \qquad (3.1)$$

The parentheses on the range letters indicate that the range is alternating, while the brackets will indicate that the range is permanent. The indices in a fixed range need not be distinct.

(4) Alternating ranges may be interchanged. For example

$$g_{[r_n](s_n)(t_n)} = g_{[r_n](t_n)(s_n)}$$

for the number of changes of sign to bring (s_n) to (t_n) is balanced by the number to bring (t_n) to (s_n), (s_n) and (t_n) being permutations of the same integers.

4. *Permanents, Transformations of Coordinates, Inner Products.* The following combinatory properties of the permanent tensor will be found useful:[*]

$$\pi^{r_1 \cdots r_n}_{\rho_1 \cdots \rho_n} \quad \pi^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} = n!\pi^{r_1 \cdots r_n}_{s_1 \cdots s_n} \tag{4.1}$$

or more generally:

$$\pi^{r_1 \cdots r_p\, s_1 \cdots s_q}_{\rho_1 \cdots \rho_p\, u_1 \cdots u_q} \quad \pi^{\rho_1 \cdots \rho_p}_{t_1 \cdots t_p} = p!\, \pi^{r_1 \cdots r_p\, s_1 \cdots s_q}_{t_1 \cdots t_p\, u_1 \cdots u_q} \tag{4.2}$$

The proofs follow readily from (2.1) and (2.3).

Analogous to the method of finding the equation of transformation of a determinant[†] the equation of transformation of a permanent may be found. For covariant elements a_{rs} the permanent is

$$\pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \quad a_{\rho_1 s_1} \cdots a_{\rho_n s_n}. \tag{4.3}$$

Any interchange of the virtual ranges of the expression leaves its value unchanged, for example, interchange $(s_1 s_2)$, follow this by interchanging $(\rho_1 \rho_2)$ and then note that $\pi^{\rho_1\rho_2 \cdots \rho_n}_{r_1 r_2 \cdots r_n} = \pi^{\rho_2\rho_1 \cdots \rho_n}_{r_1 r_2 \cdots r_n}$ and the expression has been changed to original value without a change of value. By the definition of the tensor π (4.3) may be written

$$\sum \delta^{\rho_1}_{r_1} \cdots \delta^{\rho_n}_{r_n} \; a_{\rho_1 s_1} \cdots a_{\rho_n s_n}.$$

Consider a typical term of this expression under Σ in which the ρ's have been permuted by a substitution S. Before allowing the ρ's to be summed, permute the elements by S. Then summing with respect to the ρ's this term is $a_{r_1 \sigma_1} \cdots a_{r_n \sigma_n}$ where the σ's are a permutation S of the $s_1 \cdots s_n$. The entire expression is then

$$\begin{aligned} &\pi^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \quad a_{r_1 \sigma_1} \cdots a_{r_n \sigma_n} \\ &\qquad = \pi^{\rho_1 \cdots \rho_n}_{s_1 \cdots s_n} \quad a_{r_1 \rho_1} \cdots a_{r_n \rho_n} \end{aligned} \tag{4.4}$$

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[*] For the determinant tensor see reference (3), Article I.

[†] See Paper I.

In a similar manner we arrive at the fact that any permanent range may be fixed and the others umbral if there are an even number of δ's. This may be illustrated by

$$\pi^{(\sigma_n)}_{(s_n)}\ \pi^{(\tau_n)}_{(t_n)}\ \delta^{(v_n)}_{(u_n)}\ \delta^{(\mu_n)}_{(m_n)}\ \left(a_{r_n\sigma_n\tau_n v_n\mu_n}\right). \tag{4.11}$$

Let $\Sigma\,(\delta^{\sigma_n}_{s_n})$ be substituted for the first tensor. Consider a typical term in which the permutation of the σ's is S. Permute the a's by S. Then permute all umbral letters on the elements by S^{-1}. If there is an even number of δ's the total effect is to permute the fixed range to order $S(r_n') = (\rho_n)$ say. Summing all such terms we have

$$\pi^{(\rho_n)}_{(r_n)}\ \pi^{(\tau_n)}_{(t_n)}\ \delta^{(v_n)}_{(u_n)}\ \delta^{(\mu_n)}_{(m_n)}\left(a_{\rho_n\sigma_n\tau_n v_n\mu_n}\right) \tag{4.12}$$

as equivalent to (4.11). The method of changing from a fixed alternating range to a fixed permanent range will be illustrated in (5.1) and (5.2).

The transformation of a permanent. We may represent the transformed permanent by

$$P'_{r_1\ldots r_n s_1\ldots s_n} \equiv \pi'^{\rho_1\ldots\rho_n}_{r_1\ldots r_n}\, a'_{\rho_1 s_1}\cdots a'_{\rho_n s_n} \quad \text{which we write for brevity}$$

$$P'_{[r_n][s_n]} \equiv \pi'^{[\rho_n]}_{[r_n]}\ (a'_{\rho_n s_n}) = 1/n!\ \pi'^{[\rho_n]}_{[r_n]}\ \pi^{[\sigma_n]}_{[s_n]}\ (a'_{\rho_n\sigma_n})$$

$$= 1/n!\,\pi'^{[\rho_n]}_{[r_n]}\ \pi^{[\sigma_n]}_{[s_n]}\ a_{\alpha_n\beta_n}\ \left(\frac{\partial x_{\alpha_n}}{\partial x'_{\rho_n}}\quad \frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right)$$

$$= 1/n!\ \pi^{[\rho_n]}_{[r_n]}\ \left(\frac{\partial x_{\alpha_n}}{\partial x'_{\rho_n}}\right)\ \pi^{[\sigma_n]}_{[s_n]}\ \left(\frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right)\ \cdot\ (a_{\alpha_n\beta_n}).$$

In this expression the $[\alpha_n]$ appearing on the partial derivatives is a range of a permanent and therefore permanent in character, so we can continue the equation

$$= 1/(n!)^2\,\pi^{[\rho_n]}_{[r_n]}\left(\frac{\partial x_{\gamma_n}}{\partial x'_{\rho_n}}\right)\cdot\pi^{[\sigma_n]}_{[s_n]}\ \left(\frac{\partial x_{\beta_n}}{\partial x'_{\sigma_n}}\right)\ \pi^{[\alpha_n]}_{[\gamma_n]}\ (a_{\alpha_n\beta_n})$$

$$= 1/(n!)^2\ K_{[r_n][\gamma_n]}\ \cdot\ K_{[s_n][\beta_n]}\ \cdot\ P_{[\gamma_n][\beta_n]}.$$

If the summation in this equation is extended only to distinct sets the numerical factor is removed. A bar under the umbral letters will indicate this convention. Thus

$$P'_{[r_n][s_n]} = K_{[r_n][\underline{\gamma_n}]}\ \cdot\ K_{[s_n][\underline{\beta_n}]}\ \cdot\ P_{[\gamma_n][\beta_n]}. \tag{4.5}$$

This equation is identical in form to equation (6.4) of paper I. There the J's represent the Jacobians or determinants of the transformation, while here the K's represent the permanents of the transformation. In (6.4) paper I the inner product of two alternating Jacobians reduces to the square of the Jacobian. This simplification of the inner product of permanents is not possible since the absolute values of non-vanishing permanent components differ. Unlike determinants the sets of indices in brackets need not be equal.

Inner Products of Determinants and Permanents. There is a very close similarity in the inner products of these two functions.

Let $$D^{(\gamma_n)(s_n)} = \delta^{(\gamma_n)}_{(\rho_n)} \quad (d^{\rho_n s_n})$$

and $$E_{(\gamma_n)(a_n)} = \delta^{(a_n)}_{(\gamma_n)} \; (e_{a_n a_n})$$

then the inner product of these will $= n!\ \delta^{(a_n)}_{(\rho_n)} \; (d^{\rho_n s_n})(e_{a_n a_n})$

$$= n! \sum_{\rho} \; (\delta^{a_n}_{\rho_n})(d^{\rho_n s_n} e_{a_n a_n}).$$

Now consider a typical term where the permutation on the ρ's is S. Permute the d's by S. This will leave the s's permuted by S. Sum the ρ's getting to within a sign

$$n!\ (d^{a_n \sigma_n} e_{a_n a_n}) \text{ where } S(s_n) = (\sigma_n).$$

The sum for all permutations S under Σ will be

$$n!\, \delta^{(s_n)}_{(\sigma_n)} \; (d^{a_n \sigma_n} e_{a_n a_n}).$$

If we set

$$f_a{}^{\sigma} = d^{a\sigma} e_{aa} \tag{4.6}$$

the inner product

$$D^{(\gamma_n)(s_n)}\, E_{(\gamma_n)(a_n)} = n!\, \delta^{(s_n)}_{(\sigma_n)} \; (f^{\sigma_n}_{\;a_n}) \quad = n!\, F^{(s_n)}_{(a_n)} \tag{4.7}$$

Since the (γ_n) ranges are alternating this gives the ordinary multiplication theorem for determinants

$$D^{(r_n)(s_n)} E_{(r_n)(a_n)} = \delta^{(s_n)}_{(\sigma_n)} \; (f^{\sigma_n}_{\;a_n}), \;\; (r_n) = 1 \cdot \cdot \cdot n$$

$$= F^{(s_n)}_{(a_n)} \text{ with } (4.6). \tag{4.8}$$

For permanents, every step in the proof will hold down to (4.6) and (4.7)

but the argument by which (4.8) was derived depended upon alternating indices and is not valid for permanents. We will here state the results, if P, Q and R are permanents with elements

$$p^{\rho\sigma},\ q_{\rho\sigma} \text{ and } r_{\rho}{}^{\sigma} \quad \text{where } r_{\rho}{}^{\sigma} = p^{a\sigma} q_{a\rho} \tag{4.9}$$

then

$$P^{[\gamma_n][s_n]}\, Q_{[\gamma_n][a_n]} = n!\, R^{[s_n]}_{[a_n]} \;. \tag{4.91}$$

5. *Alternative forms for determinants. Decomposition.* There are methods of writing a determinant alternative with (2.5) for example

$$n!\, g_{[r_n](s_n)(t_n)} = \pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \; \delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{\rho_1\sigma_1\tau_1} \cdots g_{\rho_n\sigma_n\tau_n} \tag{5.1}$$

or

$$g_{[r_n](s_n)(t_n)} = \pi^{\rho_1 \cdots \rho_n}_{r_1 \cdots r_n} \; \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{\rho_1 s_1 \tau_1} \cdots g_{\rho_n s_n \tau_n} \tag{5.2}$$

Equation (5.1) is obvious when it is noticed that $[\rho_n]$ is a permanent range, there being in this case an even number of inner multiplications with δ's hence all of the $n!$ term in the sum for $[\rho_n]$ are equal to $g_{[r_n](s_n)(t_n)}$. Equation (5.2) may be proved directly or verified in this manner. By making a substitution in the (s_n) and arguing as we have above, (s_n) may be seen to be alternating. By changing s to ρ in the right member of (5.2) and multiplying by $\delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n}$ the member is multiplied by $n!$ since the inner product of two alternating ranges is $n!$ times the ordinary product. But this gives the right member of (5.1) which equals $n!\, g_{[r_n](s_n)(t_n)}$. By dividing by $n!$ equation 5.2 is obtained. In these equations the first range is permanent. Corresponding equations are obtained when the other ranges are fixed.

5. Cayley originally considered higher determinants as sums of two range or ordinary determinants, and methods of decomposition are current in the literature.* A single illustration will serve to show how this is done with the present notation. In the notation of (2.5) the fixed range is virtual, so the number of ranges is one greater than the number of tensor multipliers of the elements. When the expression is written

$$\delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \; [\delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; g_{r_1\sigma_1\tau_1} \cdots g_{r_n\sigma_n\tau_n}] \tag{5.3}$$

the expression in the brackets is a class 2 determinant since but one δ is used in expressing it. The first range in this case must be considered as being the bipartite range $r_1\sigma_1, \cdots r_n\sigma_n$. When σ is summed there will be $n!$ such determinants. Either δ may be taken out of the brackets. The extension to the types of determinants classified at the close of article 4 is apparent.

6. *Multiplication of Determinants. Demonstration of the applicability of the method to more general functions than heretofore contemplated.* The muliplication of two cubic determinants of order 2 illustrates the process. However, for file multiplication the process will be carried out in its greatest generality.

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* L. H. Rice, *American Journal of Math.*, 1918, p. 242.

Element Multiplication. For the product of the two cubic determinants g and h we have

$$\begin{aligned} g \cdot h &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{\tau_1\tau_2}_{t_1t_2}\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \delta^{c_1c_2}_{\gamma_1\gamma_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2} \\ &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{\tau_1\tau_2}_{t_1t_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \delta^{c_1c_2}_{\gamma_1\gamma_2}\ \kappa^{a_1\beta_1\gamma_1}_{r_1\sigma_1\tau_1} \cdot \kappa^{a_2\beta_2\gamma_2}_{r_2\sigma_2\tau_2} = \kappa \end{aligned} \tag{6.1}$$

where

$$\kappa^{a\beta\gamma}_{r\sigma\tau} = g_{r\sigma\tau}\, h^{a\beta\gamma} \tag{6.2}$$

The class of k is found by counting the δ's and adding 1. In general then for classes p and q the product k will be of class $p-1+q-1+1=p+q-1$. In the present case r, and a are permanent indices (property 3, article 3) so can be taken identical. The point so apparent here is that in this method not a property of the δ's is utilized, we might have δ's, π's, γ's or in fact it is immaterial what they are. The class of k can be increased to $p+q$. There will by this method always be two fixed ranges, in this case the ranges r and a. The range r can be changed to $\rho_1 \cdots \rho_n$ and the determinant multiplied by $1/n!\,\pi^{\rho_1 \cdots \rho_n}_{r_1 \ldots r_n}$ where ρ is summed. If r had been alternating we should multiply by $1/n!\,\delta^{\rho_1 \cdots \rho_n}_{r_1 \ldots r_n}$ otherwise the product would vanish. Now the class will be equal to the number of ranges. Remarks such as this one on raising the class are so trivial by the analytic method that one hesitates to make them but from the geometric treatment they seem of more importance.

File Multiplication. The analytical analogue of the geometric nomenclature, "file multiplication" is "inner multiplication." Beginning with equation (6.1) and setting $t=c=\rho$ and summing (the number of choices of umbral indices is the product of the numbers of alternating ranges, each leading to a multiplication theorem) we get

$$\begin{aligned} g \cdot h &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \delta^{\tau_1\tau_2}_{\gamma_1\gamma_2}\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\, h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2} \\ &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \left[\delta^{\tau_1}_{\gamma_1}\,\delta^{\tau_2}_{\gamma_2} - \delta^{\tau_2}_{\gamma_1}\,\delta^{\tau_1}_{\gamma_2}\right]\ g_{r_1\sigma_1\tau_1}\ g_{r_2\sigma_2\tau_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2} \\ &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \left[g_{r_1\sigma_1\gamma_1}\, g_{r_2\sigma_2\gamma_2}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}\right. \\ &\qquad \left. -\ g_{r_1\sigma_1\gamma_2}\ g_{r_2\sigma_2\gamma_1}\ h^{a_1\beta_1\gamma_1}\ h^{a_2\beta_2\gamma_2}\right] \\ &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \left[\kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2} - \kappa^{a_2\beta_2}_{r_1\sigma_1}\ \kappa^{a_1\beta_1}_{r_2\sigma_2}\right] \end{aligned}$$

where

$$\kappa^{a\beta}_{r\sigma} = g_{r\sigma\gamma}\ h^{a\beta\gamma} \tag{6.3}$$

$$= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \left[\kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2} + \kappa^{a_2\beta_1}_{r_1\sigma_1}\ \kappa^{a_1\beta_2}_{r_2\sigma_2}\right]$$

from the interchange of umbral indices β_1 and β_2 and use of alternating property of $\delta^{b_1b_2}_{\beta_1\beta_2}$. Thus

$$\begin{aligned} g \cdot h &= \delta^{\sigma_1\sigma_2}_{s_1s_2}\ \delta^{b_1b_2}_{\beta_1\beta_2}\ \pi^{a_1a_2}_{a_1a_2}\ \kappa^{a_1\beta_1}_{r_1\sigma_1}\ \kappa^{a_2\beta_2}_{r_2\sigma_2} \\ &= \kappa^{[a_1a_2](b_1b_2)}_{[r_1r_2](s_1s_2)} \end{aligned}$$

where as before we exhibit the permanent and alternating ranges by brackets and parentheses.

Multiplication of a p range and a q range tensor function. The $T_{(1)} \cdots T_{(q-1)}$ are arbitrary sets of quantities which may be permanent tensors, determinant tensors γ's or otherwise. The determinants to be multiplied are

$$A = T_{(1)}{}^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \cdots T_{(p-1)}{}^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; \delta^{\upsilon_1 \cdots \upsilon_n}_{u_1 \cdots u_n} \, g_{r_1\sigma_1 \cdots \tau_1\upsilon_1} \cdots g_{r_n\sigma_n \cdots \tau_n\upsilon_n}$$

and

$$B = \pi^{b_1 \cdots b_n}_{\beta_1 \cdots \beta_n} \cdots \pi^{c_1 \cdots c_n}_{\gamma_1 \cdots \gamma_n} \; \delta^{e_1 \cdots e_n}_{\epsilon_1 \cdots \epsilon_n} \cdots \delta^{f_1 \cdots f_n}_{\eta_1 \cdots \eta_n} \; \delta^{u_1 \cdots u_n}_{\mu_1 \cdots \mu_n}$$
$$\times\; h^{a_1\beta_1 \cdots \gamma_1\epsilon_1 \;\cdots\; \eta_1\mu_1} \cdots h^{a_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n\mu_n},$$

where A contains *at least one* alternating range indicated by one δ and B contains at least one δ (but an even number of alternating ranges by property 3, art. 3) while the remaining ranges are either permanent or alternating as indicated by the π's and δ's. The total number of δ's in B will be designated by K $(\nless 1)$. The number of π's is immaterial. The last δ in both A and B contains the indices $u_1 \cdots u_n$ which must be a permutation of $1 \cdots n$ so the product of these δ's in the product AB will be

$$\delta^{\upsilon_1 \cdots \upsilon_n}_{\mu_1 \cdots \mu_n} = \sum_{\upsilon} \delta^{\upsilon_1}_{\mu_1} \cdots \delta^{\upsilon_n}_{\mu_n} \tag{1}$$

When this sum is multiplied with the product of the h's the sum of h's is

$$\sum_{\upsilon} h^{a_1\beta_1 \cdots \gamma_1\epsilon_1 \cdots \eta_1\upsilon_1} \cdots h^{a_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n\upsilon_n} \tag{2}$$

In (1) the sign of a term is ± 1 according as the class of the permutation of subscripts of υ's is even or odd—thus in (2) the sign of a term will be ± 1 according as the permutation with respect to indices of any letter say a, is even or odd. Consider a term of this sum and let S represent the permutation of the υ's in it. The υ's may be brought to former order by a permutation S^{-1} of the h's which has the effect of permuting the other indices by S^{-1}. All of the umbral indices may be brought to the original order but this will leave the lower indices on the π's and δ's permuted to order S. This is of no consequence on the π's but to bring the indices of δ's to the original orders will require a permutation S^{-1} on each of $K - 1$ δ's so if $K - 1$ is even the sign of the term is unchanged, otherwise it is changed when S^{-1} is an odd permutation. The a's are in the order S^{-1} to which they were brought by the permutation on the h's. In case the sign is unchanged it is of class S which is also of class S^{-1} which indicates that if $a_1 \cdots a_n$ is the new order of the a's the sign of the term is $\delta^{a_1 \cdots a_n}_{a_1 \cdots a_n}$. By substituting

$$C^{\alpha\beta \cdots \gamma\epsilon \cdots \eta}_{r\sigma \cdots \tau} = g^{r\sigma \cdots \tau\upsilon} h^{\alpha\beta \cdots \gamma\epsilon \cdots \eta\upsilon}$$

the product may be written

$$\begin{aligned} A \cdot B = {} & T_{(1)}{}^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \cdots T_{(p-1)}{}^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} \; \delta^{a_1 \cdots a_n}_{\alpha_1 \cdots \alpha_n} \; \pi^{b_1 \cdots b_n}_{\beta_1 \cdots \beta_n} \cdots \pi^{c_1 \cdots c_n}_{\gamma_1 \cdots \gamma_n} \\ & \times \delta^{e_1 \cdots e_n}_{\epsilon_1 \cdots \epsilon_n} \cdots \delta^{f_1 \cdots f_n}_{\eta_1 \cdots \eta_n} \cdot C^{\alpha_1\beta_1 \cdots \gamma_1\epsilon_1 \cdots \eta_1}_{r_1\sigma_1 \cdots \tau_1} \cdots C^{\alpha_n\beta_n \cdots \gamma_n\epsilon_n \cdots \eta_n}_{r_n\sigma_n \cdots \tau_n} \\ & = C. \end{aligned}$$

The class of C is found by the number of T's, π's, δ's equal to $p - 1 + q - 1$, the original number, -2 lost by inner multiplication $+1$ regained by the artifice $+1$ by which the class exceeds the umbral ranges $= p + q - 2$.

The Associated Contravariant Tensor. The following method of obtaining an associated contravariant tensor shows that the existence of an associated contravariant tensor depends upon the property of an alternating tensor. Let the three indices of g_{rst} be covariant. The only possible determinant in this case, other than a permanent, is given by (2.5). The equation of transformation may be written

$$G'_{[r_n](s_n)(t_n)} = G_{[\rho_n](s_n)(t_n)} \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \cdot J^2 \tag{6.5}$$

So
$$\begin{aligned} \frac{\partial G'_{[r_n](s_n)(t_n)}}{\partial g'_{\rho\sigma\tau}} &= J^2 \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \frac{\partial G_{[\rho_n](s_n)(t_n)}}{\partial g_{\alpha\beta\gamma}} \times \frac{\partial g_{\alpha\beta\gamma}}{\partial g'_{\rho\sigma\tau}} \\ &= J^2 \left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right) \frac{\partial G_{[\rho_n](s_n)(t_n)}}{\partial g_{\alpha\beta\gamma}} \cdot \frac{\partial x_\rho'}{\partial x_\alpha} \cdot \frac{\partial x'_\sigma}{\partial x_\beta} \cdot \frac{\partial x_\tau'}{\partial x_\gamma} \end{aligned} \tag{6.6}$$

The first partial of the determinant with respect to the element has the covariant character of the determinant and an additional contravariant character. If the determinant had no permanent range, products such as $\left(\frac{\partial x_{\rho_n}}{\partial x'_{r_n}}\right)$ would not appear in the equations of transformation. There would then be an even number of alternating ranges, hence an even power of the Jacobian of the transformation would appear in (6.5) and in (6.6) so if the first partial is divided by the determinant the resulting tensor will be of contravariant character wholly. Only an element with an even number of indices, when all represent tensor character, can have an associated tensor of opposite rank. To the writer's knowledge it is only such elements that are considered as representatives of a line element in differential geometry.

A "Bibliographie des déterminants à plus de deux dimensions" to April 1924 is given by M. Lecat, Bruxelles.

Transformations Leaving Invariant Certain Partial Differential Equations of Physics.

BY R. D. CARMICHAEL.

We shall write the general wave equation in the form

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0. \tag{1}$$

Any solution of this equation will be called a wave-function.

For the case when n is 4, the transformation $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = \sqrt{-1}\, ct$ reduces equation (1) to the classical form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

of the wave equation in theoretical physics. This is the justification for calling (1) the general wave equation.

The principal object of this paper is to determine the generators of the group of point transformations (the coordinates of the points being complex numbers) under each transformation of which a wave-function is a relative invariant. The group contains a certain linear group as a subgroup (see § 6). It is shown (see § 11) that this subgroup and two particular Cremona transformations generate the whole group when $n > 2$. When the equation is taken in the form (1), one of these Cremona transformations is real and the other is imaginary; but they are both real in the case of the classical form of the equation in mathematical physics.

It turns out that the group involved is identical with a certain group of conformal transformations (see § 3) which has been the object of several investigations. It was necessary, however, to redetermine the generators of this conformal group, since certain imaginary transformations are quite as important for our problem as the real ones which have been the object of previous investigations.

Incidental to the principal object, we have determined the corresponding groups for two other partial differential equations of physics (see § 2). (An additional equation is treated in the following paper by Mr. Goff.)

In each of the three problems, the case $n = 1$ is trivial. The case $n = 2$

(see § 7) is exceptional in character and is shown to lead to a well-known group of transformations on two variables. For this reason its exceptional features are not treated in detail.

Incidental to our main purpose we obtain new proofs of four important theorems stated below as theorems IV, V, VI, VII. The principal result is contained in theorems VIII and IX near the end of the paper.

§ 1. *The partial differential equations defining the transformations of the wave equation into itself.* Let the function $v(y_1, y_2, \cdots, y_n)$ be a solution of the wave equation

$$\sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k^2} = 0, \tag{2}$$

and let us make the transformation

$$u = w \cdot v(y_1, y_2, \cdots, y_n) \qquad w \neq 0 \tag{3}$$

where w is a function of $x_1, x_2, \cdots, x_n$ to be chosen conveniently and where $y_1, y_2, \cdots, y_n$ are taken as functionally independent functions of $x_1, x_2, \cdots, x_n$ which are to be so determined in connection with w that u shall satisfy equation (1) whenever v satisfies equation (2). Every wave-function v is thus to be transformed into a wave-function u by means of (3).

Since $1, y_i, y_i y_j$ $(i \neq j)$, $y_i^2 - y_n^2$, are solutions of (2), it follows from (3) that w and the y_i must be such that w, wy_i, wy_iy_j $(i \neq j)$, $w(y_i^2 - y_n^2)$ are wave-functions. These facts imply that $w, y_1, y_2, \cdots, y_n$ must satisfy the following system of partial differential equations:

$$\tag{4} \left\{ \begin{aligned} & \sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} = 0; \\ & \sum_{k=1}^{n} \left(2 \frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} + w \frac{\partial^2 y_i}{\partial x_k^2}\right) = 0, \quad i = 1, 2, \cdots, n; \\ & \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0, \quad i = 1, 2, \cdots, n-1, \\ & \qquad\qquad\qquad\qquad j = i+1, i+2, \cdots, n; \\ & \sum_{k=1}^{n} (\partial y_i / \partial x_k)^2 = \sum_{k=1}^{n} (\partial y_n / \partial x_k)^2, \quad i = 1, 2, \cdots, n-1. \end{aligned} \right.$$

From relation (3) we have

$$(5)\qquad \frac{\partial^2 u}{\partial x_k^2} = v\frac{\partial^2 w}{\partial x_k^2} + 2\frac{\partial w}{\partial x_k}\sum_{i=1}^{n} \frac{\partial v}{\partial y_i}\,\frac{\partial y_i}{\partial x_k}$$

$$+\,w\left\{\sum_{i=1}^{n} \frac{\partial^2 v}{\partial y_i^2}\left(\frac{\partial y_i}{\partial x_k}\right)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\partial^2 v}{\partial y_i\partial y_j}\,\frac{\partial y_i}{\partial x_k}\,\frac{\partial y_j}{\partial x_k} + \sum_{i=1}^{n} \frac{\partial v}{\partial y_i}\,\frac{\partial^2 y_i}{\partial x_k^2}\right\},$$
$$k = 1, 2, \cdots, n.$$

From these equations it is easy to see that u, as defined by (3), is a solution of (1) provided that equations (4) are satisfied.

Let us use ρ for the common value of the n sums

$$\rho = \sum_{k=1}^{n} (\partial y_i/\partial x_k)^2, \qquad i = 1, 2, \cdots, n.$$

If we employ δ_{ij} to stand for 1 or 0 according as j is or is not equal to i, then the conditions in the last two lines of (4) may be put in the form

$$(6)\qquad \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k}\,\frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho \qquad i, j = 1, 2, \cdots, n.$$

Since the Jacobian (functional determinant)

$$(7)\qquad J = \left|\,\frac{\partial y_i}{\partial x_k}\,\right|$$

is different from zero (owing to the assumed functional independence of the y's) is follows from (6) that ρ can not be identically zero.

The results so far obtained may be put into the form of the following theorem:

THEOREM I. *Let $w, y_1, y_2, \cdots, y_n$ be a set of functions of $x_1, x_2, \cdots, x_n$ such that $y_1, y_2, \cdots, y_n$ are functionally independent and $w \not\equiv 0$. Then a necessary and sufficient condition that every solution of* (2) *shall be transformed through* (3) *into a solution of* (1) *is that the functions $w, y_1, y_2, \cdots, y_n$ shall satisfy the following conditions:—a function ρ exists, not identically zero, such that*

$$(8)\qquad \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k}\,\frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho,$$

and the equations

(9)* $$\sum_{k=1}^{n} (2\frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} + w \frac{\partial y^2{}_i}{\partial x_k^2}) = 0,$$

(10) $$\sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} = 0,$$

are verified.

The totality of transformations (3), subject to the conditions just named, obviously form a group.

Corollary. A necessary and sufficient condition equivalent to that in theorem I is that each of the following functions shall be a wave-function: w; wy_i, $i = 1, 2, \cdots, n$; wy_iy_j, $i = 1, 2, \cdots, n-1$, $j = i+1, \cdots, n$; $w(y_i^2 - y_n^2)$, $i = 1, 2, \cdots, n-1$.

§ 2. *Certain related groups of transformations.* The method used in the preceding section may be employed in the proof of some results related to theorem I.

THEOREM II. *Let* $w, y_1, y_2, \cdots, y_n$ *be a set of functions of* $x_1, x_2, \cdots, x_n$ *such that* $y_1, y_2, \cdots, y_n$ *are functionally independent and* $w \neq 0$. *Then a necessary and sufficient condition that every solution of the equation*

(11) $$\sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k^2} + v = 0$$

shall be transformed through the relation

(12) $$u = w \cdot v(y_1, y_2, \cdots, y_n)$$

into a solution of the equation

(13) $$\sum_{k=1}^{n} \frac{\partial u^2}{\partial x_k^2} + u = 0$$

is that the functions $w, y_1, y_2, \cdots, y_n$ *shall satisfy the following conditions: a function* ρ *exists, not identically zero, such that*

(14) $$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho$$

and the equations

(15)† $$\sum_{k=1}^{n} \left(2\frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} + w \frac{\partial^2 y_i}{\partial x_k^2} \right) = 0$$

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[*] In § 5, these conditions are replaced by the simpler one, $w = \rho^{\frac{n-2}{4}}$. See also the latter part of § 5.

[†] See the preceding footnote.

(16) $$\sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} + w = \rho w$$

are satisfied.

The necessity of these conditions is easily proved by aid of the following solutions of (11): $\sin y_i$ and $\cos y_i$ (or $e^{y_i\sqrt{-1}}$ and $e^{-y_i\sqrt{-1}}$) taken in conjunction, and $\sin (y_i\sqrt{s}) \sin (y_j\sqrt{1-s})$ where $i \neq j$ and s is a constant different from 0 and 1. In order to show that the conditions are sufficient one may proceed as in the proof of the corresponding part of theorem I.

The partial differential equations of the group of transformations x_i into y_i and u into v such that u, as defined by the transformation $u = w \cdot v(y_1, y_2, \cdots, y_n)$, shall be a solution of the equation

(17) $$\sum_{k=1}^{n} (\partial u/\partial x_k)^2 = 0$$

whenever $v(y_1, y_2, \cdots, y_n)$ is a solution of the equation

(18) $$\sum_{k=1}^{n} (\partial v/\partial y_k)^2 = 0$$

may be determined in a similar manner. The following sufficient conditions are readily obtained:

$$\sum_{k=1}^{n} (\partial w/\partial x_k)^2 = 0;$$

$$\sum_{k=1}^{n} \frac{\partial w}{\partial x_k} \frac{\partial y_i}{\partial x_k} = 0, \quad i = 1, 2, \cdots, n;$$

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = 0, \quad i \neq j;$$

$$\sum_{k=1}^{n} (\partial y_i/\partial x_k)^2 = \sum_{k=1}^{n} (\partial y_n/\partial x_k)^2 \neq 0, \quad i = 1, 2, \cdots, n-1.$$

That these conditions are also necessary may be shown by means of the following solutions of (18): $v = 1$, $v = f(y_i + y_j\sqrt{-1})$, $v = f(y_i - y_j\sqrt{-1})$, $(i \neq j)$, $f(t)$ being an analytic function of t.

On multiplying the second of the foregoing equations by $\partial y_i/\partial x_l$ and summing as to i from 1 to n, we find (by aid of equation (21) in the next section) that $\partial w/\partial x_l = 0$, $l = 1, 2, \cdots, n$, so that w is a constant. Without loss of generality we may take it equal to unity; and this we do.

Then we have the following result:

THEOREM III. *A necessary and sufficient condition that every solution of* (18) *shall be transformed by the relation* $u=v(y_1, y_2, \cdots, y_n)$ *into a solution of* (17), *the functions* $y_1, y_2, \cdots, y_n$ of $x_1, x_2, \cdots, x_n$ *being functionally independent, is that a funtcion* ρ *shall exist, not identically zero, such that*

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}\rho. \tag{19}$$

§ 3. *The transformation group of theorem III.* The transformation group of theorem III, transforming $x_1, x_2, \cdots, x_n$ into $y_1, y_2, \cdots, y_n$ contains the corresponding transformation groups of theorems I and II. We shall therefore treat first the transformation group of theorem III.

Let J denote the functional determinant

$$J = \left| \frac{\partial y_i}{\partial x_j} \right| .$$

Then by the multiplication theorem for determinants, we have

$$J^2 = \left| \frac{\partial y_i}{\partial x_j} \right| \cdot \left| \frac{\partial y_j}{\partial x_i} \right| = \left| \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} \right| = \rho^n,$$

the last member having been obtained by use of (19). For J we have therefore the value

$$J = \rho^{n/2}. \tag{20}$$

Since $J \neq 0$, it follows that ρ can not be identically zero.

If we multiply both members of equation (19) by $\partial y_j/\partial x_l$ and sum as to j from 1 to n, we have

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \left(\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} \right) = \rho \frac{\partial y_i}{\partial x_l}, \quad i, l = 1, 2, \cdots, n.$$

On holding l fixed and allowing k and i to vary, we have a system of linear equations in the n quantities inclosed in parenthesis. The determinant of the coefficients of this system is J and is therefore different from zero. On solving the foregoing system for the indicated quantities, we have

$$\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = \delta_{kl}\rho, \qquad k, l = 1, 2, \cdots, n. \tag{21}$$

Thus equations (19) imply equations (21); it may be shown similarly that equations (21) imply equations (19).

If we take $y_1, y_2, \cdots, y_n$ to be the coordinates of a Euclidean space, then the element ds of arc length is given by the equation

$$ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2.$$

But
$$dy_m = \sum_{k=1}^{n} \frac{\partial y_m}{\partial x_k} dx_k, \qquad m = 1, 2, \cdots, n.$$

Substituting these values of the dy_m into the preceding equation and simplifying by the aid of (19), we have

$$(22) \quad ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2 = \rho(dx_1^2 + dx_2^2 + \cdots + dx_n^2).$$

Hence the mapping effected by our transformation is conformal in character. Conversely, if a transformation of the y's into the x's is such that equation (22) is satisfied, then it may readily be shown that equations (19) are satisfied. Hence the transformation group involved in theorem III is the group of conformal point transformations in a Euclidean space of n dimensions, the coordinates of the points being complex numbers.

§ 4. *Determination of ρ for the conformal transformations.* If we consider the equations

$$x_i = x_i(y_1, y_2, \cdots, y_n)$$

of the inverse transformation, then on differentiating with respect to x_j we have

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} .$$

Multiplying by $\partial y_m/\partial x_j$, summing as to j from 1 to n and reducing by aid of (19) we have

$$(23) \qquad \frac{\partial y_m}{\partial x_i} = \rho \frac{\partial x_i}{\partial y_m}, \quad i, m = 1, 2, \cdots, n.$$

These equations are useful in considering the relation of a transformation to its inverse.

From equations (19) or (21) we may obtain by elimination a system of equations involving a single one of the functions y_i alone. For this purpose we have from (19) by differentiation the equations

$$\sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial^2 y_j}{\partial x_k \partial x_l} + \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial^2 y_i}{\partial x_k \partial x_l} = \delta_{ij} \frac{\partial \rho}{\partial x_l} .$$

On multiplying by $\partial y_i/\partial x_l$, summing as to i from 1 to n, and simplifying by aid of (21), we have

$$\rho\,\frac{\partial^2 y_j}{\partial x_l^2}+\sum_{k=1}^{n}\frac{\partial y_j}{\partial x_k}\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_l}\,\frac{\partial^2 y_i}{\partial x_k\partial x_l}=\frac{\partial\rho}{\partial x_l}\,\frac{\partial y_j}{\partial x_l}.$$

The second term of this equation may be simplified by means of the relation

$$\sum_{l=1}^{n}\frac{\partial y_i}{\partial x_l}\,\frac{\partial^2 y_i}{\partial x_l\partial x_k}=\tfrac{1}{2}\,\frac{\partial\rho}{\partial x_k}$$

obtained by replacing k by l in (21) and differentiating both members of the resulting equation with respect to x_k; thus we have

$$\rho\,\frac{\partial^2 y_j}{\partial x_l^2}-\frac{\partial\rho}{\partial x_l}\,\frac{\partial y_j}{\partial x_l}+\tfrac{1}{2}\sum_{k=1}^{n}\frac{\partial\rho}{\partial x_k}\,\frac{\partial y_j}{\partial x_k}=0, \qquad l=1,2,\cdots,n. \tag{24}$$

From (24) may be derived some consequences which will be of use later. If we sum as to l from 1 to n we have

$$\rho\sum_{l=1}^{n}\frac{\partial^2 y_j}{\partial x_l^2}+\frac{n-2}{2}\sum_{k=1}^{n}\frac{\partial\rho}{\partial x_k}\,\frac{\partial y_j}{\partial x_k}=0. \tag{25}$$

For the case when $n=2$ this shows that y_1 and y_2 are both wave-functions.

On multiplying both members of (24) by $\partial y_j/\partial x_m$ and summing as to j we have

$$\sum_{j=1}^{n}\frac{\partial y_j}{\partial x_m}\,\frac{\partial^2 y_j}{\partial x_l^2}=\delta_{lm}\,\frac{\partial\rho}{\partial x_l}-\frac{1}{2}\,\frac{\partial\rho}{\partial x_m}. \tag{26}$$

On summing both members of this equation as to l we have

$$\sum_{l=1}^{n}\sum_{j=1}^{n}\frac{\partial y_j}{\partial x_m}\,\frac{\partial^2 y_j}{\partial x_l^2}=\frac{2-n}{2}\,\frac{\partial\rho}{\partial x_m}. \tag{27}$$

We may now determine* explicitly the possible values of ρ. For this purpose it is convenient to write

$$\rho=1/\sigma^2. \tag{28}$$

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* This is based on the work of R. Beez, *Zeitschrift für Mathematik und Physik*, 20 (1875), 253-270; it is necessary to repeat the argument for the sake of some additional formulae not given by Beez. Some simplifications in the computation are also effected. Moreover, in our problem, we have to examine a case not treated by Beez.

If then we substitute into (19) and (21) the values of $\partial y_i/\partial x_k$ and $\partial y_j/\partial x_k$ as given in (23) we have

$$(29)\qquad \sum_{k=1}^{n} \frac{\partial x_k}{\partial y_i}\,\frac{\partial x_k}{\partial y_j} = \delta_{ij}\sigma^2, \qquad \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k}\,\frac{\partial x_j}{\partial y_k} = \delta_{ij}\sigma^2.$$

Differentiate both members of the last equation with respect to y_l and in the result replace $\partial x_j/\partial y_k$ and $\partial x_i/\partial y_k$ by their values as given by (23) and simplify; thus we have

$$\sum_{k=1}^{n} \frac{\partial^2 x_i}{\partial y_k \partial y_l}\,\frac{\partial y_k}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial^2 x_j}{\partial y_k \partial y_l}\,\frac{\partial y_k}{\partial x_i} = \delta_{ij}\,\frac{2}{\sigma}\,\frac{\partial \sigma}{\partial y_l}\,;$$

or

$$(30)\qquad \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial x_j}{\partial y_l}\right) = \delta_{ij}\,\frac{1}{\sigma}\,\frac{\partial \sigma}{\partial y_l}.$$

From the evident relation

$$\frac{\partial}{\partial x_j}\left(\frac{\partial y_l}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial y_l}{\partial x_j}\right)$$

we have by means of (23) the relation

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma^2}\,\frac{\partial x_i}{\partial y_l}\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{\sigma^2}\,\frac{\partial x_j}{\partial y_l}\right);$$

or,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) - \frac{2}{\sigma}\,\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial x_j}{\partial y_l}\right) - \frac{2}{\sigma}\,\frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i}.$$

Combining this with (30) we have the relation

$$(31)\qquad \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = (1/\sigma)\left(\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i} + \delta_{ij}\,\frac{\partial \sigma}{\partial y_l}\right).$$

Differentiating with respect to x_m we have

$$\begin{aligned}\frac{\partial^2}{\partial x_m \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = (1/\sigma)\Bigg\{ & \frac{\partial}{\partial x_m}\left(\frac{\partial x_i}{\partial y_l}\right)\frac{\partial \sigma}{\partial x_j} \\ & + \frac{\partial x_i}{\partial y_l}\,\frac{\partial^2 \sigma}{\partial x_m \partial x_j} - \frac{\partial}{\partial x_m}\left(\frac{\partial x_j}{\partial y_l}\right)\frac{\partial \sigma}{\partial x_i} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial^2 \sigma}{\partial x_m \partial x_i} \\ & + \delta_{ij}\,\frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right)\Bigg\} - \frac{1}{\sigma^2}\,\frac{\partial \sigma}{\partial x_m}\Bigg(\frac{\partial x_i}{\partial y_l}\,\frac{\partial \sigma}{\partial x_j} - \frac{\partial x_j}{\partial y_l}\,\frac{\partial \sigma}{\partial x_i} \\ & + \tfrac{1}{2}\,\delta_{ij}\,\frac{\partial \sigma}{\partial y_l}\Bigg).\end{aligned}$$

For the first and third terms in the brace { } substitute their values as given by (31); on simplifying we have the following equation:

$$(32)\quad \frac{\partial^2}{\partial x_m \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = 1/\sigma\left(\frac{\partial x_i}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_m \partial x_j} - \frac{\partial x_j}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_m \partial x_i} + \delta_{ij}\frac{\partial}{\partial x_m}\frac{\partial \sigma}{\partial y_l}\right) + \frac{1}{\sigma^2}\frac{\partial \sigma}{\partial y_l}\left(\delta_{im}\frac{\partial \sigma}{\partial x_j} - \delta_{jm}\frac{\partial \sigma}{\partial x_i} - \delta_{ij}\frac{\partial \sigma}{\partial x_m}\right).$$

On interchanging m and j and comparing the resulting equation with (32) itself we see that

$$(33)\quad \frac{\partial x_m}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_j \partial x_i} - \frac{\partial x_j}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_m \partial x_i} - \delta_{im}\frac{\partial}{\partial x_j}\left(\frac{\partial \sigma}{\partial y_l}\right) + \delta_{ij}\frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right) = \frac{2}{\sigma}\frac{\partial \sigma}{\partial y_l}\left(\delta_{ij}\frac{\partial \sigma}{\partial x_m} - \delta_{im}\frac{\partial \sigma}{\partial x_j}\right).$$

When $n > 2$ it is possible to choose i, j, m all different; if we do so we have

$$\frac{\partial x_m}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_j \partial x_i} = \frac{\partial x_j}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_m \partial x_i}, \quad n > 2, \quad i, j, m \text{ all different.}$$

Multiplying by $\partial x_m/\partial y_l$ and summing as to l we see that

$$(34)\quad \frac{\partial^2 \sigma}{\partial x_j \partial x_i} = 0, \qquad i \neq j, \qquad n > 2.$$

From this it follows, when $n > 2$, that $\partial\sigma/\partial x_i$ is independent of each of the x's except x_i. Hence we have

$$(35)\quad \sigma = X_1(x_1) + X_2(x_2) + \cdots + X_n(x_n), \qquad n > 2,$$

where $X_i(x_i)$ is a function of x_i alone.

In (33) take $j = i$ and $m \neq i$; then we have

$$(36)\quad \frac{\partial x_m}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_i^2} = \frac{2}{\sigma}\frac{\partial \sigma}{\partial y_l}\frac{\partial \sigma}{\partial x_m} - \frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right), \qquad m \neq i.$$

For each m there exists a y_l such that $\partial x_m/\partial y_l$ is not identically zero. Taking $m = 1$ we see that $\partial^2\sigma/\partial x_i^2$ is independent of i if i belongs to the set $2, 3, \cdots, n$. Then taking $m = 2$ we see that $\partial^2\sigma/\partial x_i^2$ is independent of i if i belongs to the set $1, 3, 4, \cdots, n$. Hence when $n > 2$ it is independent of i for $i = 1, 2, \cdots, n$. But from (35) it follows that $\partial^2\sigma/\partial x_i^2$ is a function of x_i alone. It must therefore be a constant. Calling this constant $2c$ we may therefore write

$$(37)\quad \frac{\partial^2 \sigma}{\partial x_i^2} = 2c, \qquad i = 1, 2, \cdots, n, \qquad n > 2.$$

Before undertaking to evaluate σ completely it is desirable to obtain still another simple restrictive condition on it. In (32) take $j = i$, $m \neq i$, $n > 2$, and simplify by aid of (34); thus we have

$$\frac{\partial^2}{\partial x_i \partial x_m}\left(\frac{\partial x_i}{\partial y_l}\right) = \frac{1}{\sigma}\frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right) - \frac{1}{\sigma^2}\frac{\partial \sigma}{\partial x_m}\frac{\partial \sigma}{\partial y_l}, \qquad m \neq i, \qquad n > 2.$$

In (32) take $m = i$, $j \neq i$, $n > 2$, and simplify by use of (34); thus we have

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = -\frac{1}{\sigma}\frac{\partial x_j}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_i^2} + \frac{1}{\sigma^2}\frac{\partial \sigma}{\partial y_l}\frac{\partial \sigma}{\partial x_j}, \qquad j \neq i, \qquad n > 2.$$

In the last equation change j to m, subtract the resulting equation member by member from the preceding one, and simplify; then by the aid of (34) we have

$$\begin{aligned}
\frac{2}{\sigma}\frac{\partial \sigma}{\partial y_l}\frac{\partial \sigma}{\partial x_m} - \frac{\partial x_m}{\partial y_l}\frac{\partial^2 \sigma}{\partial x_i^2} &= \frac{\partial}{\partial x_m}\left(\frac{\partial \sigma}{\partial y_l}\right) \\
&= \frac{\partial}{\partial x_m}\sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial x_k}{\partial y_l} \\
&= \frac{\partial^2 \sigma}{\partial x_m^2}\frac{\partial x_m}{\partial y_l} + \sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial}{\partial x_m}\left(\frac{\partial x_k}{\partial y_l}\right) \\
= \frac{\partial 2\sigma}{\partial x_m^2}\frac{\partial x_m}{\partial y_l} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial x_m}\sum_{k=1}^{n}\frac{\partial \sigma}{\partial x_k}\frac{\partial x_k}{\partial y_l} &- \frac{1}{\sigma}\frac{\partial x_m}{\partial y_l}\sum_{k=1}^{n}(\partial \sigma/\partial x_k)^2 \\
&+ \frac{1}{\sigma}\frac{\partial \sigma}{\partial x_m}\frac{\partial \sigma}{\partial y_l},
\end{aligned}$$

the last line being obtained by aid of (31). Hence, since $\partial x_m/\partial y_l$ is certainly not zero for every l, we have by aid of (37)

$$(38) \qquad \sum_{k=1}^{n}(\partial \sigma/\partial x_k)^2 = \frac{\partial^2 \sigma}{\partial x_i^2} + \frac{\partial^2 \sigma}{\partial x_m^2} = 4c\sigma, \qquad n > 2.$$

For determining σ there are two cases to be examined. Suppose first that c is different from zero. Then from (35), (37), and (38), it follows readily that σ has the form

$$(39) \qquad \sigma = c\sum_{k=1}^{n}(x_k - a_k)^2, \qquad c \neq 0, \qquad n > 2,$$

where $a_1, a_2, \cdots, a_n$ are constants. Later we shall show that this value of σ does indeed belong to a possible transformation.

Suppose* next that $c = 0$. Then from (37) and (35) it follows that σ is a linear function of $x_1, x_2, \cdots, x_n$. If we write

$$(40) \qquad \sigma = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \beta, \qquad n > 2,$$

we see from (38) that we must have

$$(41) \qquad \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 = 0.$$

The transformations which arise for such values of σ will be considered later. In particular we must consider the case when σ is a constant, that is, when $\beta_1, \beta_2, \cdots, \beta_n$ are all zero.

§ 5. *The determination of w.* By means of conditions (8) and (9) in theorem I or the equivalent conditions (14) and (15) in theorem II it is possible to find the value of w explicitly in terms of ρ. For this purpose we multiply both members of (9) by $\partial y_i / \partial x_l$, sum as to i from 1 to n, simplify by aid of (21) and reduce; thus we have

$$\frac{1}{w} \frac{\partial w}{\partial x_l} = -\frac{1}{2\rho} \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_l} \sum_{k=1}^{n} \frac{\partial^2 y_i}{\partial x_k^2}, \qquad l = 1, 2, \cdots, n.$$

On simplifying this equation by aid of (27) we have

$$\frac{\partial}{\partial x_l} \log w = -\frac{1}{2\rho} \frac{2-n}{2} \frac{\partial \rho}{\partial x_l} = \frac{n-2}{4} \frac{\partial}{\partial x_l} (\log \rho), \quad i = 1, 2, \cdots, n.$$

Hence except for a trivial constant factor (taken equal to unity) we see from this equation and (28) that

$$(42) \qquad w = \rho^{\frac{n-2}{4}} = \sigma^{\frac{2-n}{2}}.$$

Thus w is completely determined in terms of ρ except for a constant factor. When $n = 2$ we have $w = 1$.

We shall now show that this value of w may be used to replace equation (9) in theorem I and equation (15) in theorem II. For this purpose it is sufficient to show that (9) is implied by (8) and (42). In this proof we may use equations (21) to (27) since they are all implied by (8). Differentiating both members of (42) logarithmically we obtain the equation preceding (42); thence by the aid of (27) we pass to the first equation of this section. On multiplying both members of that equation by $\partial y_j / \partial x_l$ and summing as to

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* The case $c = 0$ is not treated by Beez in the article already cited.

l, we obtain an equation which reduces readily to the form of (9). Hence, *in theorems I and II equations* (9) *and* (15) *may each be replaced by* (42).

Equation (42) gives the value of w for the transformation groups involved in theorems I and II. In the case of theorem III we saw that the corresponding quantity w must be a constant, whence it was taken to be unity. When $n = 2$ it is unity even in the cases of theorems I and II. When w is unity the wave function is an absolute invariant in the usual sense of the term in the theory of invariants. The same is true for solutions of equation (13). Also we have absolute invariants in all cases when $\rho = 1$. In general, the solutions of equations (1) and (13) are relative invariants, the multiplying factor w being a power of the Jacobian J of the transformation, since as we see from (20) and (42) we have

$$w = J^{\frac{n-2}{2n}}. \tag{43}$$

By aid of the values of σ given in (39) and (40) it is seen that w, as defined by (42), satisfies equation (10). From this it follows that conditions (8) and (9) in theorem I are superfluous, except as they are involved in finding the value of w in terms of ρ. Hence *the groups involved in theorems I and III are the same.* With the given possible values of σ, and hence of ρ, and by aid of (42), it is seen from equation (16) that ρ must be unity in the case of the group in theorem II.

§ 6. *The case when ρ is a constant C.* When ρ is a constant C, whence σ is a constant, the second member of equation (31) is zero. Hence with the aid of (23) we see that

$$0 = \frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial y_l}\right) = \frac{\partial}{\partial x_j}\left(\frac{1}{c}\,\frac{\partial y_l}{\partial x_i}\right) = \frac{1}{c}\,\frac{\partial^2 y_l}{\partial x_i \partial x_j}.$$

From this it follows that each function y_l is linear in the variables $x_1, x_2, \cdots, x_n$. Then we must have y_j of the form

$$y_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + a_j, \qquad j = 1, 2, \cdots, n. \tag{44}$$

Necessary and sufficient conditions that equations (19) are satisfied are then the following:

$$\sum_{k=1}^{n} a_{ik}a_{jk} = \delta_{ij}C. \tag{45}$$

When these conditions are satisfied we have in (44) transformations belonging to the groups involved in theorems I and III. In order that the transformations shall belong to the group involved in theorem II it is necessary to add the further condition that C shall be unity, as one sees by the aid of equation (16). In view of the result at the end of § 5 it is then seen that the transformations (44), with the named conditions, constitute the whole of the group defined in theorem III.

§ 7. *The case when* $n=2$. When $n=2$ it follows from equation (42) that the function w of theorem I is unity. Hence, when $n=2$, the transformation groups of theorems I and III coincide and each of them is the group of conformal transformations in the Euclidean plane. This group is sufficiently well known to justify us in not treating it further. The results given in the next section, however, are valid for all values of n.

§ 8. *On the class of possible solutions when* ρ *is given.* From (23), (19) and (21) we see that

$$\sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} = (1/\rho^2) \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \frac{\partial y_k}{\partial x_j} = (1/\rho^2) \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}/\rho.$$

Hence the value of ρ for a given transformation is the reciprocal of its value for the inverse transformation.

Again, if we have a second transformation of the y's into z's we have an equation of the form

$$\sum_{k=1}^{n} \frac{\partial z_i}{\partial y_k} \frac{\partial z_j}{\partial y_k} = \delta_{ij}\,\bar{\rho}.$$

Then we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{\partial z_i}{\partial x_k} \frac{\partial z_j}{\partial x_k} &= \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_l} \frac{\partial y_l}{\partial x_k} \frac{\partial z_j}{\partial y_m} \frac{\partial y_m}{\partial x_k} \\ &= \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_l} \frac{\partial z_j}{\partial y_m} \sum_{k=1}^{n} \frac{\partial y_l}{\partial x_k} \frac{\partial y_m}{\partial x_k} \\ &= \rho \sum_{m=1}^{n} \frac{\partial z_i}{\partial y_m} \frac{\partial z_j}{\partial y_m} \\ &= \delta_{ij}\,\rho\,\bar{\rho}. \end{aligned}$$

Hence if two transformations have the characteristic functions ρ and $\bar{\rho}$ then the product of the two have the characteristic function $\rho\bar{\rho}$.

Now let S and T be two transformations with the same characteristic function ρ. Then ρ^{-1} is the characteristic function of the inverse S^{-1} of S. Hence the transformation $S^{-1}T$ has the characteristic function 1. If we write

$$S^{-1}T = I$$

then I is a transformation with the characteristic function 1. But from the last equation it follows that $T = SI$. Hence if two transformations have the same characteristic function ρ then either one of them is equal to the other multiplied by a transformation with the characteristic function 1. In § 6 we have already determined all such transformations and have seen that they are linear. It follows that if we have a particular transformation associated with a given characteristic function ρ we can determine at once all the transformations having the same characteristic function. Therefore it will be sufficient henceforth to determine only a single transformation having a particular characteristic function ρ.

§ 9. *The case when σ is a quadratic function and $n > 2$.* The necessary form of σ in this case is given by equation (39). It remains to construct a particular transformation associated with this value of σ. Now we have

$$(46) \qquad \sum_{l=1}^{n} \left(\frac{\partial(1/\sigma)}{\partial y_l}\right)^2 = (1/\sigma^4) \sum_{l=1}^{n} \left(\frac{\partial \sigma}{\partial y_l}\right)^2 = (1/\sigma^4) \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial \sigma}{\partial x_k} \frac{\partial x_k}{\partial y_l}\right)^2$$

$$= (1/\sigma^4) \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \sigma}{\partial x_k}\right)^2 \left(\frac{\partial x_k}{\partial y_l}\right)^2 + (1/\sigma^4) \sum_{l=1}^{n} \sum_{\substack{k,m=1 \\ k \neq m}}^{n} \frac{\partial \sigma}{\partial x_k} \frac{\partial \sigma}{\partial x_m} \frac{\partial x_k}{\partial y_l} \frac{\partial x_m}{\partial y_l}$$

$$= (1/\sigma^2) \sum_{k=1}^{n} \left(\frac{\partial \sigma}{\partial x_k}\right)^2 = 4c/\sigma,$$

the last two members being gotten by aid of (23), (21) and (38). On comparing with the paragraph containing equation (39) we see that constants β must exist such that

$$1/\sigma = c \sum_{k=1}^{n} (y_k - \beta_k)^2.$$

Comparing with (39) we see that we must have

$$\sum_{k=1}^{n} (x_k - a_k)^2 \cdot \sum_{k=1}^{n} (y_k - \beta_k)^2 = 1/c^2.$$

This suggests the inversion transformation with respect to an n-dimensional sphere and hence the solution

$$y_i - \beta_i = \frac{x_i - a_i}{c\{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2\}}, \qquad i = 1, 2, \cdots, n.$$

It is readily verified that these values of the y's satisfy equation (19) for the given value of σ. Hence they afford a particular transformation associated with the given value of σ. The remaining transformations associated with σ are then determined by means of the results in § 8.

On taking appropriate translations in the x-space and the y-space respectively and adopting a convenient magnification in one or both spaces we may reduce the preceding transformation to the case in which $c = 1$ and the a's and β's are all zero. This change can be effected by means of linear transformations belonging to our group. Hence, except for suitable linear transformations, the preceding transformation may be written in the form

$$(47) \qquad y_i = x_i/s^2 \qquad \text{where } s^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

This is a real Cremona transformation of period two. It leads us to the following known theorems * (in the first case by the aid of the value of w given in § 5):

THEOREM IV. *If* $u = f(x_1, x_2, \cdots, x_n)$ *is any solution of the wave equation* (1), *then*

$$u = \frac{1}{s^{n-2}} f\,(x_1/s^2, x_2/s^2, \cdots, x_n/s^2).$$

is also a solution.

THEOREM V. *If* $u = f(x_1, x_2, \cdots, x_n)$ *is any solution of equation* (17) *then*

$$u = f\,(x_1/s^2, x_2/s^2, \cdots, x_n/s^2).$$

is also a solution.

§ 10. *The case when σ is a linear function and $n > 2$.* In this case σ has the form given in equation (40). From (46) and the results at the end of § 4, the latter carried over to the inverse transformation, it follows that we have

$$(48) \qquad 1/\sigma = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n + \gamma.$$

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* For the case $n = 3$ see Kelvin, *Liouville's Journal*, 12 (1847): 259. For general n see Böcher, *Bull. Amer. Math. Soc.*, 9 (1903): 459. For $n = 4$ see Bateman, *Proc. Lond. Math. Soc.*, (2) 7 (1908): 75. The methods of the present paper are different from those of these authors.

where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are constants such that

$$\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_n^2 = 0. \tag{49}$$

From (40) and (48) it follows that

$$(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \beta) (\gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n + \gamma) = 1. \tag{50}$$

By aid of (23) equation (31) may be put in the form

$$\frac{\partial}{\partial x_j}\left(\sigma^2 \frac{\partial y_l}{\partial x_i}\right) = \sigma\left(\frac{\partial y_l}{\partial x_i}\frac{\partial \sigma}{\partial x_j} - \frac{\partial y_l}{\partial x_j}\frac{\partial \sigma}{\partial x_i} - \delta_{ij}\frac{\partial(1/\sigma)}{\partial y_l}\right).$$

In the present case, by aid of (40) and (48), this equation reduces to

$$\sigma \frac{\partial^2 y_l}{\partial x_i \partial x_j} + \beta_j \frac{\partial y_l}{\partial x_i} + \beta_i \frac{\partial y_l}{\partial x_j} + \delta_{ij}\gamma_l = 0.$$

On making the transformation

$$y_l = \sigma^{-1} t_l \tag{51}$$

the equation takes the form

$$\frac{\partial^2 t_l}{\partial x_i \partial x_j} + \delta_{ij}\gamma_l = 0.$$

On taking i and j different in this equation one sees that t_l has the form

$$t_l = T_{l1}(x_1) + T_{l2}(x_2) + \cdots + T_{ln}(x_n),$$

where $T_l(x_1)$ is a function of x_i alone. Moreover, on taking $j = i$ we see that each $T_{li}(x_i)$ has its second derivative with respect to x_i equal to $-\gamma_l$. Hence t_l can be written in the form

$$t_l = -\tfrac{1}{2}\gamma_l(x_1^2 + x_2^2 + \cdots + x_n^2) + \epsilon_{l1}x_1 + \epsilon_{l2}x_2 + \cdots + \epsilon_{ln}x_n + \eta_l,$$

where the ϵ's and η's are constants. Then from (51) it follows that we must have

$$\sigma y_i = -\tfrac{1}{2}\gamma_i(x_1^2 + x_2^2 + \cdots + x_n^2) + \epsilon_{i1}x_1 + \epsilon_{i2}x_2 + \cdots + \epsilon_{in}x_n + \eta_i, \quad i = 1, 2, \cdots, n, \tag{52}$$

where the constants β_i and γ_i satisfy conditions (41) and (49). Since it is necessary and sufficient that the y_i shall satisfy equations (19), we now impose the conditions stated in these equations. Thus we have

$$\begin{aligned}\delta_{ij} &= \sigma^2 \sum_{k=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} \\ &= \sum_{k=1}^{n} (\gamma_i x_k + \beta_k y_i - \epsilon_{ik}) \; (\gamma_j x_k + \beta_k y_j - \epsilon_{jk}) \\ &= \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} + \gamma_i \eta_j + \gamma_j \eta_i - y_i \; [\sum_{k=1}^{n} \beta_k \epsilon_{jk} + \beta\gamma_j] - y_j \; [\sum_{k=1}^{n} \beta_k \epsilon_{ik} + \beta\gamma\end{aligned}$$

the last member being gotten by aid of equation (41). Hence, in order that the transformation defined in (52) shall be effective, it is necessary and sufficient that the constants involved shall satisfy the following conditions additional to that expressed in equation (41):

$$(53) \qquad \sum_{k=1}^{n} \epsilon_{ik}\epsilon_{jk} + \gamma_i \eta_j + \gamma_j \eta_i = \delta_{ij}, \quad \sum_{k=1}^{n} \beta_k \epsilon_{ik} + \beta\gamma_i = 0, \qquad i, j = 1, 2, \cdots, n.$$

It thus appears that condition (49) must be implied by (41) and (53), since there was no need of employing (49) in the derivation just given of the necessary and sufficient conditions that an effective transformation shall be afforded by equations (52).

We have seen that the γ's must satisfy (49). From (40), (50), (52) we see that the following conditions also must be satisfied:

$$(54) \qquad \sum_{k=1}^{n} \gamma_k \eta_k + \gamma\beta = 1, \quad \sum_{k=1}^{n} \gamma_k \epsilon_{ki} + \gamma\beta_i = 0, \quad i = 1, 2, \cdots, n.$$

It is clear, however, that these conditions on the constants are not independent of those previously derived.

Let us apply the linear transformations of § 6 to simplify transformation (52). By a translation of the x-space we may reduce β to zero. Similarly by a translation of the y-space we may reduce γ to zero. Now at least two of the quantities $\beta_1, \beta_2, \cdots, \beta_n$ are different from zero, since the sum of their squares is zero and we are supposing that σ is not a constant. Without loss of generality we suppose that $\beta_{n-1} \neq 0$, $\beta_n \neq 0$. If we now replace x_i by $\sqrt{-1}\, x_i'/\beta_n$, $i = 1, 2, \cdots, n$, we have a linear transformation belonging to those in § 6. In terms of these variables σ has the form

$$\sigma = \bar{\beta}_1 x_1' + \bar{\beta}_2 x_2' + \cdots + \bar{\beta}_{n-1} x'_{n-1} + \sqrt{-1}\, x'_n .$$

where

$$\bar{\beta}_1^2 + \bar{\beta}_2^2 + \cdots + \bar{\beta}^2_{n-1} = 1.$$

Now if we revolve the $(n-1)$-space $x'_{n-1}=0$ in such a way as to leave the $(n-1)$-space $x_n'=0$ invariant, we can bring the former into coincidence with the $(n-1)$-space $\bar{\beta}_1 x_1' + \cdots + \bar{\beta}_{n-1}x'_{n-1}=0$. Let x_1'', x_2'', $\cdots$, x''_{n-1}, $x_n''=x_n'$ be the new coordinates after this revolution; then σ has the form $\sigma = x''_{n-1} + \sqrt{-1}\; x_n''$. The transformations employed all belong to the group of § 6. Hence, by means of transformations belonging to the linear group of § 6, we can reduce σ to a normal form in which the coefficients β are

$$\beta=\beta_1=\beta_2=\cdots=\beta_{n-2}=0, \qquad \beta_{n-1}=1, \quad \beta_n=\sqrt{-1}.$$

Let us now take the original value of σ in this normal form. In view of the results in § 8 it is sufficient to obtain a single transformation of the type of (52). In order to find a convenient one we may also similarly normalize the y-coordinates so that $\gamma=\gamma_1=\gamma_2=\cdots=\gamma_{n-2}=0$, $\gamma_{n-1}=1$, $\gamma_n=\sqrt{-1}$. Then it is easy to obtain a particular solution of equations (53). Thus we have for a particular form of transformation (52) the following:

$$(55) \qquad \left\{ \begin{aligned} y_i &= \frac{x_i}{x_{n-1}+\sqrt{-1}\; x_n}, \qquad i=1,2,\cdots,n-2, \\ y_{n-1} &= \frac{1-s^2}{2(x_{n-1}+\sqrt{-1}\; x_n}, \\ y_n &= \frac{1+s^2}{2\sqrt{-1}(x_{n-1}+\sqrt{-1}\; x_n)}, \end{aligned} \right.$$

where $s^2=x_1^2+x_2^2+\cdots+x_n^2$. For this transformation and the resulting theorems in the special case when $n=4$ see Bateman, *l. c.*, p. 77.

Then we have the following theorems:

THEOREM VI. *If* $u=f(x_1, x_2, \cdots, x_n)$ *is any solution of the wave equation* (1) *then*

$$u=(x_{n-1}+\sqrt{-1}\; x_n)^{\frac{1}{2}(2-n)} f(y_1, y_2, \cdots, y_n)$$

is also a solution, the functions y_i *being defined by equations* (55).

THEOREM VII. *If* $u=f(x_1, x_2, \cdots, x_n)$ *is any solution of equation* (17) *then* $u=f(y_1, y_2, \cdots, y_n)$ *is also a solution, the functions* y_i *being defined by equations* (55).

The transformation defined by equations (55) is an imaginary Cremona transformation of period two. A real Cremona transformation of period two in $2n$ variables may be obtained from it by separating the given variables into real and imaginary parts and equating the real and imaginary parts of the two members of each of the resulting equations.

§ 11. *Summary of results.* Some results proved but not explicitly stated in the foregoing sections may be summarized in the form of the following theorems:

THEOREM VIII. *The transformation group involved in Theorem I and that involved in theorem III are the same. The group is that of the conformal transformations in space of n dimensions characterized at the end of* § 3, *imaginary transformations being allowed. The transformation group of theorem II is that subgroup of the foregoing in which* $\rho = 1$. (It is completely characterized in § 6.)

THEOREM IX. *The conformal transformation group of theorem I or III* (see end of § 3), *when $n > 2$, is generated by the linear transformations of* § 6 *and the two particular transformations* (47) *and* (55).

By means of the transformation indicated in the second paragraph of the introduction, it is seen that the two particular transformations (47) and (55) are real when they are taken in the form applicable to the wave equation as it is generally written by the mathematical physicists.

Transformations Leaving Invariant the Heat Equation of Physics.*

BY JOHN A. GOFF.

We shall write the general heat equation in the form

$$(1) \qquad \frac{\partial v}{\partial y_0} + \sum_{k=1}^{n} \frac{\partial^2 v}{\partial y_k^2} = 0.$$

For the case when $n = 3$, the transformation $y_0 = - c\rho t/k$, $y_1 = x$, $y_2 = y$, $y_3 = z$ reduces equation (1) to the classical form

$$c\rho \frac{\partial v}{\partial t} = k \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) .$$

of the heat equation of theoretical physics.

§ 1. *The Partial Differential Equations Defining the Transformation of the Heat Equation into Itself.* Let the function $v(y_0, y_1, y_2, \cdots, y_n)$ be a solution of the heat equation (1) and consider the transformation,

$$(2) \qquad u = w \cdot v(y_0, y_1, \cdots, y_n), \qquad w \not\equiv 0,$$

where w is a funcion of $x_0, x_1, \cdots, x_n$. Also the functions $y_0, y_1, \cdots, y_n$ are to be functionally independent functions of $x_0, x_1, \cdots, x_n$ so determined, in connection with w, that whenever v satisfies equation (1), u will satisfy the following equation:

$$(3) \qquad \frac{\partial u}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0.$$

We shall prove the following theorem:

THEOREM I. *Let $w, y_0, y_1, \cdots, y_n$ be a set of functions of $x_0, x_1, \cdots, x_n$ such that $y_0, y_1, \cdots, y_n$ are functionally independent and $w \not\equiv 0$. Then a necessary and sufficient condition that every solution of* (1) *shall be transformed through* (2) *into a solution of* (3) *is that the functions $w, y_0, y_1, \cdots, y_n$ shall satisfy the following conditions: a function ρ exists, not identically zero, such that*

$$(4) \qquad \frac{\partial w}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2 w}{\partial x_k^2} = 0,$$

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* Written with the aid of the foregoing paper by Professor Carmichael. References to Carmichael's paper are indicated by prefixing the letter C; thus, (C30) refers to Carmichael's equation (30).

$$(5)\qquad w\sum_{k=1}^{u}\frac{\partial^2 y_j}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_j}{\partial x_k}+w\frac{\partial y_j}{\partial x_0}=0,\qquad j=1,2,\cdots,n,$$

$$(6)\qquad \sum_{k=1}^{n}\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}=\delta_{ij}\,\rho,\qquad i,j=1,2,\cdots;n,$$

$$(7)\qquad w\sum_{k=1}^{n}\frac{\partial^2 y_0}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_0}{\partial x_k}+w\frac{\partial y_0}{\partial x_0}=w\,\rho,$$

$$(8)\qquad \sum_{k=1}^{n}\frac{\partial y_0}{\partial x_k}\frac{\partial y_j}{\partial x_k}=0,\qquad j=0,1,2,\cdots,n.$$

Since 1, y_j, y_iy_j $(i\neq j)$, and $y_j^2-y_n^2$ $(i,j=1,2,\cdots,n)$ are solutions of (1), it follows that the functions w and y_j must be such that w, wy_j, wy_iy_j $(i\neq j)$, and $w(y_j^2-y_n^2)$ for $i,j=1,2,\cdots,n$ are solutions of (3). These facts imply equations (4), (5), and (6). Substituting the value of u from (2) into (3) and simplifying by aid of (1), (4), (5), (6), we have

$$\begin{aligned}\frac{\partial v}{\partial y_0}&\Big\{w\sum_{k=1}^{n}\frac{\partial^2 y_0}{\partial x_k^2}+2\sum_{k=1}^{n}\frac{\partial w}{\partial x_k}\frac{\partial y_0}{\partial x_k}+w\frac{\partial y_0}{\partial x_0}-w\rho\Big\}\\&+2w\sum_{j=1}^{n}\frac{\partial^2 v}{\partial y_0\partial y_j}\sum_{k=1}^{n}\frac{\partial y_0}{\partial x_k}\frac{\partial y_j}{\partial x_k}\\&+w\frac{\partial^2 v}{\partial y_0^2}\sum_{k=1}^{n}\left(\frac{\partial y_0}{\partial x_k}\right)^2=0.\end{aligned}$$

The particular solutions of (1), namely, $y_0-\frac{1}{2}y_j^2$ and $y_0y_j-\frac{1}{6}y_j^3$ $(j=1,2,\cdots,n)$, show that (7) must be satisfied and that (8) must be satisfied for $j=1,2,\cdots,n$. That (8) must also be satisfied for $j=0$ then follows immediately. The conditions of the theorem, with the exception of the condition that ρ be not identically zero, are thus shown to be necessary.

Formulae (C19) and (C21) can be used since both of these are implied by (6). Let us multiply (8) by $\partial y_j/\partial y_l$, sum as to j from 1 to n, and simplify by means of (C21) getting

$$(9)\qquad \frac{\partial y_0}{\partial x_l}=0,\qquad l=1,2,\cdots,n.$$

Then (7) becomes

$$(10)\qquad \frac{\partial y_0}{\partial x_0}=\rho.$$

The Jacobian (functional determinant) is

$$J=\left|\frac{\partial y_j}{\partial x_k}\right|,\qquad j,k=0,1,2,\cdots,n,$$

which becomes, by virtue of (9) and (10),

$$J = \rho \left| \frac{\partial y_j}{\partial x_k} \right| \qquad j, k = 1, 2, \cdots, n,$$

so that

(11) $$J = \rho^{\frac{n+2}{2}} \qquad \text{by (C20)}$$

Consequently it is necessary and sufficient that ρ be not identically zero in order that the functions $y_0, y_1, \cdots, y_n$ be functionally independent functions of $x_0, x_1, \cdots, x_n$. Moreover conditions (4) to (8) can easily be seen to be sufficient conditions. Hence the theorem.

§ 2. *Determination of $w, y_0, y_1, \cdots, y_n$.* From equations (9) and (10) we get

(12) $$\frac{\partial \rho}{\partial x_l} = 0, \qquad l = 1, 2, \cdots, n,$$

so that ρ is a function of x_0 alone. If we consider the equations

$$x_i = x_i(y_0, y_1, \cdots, y_n), \qquad i = 1, 2, \cdots, n,$$

of the inverse transformation, then on differentiating with respect to x_j we get

$$\delta_{ij} = \sum_{k=0}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \qquad i, j = 1, 2, \cdots, n.$$

Then by virtue of (9) we have

$$\delta_{ij} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \qquad i, j = 1, 2, \cdots, n.$$

Hence we can use equations (C31) and (C23) together with (12) to show that

(13) $$\frac{\partial^2 y_j}{\partial x_k \partial x_l} = 0, \qquad j, k, l = 1, 2, \cdots, n.$$

The functions y_j are then of the form

(14) $$y_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + a_j, \qquad j = 1, 2, \cdots, n.$$

where the a's are functions of x_0 alone.

In view of (13), equation (5) becomes

(15) $$\frac{\partial y_j}{\partial x_0} = -2 \sum_{m=1}^{n} \frac{\partial(\log w)}{\partial x_m} \frac{\partial y_j}{\partial x_m}, \qquad j = 1, 2, \cdots, n.$$

Now (6) implies that

(C21) $$\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = \delta_{kl}\rho, \qquad k, l = 1, 2, \cdots, n.$$

Differentiate both sides of this equation with respect to x_0 and then by means of (15) obtain

$$\sum_{m=1}^{n} \frac{\partial^2 (\log w)}{\partial x_m \partial x_l} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_m} + \sum_{m=1}^{n} \frac{\partial^2 (\log w)}{\partial x_m \partial x_k} \sum_{j=1}^{u} \frac{\partial y_j}{\partial x_l} \frac{\partial y_j}{\partial x_m}$$

$$+ \sum_{m=1}^{n} \frac{\partial (\log w)}{\partial x_m} \frac{\partial}{\partial x_m} \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_k} \frac{\partial y_j}{\partial x_l} = -\tfrac{1}{2} \delta_{kl} \frac{\partial \rho}{\partial x_0}, \qquad k, l = 1, 2, \cdots, n.$$

When this is simplified by means of (C21) and (12), the result is

$$(16) \qquad \frac{\partial^2 (\log w)}{\partial x_k \partial x_l} = -\tfrac{1}{4} \delta_{kl}\, \rho'/\rho = -\tfrac{1}{4} \delta_{kl}\, \zeta, \qquad k, l = 1, 2, \cdots, n.$$

where ρ' denotes the derivative of $\rho(x_0)$ with respect to its argument. For convenience we have denoted the ratio ρ'/ρ by ζ.

From (15) with the aid of (13), (14) and (16), we get

$$\frac{\partial^2 y_j}{\partial x_0 \partial x_l} = \tfrac{1}{2} \zeta \cdot a_{jl}, \qquad j, l = 1, 2, \cdots, n.$$

But from (14) alone we can obtain directly,

$$\frac{\partial^2 y_j}{\partial x_0 \partial x_l} = a'_{jl}, \qquad j, l = 1, 2, \cdots, n.$$

where the prime denotes the derivative with respect to x_0. Consequently $a'_{jl} = \tfrac{1}{2} a_{jl}\, \rho'/\rho$, whence

$$(17) \qquad a_{jl} = C_{jl}\, \rho^{\frac{1}{2}}, \qquad j, l = 1, 2, \cdots, n,$$

where the quantities C_{jl} are constants. These results show that the y_j can be written in the form,

$$(18) \qquad y_j = \rho^{\frac{1}{2}} \{ \sum_{m=1}^{n} C_{jm} x_m + \gamma_j \}.$$

where, by virtue of (6) and (C21) the constants satisfy the conditions

$$(19) \qquad \sum_{k=1}^{n} C_{jk} C_{ik} = \delta_{ij}, \qquad i, j = 1, 2, \cdots, n,$$

and the γ_j are functions of x_0 alone.

From (16) we see that log w is of the form

$$(20) \qquad \log w = -\tfrac{1}{8} \zeta \sum_{m=1}^{n} (x_m - \beta_m)^2 + \beta,$$

where the functions β are functions of x_0 alone. Moreover, since w must satisfy (4), log w must satisfy the relation

$$(21) \qquad \frac{\partial (\log w)}{\partial x_0} + \sum_{k=1}^{n} \frac{\partial^2 (\log w)}{\partial x_k^2} + \sum_{k=1}^{n} \{ \frac{\partial (\log w)}{\partial x_k} \}^2 = 0.$$

From this, with the aid of (20), we obtain the following equation:

$$(22) \qquad \tfrac{1}{4}\zeta \sum_{m=1}^{n} (x_m - \beta_m)\beta'_m + \beta' - (n/4)\,\zeta + (1/16\,\zeta^2 - \tfrac{1}{8}\zeta') \sum_{m=1}^{n} (x_m - \beta_m)^2 = 0.$$

This equation implies that

$$(23) \qquad \zeta^2 = 2\zeta' \qquad \text{and} \qquad \beta' = (n/4)\,\zeta;$$

and, if $\zeta \not\equiv 0$, that

$$(24) \qquad \beta'_m = 0, \qquad m = 1, 2, \cdots, n.$$

Now form $\partial x_j/\partial x_0$ from equation (18) and compare with $\partial y_j/\partial x_0$ obtained from (15) with the aid of (18) and (20). This leads to the following differential equation:

$$(25) \qquad \gamma'_j + \tfrac{1}{2}\zeta\gamma_j + \tfrac{1}{2}\zeta \sum_{m=1}^{n} C_{jm}\beta_m = 0, \qquad j = 1, 2, \cdots, n.$$

Remembering that $\zeta = \rho'/\rho$ and supposing for the moment that ρ is not a constant, we then see, by aid of (24), that

$$(26) \qquad \gamma_j = -\sum_{m=1}^{n} C_{jm}\beta_m + C_j/\rho^{\frac{1}{2}}, \qquad j = 1, 2, \cdots, n$$

where the C_j are constants. In this case the β_m are also constants. Consequently (18) becomes

$$(27) \qquad y_j = \rho^{\frac{1}{2}} \sum_{m=1}^{n} C_{jm}(x_m - \beta_m) + C_j, \qquad j = 1, 2, \cdots, n,$$

when ρ is not a constant.

§ 3. *The Case when $\zeta \equiv 0$.* In this case $\rho = k^2 =$ constant. From (25) we see that each γ_j is a constant which we may choose in such a way as to enable us to write

$$(28) \qquad y_j = k \sum_{m=1}^{n} C_{jm}(x_m - \beta_m) \qquad j = 1, 2, \cdots, n.$$

where the constants C_{jm} must satisfy conditions (19); namely,

$$(19) \qquad \sum_{m=1}^{n} C_{jm}C_{im} = \delta_{ij}, \qquad j = 1, 2, \cdots, n.$$

From equation (10) we see that y_0 is of the form

$$(29) \qquad y_0 = k^2(x_0 - \beta_0).$$

Furthermore from (2) and the last of (23) (which becomes $\beta' = 0$) we see that w is a constant and without loss of generality we may take

$$(30) \qquad w = 1.$$

If $\rho^{\frac{1}{2}} = 1$, we may write (28) and (29) as follows

$$y_j = \sum_{m=0}^{n} C_{jm}(x_m - \beta_m), \qquad j = 0, 1, 2, \cdots, n,$$

where $C_{0m} = C_{m0} = \delta_{0m}$. Then

$$\sum_{k=0}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} = \delta_{ij}, \qquad i, j = 0, 1, 2, \cdots, n;$$

for if $j = 0$ and $i \neq 0$, it reduces to 0; if $j = 0$ and $i = 0$, it reduces to 1; while if $i \neq 0$ and $j \neq 0$ it becomes simply δ_{ij} by (6). These results show that the transformation described by equations (28), (29), (19), and (30) is conformal when $\rho^{\frac{1}{2}} = 1$.

§ 4. *The Case when* $\zeta \neq 0$. By using equations (23) we get the following particular solutions:

$$(31) \qquad \zeta = -2/x_0, \qquad \rho = 1/x_0^2, \qquad \beta = \log x_0^{-(n/2)};$$

whence by aid of (20) and (27) we have

$$(32) \qquad w = x_0^{-(n/2)} \exp \left\{ \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{4x_0} \right\} \quad ; \quad y_j = x_j/x_0, \quad j = 1, 2, \cdots, n.$$

By (10) we have

$$(33) \qquad y_0 = -1/x_0.$$

The constants which have been omitted in obtaining these particular solutions can be removed by a transformation described under § 3. Thus we can state the following theorem:*

THEOREM II. *If* $u = f(x_0, x_1, \cdots, x_n)$ *is any solution of the heat equation* (3), *then*

$$u = x_0^{-(n/2)} \exp \left\{ \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{4x_0} \right\} f\,(-1/x_0, x_1/x_0, x_2/x_0, \cdots, x_n/x)$$

is also a solution of (3).

§ 5. *Summary. The results show that the transformation of theorem II together with the transformations described by equations* (28), (29), (19), *and* (30) *generate all the transformations of the form of equation* (2) *which leave the heat equation invariant.*

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[*] For the case $n = 3$ see J. Brill, *Messenger of Mathematics,* 2 (1891), p. 137; for the case $n = 1$ see P. Appell, *Journal de Mathématiques,* (4) 8 (1892), p. 187.

The Application of Fractional Operators to Functional Equations.[*]

BY H. T. DAVIS.

1. *Fractional Operations.* In a recent paper [†] the author showed how it was possible to express certain types of integral equations as differential equations of fractional order and by operating on the latter to obtain solutions of the original equations. Since that time several examples of such equations have come to his attention, but the methods of solution employed in these cases made no use of the idea of fractional operations. For example, John R. Carson in a series of papers [‡] has made an analysis of the calculus which Oliver Heaviside developed and used so effectively in the circuit problems of electrical theory.[§] Carson shows very elegantly how the Heaviside calculus is formally equivalent to the process of solving an integral equation of the form

$$f(x) = \int_0^\infty e^{-xt} u(t)\, dt,$$

but he does not point out the interesting fact that the most striking feature of the Heaviside calculus is also a special application of the fractional operators of Riemann.[‖] In particular, the series expansion of the Heaviside operator for the general case of the non-inductive cable which Carson characterizes as "hopelessly complicated to either interpret or compute" is reduced with considerable simplicity by the methods of this paper.[¶]

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[*] Presented to the Indiana Section of the Mathematical Association of America, May 8, 1926.

[†] "Fractional Operations as Applied to a Class of Volterra Integral Equations," *American Journal of Mathematics*, Vol. 46 (1924), pp. 95-109.

[‡] "Theory of the Transient Oscillations of Electrical Networks and Transmission Systems," *Trans. A. I. E. E.* (1919); "The Heaviside Operational Calculus," *The Bell System Tech. Journal*, (1922); "Electric Circuit Theory and the Operational Calculus," *Ibid.*, Vol. 4 (1925), pp. 685-761; Vol. 5 (1926), pp. 50-95; *Bull. Amer. Math. Soc.*, Vol. 32 (1926), pp. 43-63.

[§] *Electromagnetic Theory*, Vol. II.

[‖] A similar observation has recently been made by P. Levy: "Le calcul symbolique d'Heaviside," *Bulletin des sciences Mathématiques*, Vol. 50, 2nd series (June, 1926), pp. 174-192.

[¶] *Bell System Tech. Journal*, Vol. 5 (1926), p. 59.

It seems important, therefore, that some effort should be made to give wider introduction to these fractional operators and to suggest that a standard symbolism be adopted for them. The great elegance that can be secured by proper use of fractional operators and the power which they have in the solution of complicated functional equations should more than justify a more general recognition and use. The aim of the present paper is to present the case of the "fractional calculus."

The idea of fractional differentiation was clearly in the mind of Leibnitz who said in a letter to G. F. A. de l'Hospital in 1695 that he was considering the meaning to be attached to $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$. The first real use of fractional operations was apparently made by Abel who, as early as 1823 used both the symbols $d^{-\frac{1}{2}}/dx^{-\frac{1}{2}}\ \psi(x) = 1/\sqrt{\pi}\ \int^{\frac{1}{2}} \psi(x)\ dx$ and $\psi(x) = \sqrt{\pi}\ d^{\frac{1}{2}}s/dx^{\frac{1}{2}}$ in solving the problem of the tautochrone.*

Although there is mention of fractional operators in the works of Euler, Laplace and Fourier, it was J. Liouville who first made an extensive study of the new calculus. Altogether he devoted eight memoirs totaling about three hundred pages to the subject making application to problems in geometry and mechanics.† The effectiveness of this work was somewhat limited, however, by the fact that a specialized definition of fractional operations was employed. Most of the modern work in this field goes back to Riemann who, in a paper developed during his student days but posthumously published, gave general definitions of fractional differentiation and integration.‡

Slightly generalized these definitions may be stated as follows:

By the *fractional integration* of a function $u(x)$ we shall mean the operation

$$ {}_cD_x^{-\nu}u(x) = \int_c^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)}\, u(t)\, dt, \qquad \nu > 0, \tag{1} $$

and by *fractional differentiation*,

$$ {}_cD_x^{m+\nu}u(x) = \frac{d^{m+1}}{dx^{m+1}} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(1-\nu)}\, u(t)\, dt, \quad m = 0, 1, 2, \cdots, \quad 0 < \nu < 1. \tag{2} $$

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* "Solution de quelques problèmes a l'aide d'integrales définies." *Werke* (1881), Vol. 1, pp. 10-27.

† *Journal de l'école polytechnique*, cah. 21 (1832), pp. 1-69; 71-162; 163-186; cah. 24 (1835), pp. 17-54; cah. 25 (1836), pp. 56-84; *Journal für Mathematik*, Vol. 11 (1834), pp. 1-19; Vol. 12 (1834), pp. 273-287; Vol. 13 (1835), pp. 219-232.

‡ "Versuch einer allgemeinen Auffassung der Integration und Differentiation." *Gesammelte Werke*, Leipzig (1892), pp. 353-366.

It is important to note that the constant "c" is a vital part of the operational symbol. Thus Riemann let $c = 0$ in his original definition and Liouville let $c = -\infty$.

Various symbols have been suggested for these operators. We have already mentioned d^{ν}/dx^{ν} and $d^{-\nu}/dx^{-\nu}$, but they are somewhat cumbersome and do not convey information regarding the lower limit of the integral. This symbolism was effectively used by Liouville who, as has been stated, employed a fixed limit. Emil Post in discussing the solution of the problem $d^{\frac{1}{2}}y/dx^{\frac{1}{2}} - y/x = 0$ in the *Mathematical Monthly* several years ago [*] suggested the use of the symbol $\{d^{-\nu}/dx^{-\nu}\}_c^x$. Riemann used $\partial_x^{-\nu}$ and ∂_x^{ν} for his operators and P. Levy in a recent paper [†] employed the symbols $I_x^{\nu}[\phi(x)]$ for integration and $D_x^{\nu}[\phi(x)]$ for differentiation where the lower limit of the integral was zero. In a contribution to the subject by H. Laurent in 1883 we find the operation expressed by $\left.\vphantom{\frac{d^{\nu}f}{dx^{\nu}}}\right|_{c}^{x} \dfrac{d^{\nu}f}{dx^{\nu}}$.[‡] J. Hadamard, who made fundamental use of Riemann's operator and gave some generalization to it in his work on analytic functions, employed $D_x^{\nu}\phi(x)$.[§] The mathematical encyclopedia, to which the reader is referred for a longer bibliography on the subject, uses $D_c^{x\nu} f(x)$ as a symbol for the general operator.[‖] Since all of the information is conveyed in a simple manner by the symbols ${}_cD_x^{-\nu}u(x)$ and ${}_cD_x^{\nu}u(x)$ and since they are only slight variations of those which appear to have had the largest usage, the author has adopted them in his work. It is occasionally convenient to use the easily understood abbreviations $u^{(-\nu)}(x)$ and $u^{(\nu)}(x)$ when there is no question as to the lower limit of integration.

2. *Some Properties of the Operators.* For convenience in reference a few of the properties of fractional operators will be stated. Where proofs have been omitted they will be found elsewhere.[¶]

The significance of the definitions given in the preceding section attaches to the fact that

$$\begin{aligned} {}_cD_x^{\mu}[u(x) + v(x)] &= {}_cD_x^{\mu}\, u(x) + {}_cD_x^{\mu}\, v(x), \\ {}_cD_x^{\mu} k\, u(x) &= k\, {}_cD_x^{\mu}\, u(x), \\ {}_cD_x^{\mu}\, {}_cD_x^{\nu}\, u(x) &= {}_cD_x^{(\mu+\nu)} u(x). \end{aligned} \tag{3}$$

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[*] *Mathematical Monthly*, Vol. 26 (1919), p. 37; see also Vol. 25 (1918), p. 172, problem 433.

[†] *Bulletin des Sciences Mathématiques*, Vol. 47 (1923), pp. 307 and 343.

[‡] *Nouvelles Annales*, series 3, Vol. 3 (1883), p. 240.

[§] *Journal de mathématiques*, Vol. 57 (1892), p. 154.

[‖] *Encyklopädie der Mathematischen Wissenschaften*, II, A, 1, pp. 116-119; II, A, 11, pp. 770-772. Also *Encyclopédie des Sciences Mathématiques*, T. 11, V. 5, fas. 1, 8.

[¶] Davis, *loc. cit.*

The analogue of Taylor's theorem is found in the following development of the integration operator:

$$
{}_cD_x^{-\nu}\,u(x) = \frac{(x-c)^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (-1)^m \frac{\nu}{\nu+m}\,\frac{(x-c)^m}{m!}\,u^{(m)}(x). \qquad (4)
$$

This is easily proved by replacing $u(t)$ in (1) by its expansion about the point x. We thus get,

$$
{}_cD_x^{-\nu}\,u(x) = \int_c^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} \sum_{m=0}^{\infty} u^{(m)}(x)\,\frac{(t-x)^m}{m!}\,dt,
$$

which easily reduces to (4).

Another useful expansion is that analogous to the formula of Leibnitz which may be stated as follows:

$$
{}_cD^{x-\nu}\,[U(x)\,V(x)] = U(x)\,V^{(-\nu)}(x) - \nu U'(x)\,V^{(-\nu-1)}(x) + \frac{\nu(\nu+1)}{2!}\,U''(x)\,V^{(-\nu-2)}(x) - \cdots. \qquad (5)
$$

In the application of these formulas it is often useful to note the following derivative:

If $v(x) = \displaystyle\int_c^x \frac{g(x,s)\,u(s)}{(x-s)^\nu}\,ds$, we then have

$$
v'(x) = \frac{g(x,c)\,u(c)}{(x-c)^\nu} + \int_c^x \frac{D_xg(x,s) + D_sg(x,s)}{(x-s)^\nu}\,u(s)\,ds + \int_c^x \frac{g(x,s)}{(x-s)^\nu}\,u'(s)\,ds. \qquad (6)
$$

This formula can be established by the method of "partie finie" due simultaneously to R. D'Adhemar and J. Hadamard in 1904.*

Let us write

$$
v_\epsilon(x) = \int_c^{x-\epsilon} \frac{g(x,s)\,u(s)}{(x-s)^\nu}\,ds;
$$

since there is no longer a singularity in the integrand for the upper limit of the integral, we can apply the ordinary rule for the differentiation of a function defined by an integral. We then have

$$
\frac{dv_\epsilon(x)}{dx} = \int_c^{x-\epsilon} \left\{ \frac{\partial/\partial x\; g(x,s)\,u(s)}{(x-s)^\nu} + g(x,s)\,u(s)\;\partial/\partial x\,(x-s)^{-\nu} \right\} ds + \frac{g(x,x-\epsilon)\,u(x-\epsilon)}{\epsilon^\nu}.
$$

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* R. D'Adhemar, Thesis, *Journal de Math.* (1904); see also *Exercices et leçons d'analyse*, Paris (1908), p. 150; J. Hadamard, *Congres de Mathém.* (1904), pp. 265-271.

Since $\partial/\partial x\,(x-s)^{-\nu} = -\,\partial/\partial s\,(x-s)^{-\nu}$, we can substitute this value in the integrand of the preceding integral and integrate by parts. We then get,

$$\frac{dv_\epsilon(x)}{dx} = \int_c^{x-\epsilon} \frac{\partial/\partial x\; g(x,s)\,u(s) + \partial/\partial s\,[g(x,s)\,u(s)]}{(x-s)^\nu}\,ds + \frac{g(x,c)\,u(c)}{(x-c)^\nu}.$$

and in the limit we have formula (6).

It will be useful in another section of the paper to know that the well-known operator:*

$$(d/dx + k)^n f(x) = e^{-kx}\, d^n/dx^n\,\{e^{kx} f(x)\}, \quad n \text{ a positive integer},$$

is also valid for fractional values of n. This can be proved by means of formula (5).

We first write,

$$e^{-kx}\, d^n/dx^n\,\{e^{kx} f(x)\} = e^{-kx}\,\frac{d^{m-\mu}}{dx^{m-\mu}}\,\{e^{kx} f(x)\},$$

where m is an integer and μ a fraction between 0 and 1. Then in (5), setting $U(x)$ equal to e^{kx} and $V(x)$ equal to $f(x)$, we get,

$$e^{-kx}\,\frac{d^{m-\mu}}{dx^{m-\mu}}\,\{e^{kx} f(x)\} =$$

$$e^{-kx}\, d^m/dx^m \Big[e^{kx}\,\{f^{(-\mu)}(x) - \mu k f^{(-\mu-1)} + \frac{\mu(\mu+1)}{2!}\,k^2 f^{(-\mu-2)} + \cdots\}\Big] =$$

$$e^{-kx}\, d^m/dx^m\,[e^{kx}\,\{1 - \mu k z^{-1} + \frac{\mu(\mu+1)}{2!}\,k^2 z^{-2} + \cdots\} f^{(-\mu)}(x)],$$

where we have used the abbreviation $z = d^{-1}/dx^{-1}$. This expression then reduces to

$$e^{-kx}\, d^m/dx^m\,[e^{kx}\,(1 + k/z)^{-\mu} f^{(-\mu)}(x)] =$$

$$(d/dx + k)^m\,(1 + k/z)^{-\mu} f^{(-\mu)}(x) = (d/dx + k)^{m-\mu} f(x).$$

3. *The Solution of Fractional Equations.* One of the simplest cases of fractional equations is that of the linear equation with constant coefficients involving derivatives (or integrals) of orders which differ by multiples of $1/n$, n an integer, from one another. Thus let us consider the equation,

$$L(u) \equiv A_0 u + A_1 u^{(1/n)} + A_2 u^{(2/n)} + \cdots + A_{n-1} u^{(1-1/n)} = f(x), \qquad (7)$$

---

* J. Edwards, *Differential Calculus*, 3rd ed., London (1904), p. 70.

where we have designated by $u^{(\nu)}$ the derivative ${}_cD_x^\nu u(x)$. The function $f(x)$ will be characterized later.

If we replace $u(x)$ by $d/dx \int_c^x u(s)\, ds$ and the fractional derivatives by their formal values we see that (7) is equivalent to the following functional equation:

$$\frac{d}{dx}\int_c^x \frac{G(x,s)\,u(s)}{(x-s)^{1-1/n}}\,ds = f(x),$$

where

$$G(x,s) \equiv A_0(x-s)^{1-1/n} + \frac{A_1}{\Gamma(1-1/n)}(x-s)^{1-2/n} + \cdots + \frac{A_{n-1}}{\Gamma(1/n)};$$

Upon integrating both sides we arrive at the integral equation:

$$\int_c^x \frac{G(x,s)}{(x-s)^{1-1/n}}\,u(s)\,ds = \int_c^x f(s)\,ds.$$

The existence theorem is well known and has been stated under a variety of conditions for $f(x)$.[*] It is sufficient to assume that $f(x)$ is analytic in the neighborhood of $x = c$. Since the first equation is derived from the latter by differentiation, it is clear that a solution of the integral equation will also be a solution of its derivative.

Having thus assured ourselves of the existence of at least one solution of equation (7) we can proceed to find this solution by the following formal method. This method will be found to apply equally well to an equation involving fractional integrals differing by multiples of $1/n$ from one another and since the details are essentially the same no separate discussion will be made of this case.

Representing (7) symbolically by P,

$$P \equiv A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots + A_{n-1} z^{1-1/n},$$

we can rationalize this algebraic equation in the following manner. Multiplying P successively by $z^{1/n}$, $z^{2/n}$, etc. we get the system of equations,

$$\begin{aligned}
P &= A_0 + A_1 z^{1/n} + A_2 z^{2/n} + \cdots + A_{n-1} z^{1-1/n},\\
z^{1/n}P &= zA_{n-1} + A_0 z^{1/n} + A_1 z^{2/n} + \cdots + A_{n-2} z^{1-1/n},\\
z^{2/n}P &= zA_{n-2} + zA_{n-1} z^{1/n} + A_0 z^{2/n} + \cdots + A_{n-3} z^{1-1/n},\\
&\cdots\cdots\cdots\cdots\cdots\\
z^{1-1/n}P &= zA_1 + zA_2 z^{1/n} + zA_3 z^{2/n} + \cdots + A_0 z^{1-1/n}.
\end{aligned}$$

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[*] T. Lalesco, *Journal de Mathématique*, Vol. 73 (1908), pp. 125-202.

Eliminating $z^{1/n}$, $z^{2/n}$, $\cdots$ from the right hand side of these equations we obtain

$$\begin{vmatrix} A_0 - P, & A_1, & A_2, & \cdots, & A_{n-1} \\ zA_{n-1} - z^{1/n}P, & A_0, & A_1, & \cdots, & A_{n-2} \\ zA_{n-2} - z^{2/n}P, & zA_{n-1}, & A_0, & \cdots, & A_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ zA_1 - z^{1-1/n}P, & zA_2, & zA_3, & \cdots, & A_0 \end{vmatrix} = 0.$$

From this it follows that

$$P\Delta(z) = A(z),$$

where

$$\Delta(z) = \begin{vmatrix} 1, & A_1, & A_2, & \cdots, & A_{n-1} \\ z^{1/n}, & A_0, & A_1, & \cdots, & A_{n-2} \\ z^{2/n}, & zA_{n-1}, & A_0, & \cdots, & A_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z^{1-1/n}, & zA_2, & zA_3, & \cdots, & A_0 \end{vmatrix} \qquad (8)$$

and $A(z)$ is the same determinant with the first column replaced by A_0, zA_{n-1}, zA_{n-2}, $\cdots$, zA_1.

Since $A(z)$ is a polynomial in z of degree less than or equal to $n-1$, it is clear that Δ is a rationalizing factor and equation (7) can be formally replaced by the ordinary linear equation

$$A(d/dx)\, u = \Delta(d/dx)\, f(x). \qquad (9)$$

To this equation there corresponds an integral of the form

$$u(x) = \int_c^x W(x,t)\, \Delta(d/dt)\, f(t)\, dt, \qquad (10)$$

where $W(x,t)$ is the Cauchy function familiar from the method of variation of parameters, provided $W(x,t)\,\Delta(d/dt)\,f(t)$ does not become infinite to a higher order than $\mu < 1$ at $t = c$.

To show that (10) is the unique solution of (7) we shall need the following theorem:

THEOREM 1. *The homogeneous equation* $L(u) = 0$ has no solution ex*cept the trivial one* $u \equiv 0$, *provided* $c \neq \infty$.

Proof: Equation (7) is equivalent to the equation

$$A_0 u(x) + d/dx \int_c^x \left\{ \frac{A_1 (x-s)^{-(1/n)}}{\Gamma(1-1/n)} + \frac{A_2 (x-s)^{-(2/n)}}{\Gamma(1-2/n)} \right.$$
$$\left. + \cdots + \frac{A_{n-1}(x-s)^{-(1-1/n)}}{\Gamma(1/n)} \right\} u(s)\, ds = 0, \tag{11}$$

which may be written from (6) in the form

$$L(u) \equiv A_0 u(x) + \int_c^x I(x,s)\, u'(s)\, ds + u(c)\, I(x,c) = 0,$$

where $I(x,s)$ is the integrand of (11). Now unless $u(c) = 0$ we should have a contradiction since $\lim\limits_{x=c} L(u)$ would then be infinity instead of zero. Similarly we may prove that $u(c) = u'(c) = \cdots = u^{(n)}(c) = 0$, which would not be the case unless c were a singular point of the equation $A(d/dx) = 0$. But since $c = \infty$ is the only singular point we conclude that $L(u) \doteq 0$ can have no solution except the trivial one.

As illustration we shall consider two special equations which are of frequent occurrence in application.

I. As a special case of (7) we have

$${}_cD^{\frac{1}{2}}{}_x\, u + \lambda u = f(x). \tag{12}$$

Rationalizing we obtain the ordinary equation

$$d/dx\, u - \lambda^2 u = {}_cD_x^{\frac{1}{2}} f(x) - \lambda f(x), \tag{13}$$

whose solution is

$$u(x) = e^{\lambda^2 x} \int_c^x \{ ({}_cD_t^{\frac{1}{2}} - \lambda)\, f(t) \} e^{-\lambda^2 t}\, dt. \tag{14}$$

In order to see that this solution behaves properly at the point $x = c$, write (12) in the expanded form

$$\frac{u(c)}{\sqrt{\pi}\,(x-c)^{\frac{1}{2}}} + 1/\sqrt{\pi} \int_c^x \frac{u'(t)}{(x-t)^{\frac{1}{2}}}\, dt + \lambda u(x) = f(x).$$

If $f(x)$ is bounded at the point $x = c$, then it is clear that $u(c)$ must be zero if we are to avoid a contradiction. Since (14) is the only solution of (13) which satisfies this condition it must be the solution of (12).

Example: Solve ${}_0D_x^{\frac{1}{2}}\, u - u = -1$.

We get as the formal equivalent of this equation the ordinary equation

$$u' - u = -(1 + 1/\sqrt{\pi x}),$$

whose solution is

$$u(x) = e^{x} \int_0^x e^{-t} \left(-1 - 1/\sqrt{\pi t}\right) dt,$$

$$= 1 - e^{x} - 1/\sqrt{\pi x}\, \left\{ (2x) + \frac{(2x)^2}{1 \cdot 3} + \frac{(2x)^3}{1 \cdot 3 \cdot 5} + \cdots \right\}.$$

This function can be shown by direct substitution to satisfy the original equation.

II. Consider also

$$ {}_cD_x^{-\frac{1}{2}} u + \lambda u = f(x), \tag{15}$$

whose rational equivalent is the equation

$$\lambda^2 \; d/dx \; u - u = \lambda f'(x) - {}_cD_x^{\frac{1}{2}} f(x). \tag{16}$$

The solution of this equation is

$$u(x) = C\, e^{1/\lambda^2 x} + e^{1/\lambda^2 x} \int_c^x \left\{ 1/\lambda\, f'(t) - 1/\lambda^2 \; {}_cD_t^{\frac{1}{2}} f(t) \right\} e^{-1/\lambda^2 t}\, dt. \tag{17}$$

In order to determine C consider equation (15) in the form

$$1/\sqrt{\pi} \int_c^x \frac{u(t)}{(x-t)^{\frac{1}{2}}}\, dt + \lambda u(x) = f(x).$$

It is clear that for $x = c$ we have $\lambda u(c) = f(c)$. Then from (17) we get $u(c) = C\, e^{1/\lambda^2 c} = 1/\lambda\, f(c)$ and (17) can be written in the form

$$u(x) = 1/\lambda\, f(c)\, e^{1/\lambda^2 (x-c)} + \int_c^x \left\{ 1/\lambda\, f'(t) - 1/\lambda^2 \; {}_cD_t \; f(t) \right\} e^{1/\lambda^2 (x-t)}\, dt \tag{18}$$

Example: Solve

$$ {}_0D_x^{-\frac{1}{2}} u + u = 1.$$

This rationalizes to the equation

$$u' - u = - \frac{1}{\sqrt{\pi x}},$$

whose solution is

$$u(x) = e^{x} - e^{x} \int_0^x e^{-t} \; 1/\sqrt{\pi t}\, dt.$$

This solution can be expanded in two ways:

$$u_1 = e^{x} - \frac{1}{\sqrt{\pi x}} \left\{ (2x) + \frac{(2x)^2}{1 \cdot 3} + \frac{(2x)^3}{1 \cdot 3 \cdot 5} + \cdots \right\},$$

and $u_2 = \dfrac{1}{\sqrt{\pi x}} \left\{ 1 - \dfrac{1}{2x} + \dfrac{1 \cdot 3}{(2x)^2} - \dfrac{1 \cdot 3 \cdot 5}{(2x)^3} + \cdots \right\}.$

The second, although everywhere divergent, is the asymptotic expansion of the first.

4. *Generalization of Abel's Integral Equation.* Among special types of integral equations we find the following:

$$k\,x^a = \int_0^x \sqrt{x-t}\;u(t)\,dt, \tag{19}$$

which was first solved by N. Hirakawa[*] and subsequently discussed by T. Hayashi[†] on the basis of an elegant method due to P. Humbert.[‡]

The generalization

$$f(x) = \int_c^x (x-t)^{m+\nu}\,u(t)\,dt, \qquad -1 < \nu < 0, \quad m = 0, 1, 2, \cdots \tag{20}$$

which includes both Hirakawa's and Abel's equations[§] as special cases is easily solved by fractional operators. This can be written

$$f(x) = \Gamma(m+\nu+1)\;{}_cD_x^{-(m+\nu+1)}u(x).$$

Operating on both sides with the symbol ${}_cD_x^{m+\nu+1}$ we have

$$\begin{aligned} u(x) &= \frac{{}_cD_x^{m+\nu+1}f(x)}{\Gamma(m+\nu+1)}, \\ &= \frac{d^{m+2}}{dx^{m+2}}\int_c^x \frac{(x-t)^{-\nu}}{\Gamma(m+\nu+1)\,\Gamma(1-\nu)}\,f(t)\,dt. \end{aligned} \tag{21}$$

Recalling elementary properties of the gamma function and making use of (6) we can write (21) as

$$\begin{aligned} u(x) = \frac{1}{\Gamma(m+\nu+1)\,\Gamma(1-\nu)}\Big\{&(-1)^{m+1}\frac{\Gamma(m+\nu+1)}{\Gamma(\nu)}\,\frac{f(c)}{(x-c)^{m+\nu+1}} \\ &+\cdots+(-1)^m\frac{\Gamma(m+\nu)}{\Gamma(\nu)}\,\frac{f'(c)}{(x-c)^{m+\nu}} \\ &+\cdots+\frac{\Gamma(\nu)}{\Gamma(\nu)}\,\frac{f^{(m+1)}(c)}{(x-c)^{\nu}}+\Gamma(1-\nu)\;{}_cD_x^{-(1-\nu)}\,f^{(m+2)}(x)\Big\}. \end{aligned} \tag{22}$$

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[*] *The Tôhoku Math. Journal.* Vol. 8 (1915), pp. 38-41.

[†] *The Tôhoku Math. Journal,* Vol. 10 (1916), pp. 56-59.

[‡] *Proc. Edinburgh Math. Soc.,* Vol. 32 (1914), pp. 19-29; Vol. 33 (1915), pp. 35-41.

[§] Abel's equation is the special case $m = 0$, $-1 < \nu < 0$.

But it is clear that if the value of $u(x)$ given by equation (22) is substituted in (20), the integral will not be convergent unless

$$f(c) = f'(c) = \cdots = f^{(m)}(c) = 0.$$

Under these conditions $u(x)$ defined by (22) will be the desired solution. These results can be summarized as follows:

THEOREM 2. *If in equation* (20), $f(x)$ *exists together with its first* $m+2$ *derivatives in the interval* $c \leqq x \leqq x_0$, *and if* $f(c) = f'(c) = \cdots = f^{(m)}(c) = 0$, $f^{(m+1)}(c) \neq 0$, *then the solution of equation* (20) *exists in the open interval* $c < x \leqq x_0$. The solution is given by formula (22).

We can apply these results to equation (19), assuming α to be a positive integer, and obtain the solution with the greatest ease. Since we have ${}_0D_x^{-\frac{1}{2}}\,\alpha(\alpha-1)\,kx^{\alpha-2} = \alpha(\alpha-1)k\,\dfrac{\Gamma(\alpha-1)}{\Gamma(\alpha-\frac{1}{2})}x^{\alpha-3/2}$, it follows that, for $\alpha > 1$,

$$u(x) = k\,\frac{\Gamma(\alpha+1)}{\Gamma(3/2)\,\Gamma(\alpha-\frac{1}{2})}\,x^{\alpha-3/2},$$

and for $\alpha = 1$,

$$u(x) = \frac{k}{\Gamma(3/2)\,\Gamma(\frac{1}{2})}\,\frac{1}{\sqrt{x}} = \frac{2k}{\pi}\,\frac{1}{\sqrt{x}}.$$

5. *Lalesco's Integral Equation.* Lalesco's integral equation is a generalization of the problem that we have just solved.*

Thus let us consider

$$f(x) = \int_c^x \frac{G(x,t)}{(x-t)^\nu}\,u(t)\,dt, \tag{23}$$

where we assume that

$$G(x,t) = g_0(t) + g_1(t)\,(x-t) + \cdots + g_n(t)\,(x-t)^n.$$

It is then apparent that (23) can be written in the form

$$f(x) = \Gamma(1-\nu)\,{}_cD_x^{-(1-\nu)}(g_0u) + \Gamma(2-\nu)\,{}_cD_x^{-(2-\nu)}(g_1u) \tag{24}$$
$$+ \Gamma(3-\nu)\,{}_cD_x^{-(3-\nu)}(g_2u) + \cdots + \Gamma(n+1-\nu)\,{}_cD_x^{-(n+1-\nu)}(g_nu).$$

We can now operate on both sides of equation (24) with the operator ${}_cD_x^{n+1-\nu}$ and obtain the following ordinary differential equation of nth order:

$$\Gamma(1-\nu)\,d^n/dx^n\,(g_0u) + \Gamma(2-\nu)\,d^{n-1}/dx^{n-1}\,(g_1u) + \cdots$$
$$+ \Gamma(n+1-\nu)\,g_nu = {}_cD_x^{n+1-\nu}f. \tag{25}$$

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* T. Lalesco, "Sur l'équation de Volterra," *Journal de Mathématique*, Vol. 73 (1908), pp. 125-202. In particular § 9 of part I.

Since equation (25) is a linear non-homogeneous one, its solution will be given by the method of variation of parameters in the form

$$u(x) = \sum_{i=1}^{n} c_i u_i(x) + \int_c^x W(x, t)\ {}_cD_t^{n+1-\nu} f(t)\, dt, \tag{26}$$

where $u_1(x), \cdots, u_n(x)$ are n linearly independent solutions of the homogeneous equation, $W(x, t)$ is the Cauchy function, and the c_i are arbitrary constants.*

Returning to equation (24) let us operate on both sides with ${}_cD_x^{-\nu}$. We shall then have

$$\begin{gathered} {}_cD_x^{1-\nu} f(x) = \Gamma(1-\nu)\, g_0(x) u(x) + \Gamma(2-\nu)\ {}_cD_x^{-1}(g_1 u) + \\ \Gamma(3-\nu)\ {}_cD_x^{-2}(g_2 u) + \cdots + \Gamma(n+1-\nu)\ {}_cD_x^{-n}(g_n u). \end{gathered}$$

Then if ${}_cD_c^{1-\nu} f(x) = 0$, we must have

$$g_0(c)\, u(c) = 0.$$

Operating again with ${}_cD_x^1 = d/dx$ we get in similar fashion

$$[g_0(x)\, u(x)]'_{x=c} + (1-\nu)\, g_1(c)\, u(c) = 0.$$

Repeating this n times we have as boundary conditions the system:

$$\begin{aligned} & g_0(c)\, u(c) = 0, \\ & [g_0(x)\, u(x)]'_{x=c} + (1-\nu)\, g_1(c)\, u(c) = 0, \\ & [g_0(x)\, u(x)]''_{x=c} + (1-\nu)\, [g_1 u]'_{x=c} + (1-\nu)(2-\nu)\, g_2(c)\, u(c) = 0, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & [g_0 u]_{x=c}^{(n-1)} + (1-\nu)[g_1 u]_{x=c}^{(n-2)} + \cdots + (1-\nu) \cdots \\ & \qquad\qquad (n-1-\nu)\, g_{n-1}(c)\, u(c) = 0. \end{aligned} \tag{27}$$

The determinant of this linear system in $u(c); u'(c), \cdots, u^{(n-1)}(c)$ is seen to be equal to $[g_0(c)]^n$. Hence if $g_0(c) \neq 0$, we shall have $u(c) =$

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* An important example is found in Whittaker's treatment of Abel's integral equation: "On the Numerical Solution of Integral Equations," *Proc. Royal Soc. of London*, Vol. 94(A) (1917-18), pp. 367-383. In Whittaker's paper the $g_i(t)$ are constants, a_i. If the roots of $F(z) \equiv a_0 z^n + (1-a) a_1 z^{n-1} + (1-a)(2-a) z^{n-2} + \ldots + (1-a)(2-a) \ldots (n-a) a_n = 0$, are $A, B, \ldots, N$, the Cauchy function will be

$$W(x, t) = \frac{e^{A(x-t)}}{F'(A)} + \frac{e^{B(x-t)}}{F'(B)} + \cdots + \frac{e^{N(x-t)}}{F'(N)}.$$

By simple considerations Whittaker's form of solution is then obtained from (26).

$u'(c) = \cdots = u^{(n-1)}(c) = 0$, and it follows that in (26) we get after n successive differentiations

$$\sum_{i=1}^{n} c_i u_i^{(j)}(c) = 0, \qquad (j = 0, 1, \cdots, n-1).$$

Since the determinant of this system is the Wronskian, $W = (u_1, u'_2, \cdots, u_n^{(n-1)})$, which is different from zero for $x = c$, provided $g_0(c) \neq 0$, it follows that $c_1 = c_2 = \cdots = c_n = 0$. *In this case the solution of* (23) *is unique.*

We next turn to the problem of solving (23) when $g_0(x) = g_1(x) = \cdots = g_{r-1}(x) \equiv 0$, $g_r(c) \neq 0$. In this case we replace equation (24) by the following:

$$\begin{aligned} f(x) = {} & \Gamma(r+1-\nu)\, {}_cD_x^{-(r+1-\nu)}[g_r(x)\, u(x)] \\ & + \Gamma(r+2-\nu)\, {}_cD_x^{-(r+2-\nu)}[g_{r+1}(x)\, u(x)] \\ & + \cdots + \Gamma(n+1-\nu)\, {}_cD_x^{-(n+1-\nu)}[g_n(x)\, u(x)], \end{aligned}$$

and equations (27) by the system:

$$\begin{aligned} & g_r(c)\, u(c) = 0, \\ & [g_r(x)\, u(x)]'_{x=c} + (r+1-\nu) g_{r+1}(c)\, u(c) = 0, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & [g_r(x)\, u(x)]_{x=c}^{(n-r-1)} + (r+1-\nu)[g_{r+1}u]_{x=c}^{(n-r-2)} \\ & \qquad + \cdots + (r+1-\nu) \cdots (n-1-\nu) \\ & \qquad\qquad g_{n-1}(c)\, u(c) = 0. \end{aligned}$$

By reasoning similar to that in the first case it is clear that equation (23) will have a unique solution if $g_r(c) \neq 0$.

It is next of interest to consider the case where

$$g_0(x) = (x-c)^n G_0(x), \qquad g_1(x) = (x-c)^{n-1} G_1(x), \cdots,$$
$$g_{n-1}(x) = (x-c)\, G_{n-1}(x).$$

It is further assumed that $G_0(c), G_1(c), \cdots, G_{n-1}(c), g_n(c)$ are all different from zero.

Then system (27) is satisfied for any value of $u(c), u'(c), \cdots, u^{(n-1)}(c)$ whatsoever. The solution of (23) is therefore the general one (26) with n arbitrary constants.

Two examples follow:

Example 1. $f(x) = \int_0^x \frac{A_1 t\, u(t)}{(x-t)^{\frac{1}{2}}}\, dt + \int_0^x A_2 (x-t)^{\frac{1}{2}} u(t)\, dt.$

This can be reduced to

$$A_1 \Gamma(\tfrac{1}{2})\, d/dx\, (xu) + A_2 \Gamma(3/2) u = {}_0D_x^{3/2} f(x).$$

From $A_1\, d/dx\, (x\bar{u}) + \frac{1}{2}A_2\bar{u} = 0$, we get $\bar{u}(x) = k\, t^c$ where $c = -(A_1 + \frac{1}{2}A_2)/A_1$. Hence we have

$$u(x) = k\, x^c + 1/A_1 \int_0^x (x/t)^c\, {}_0D_t^{3/2} f(t)\, dt,$$

provided the integral exists and $c > -1$.

Example 2. Consider the homogeneous equation

$$A_1 \int_0^x \frac{t^2}{(x-t)^\nu}\, u(t)\, dt + A_2 \int_0^x t\, (x-t)^{1-\nu} u(t)\, dt + A_3 \int_0^x (x-t)^{2-\nu} u(t)\, dt = 0,$$

which reduces to

$$a_0 x^2 u''(x) + a_1 x u'(x) + a_2 u(x) = 0$$

where $a_0 = A_1$, $a_1 = 4A_1 + A_2(1-\nu)$, $a_2 = 2A_1 + A_2(1-\nu) + A_3(2-\nu)(1-\nu)$.

The solution will be

$$u(x) = c_1 x^{r_1} + c_2 x^{r_2},$$

where r_1 and r_2 are roots of the equation

$$a_0 r(r-1) + a_1 r + a_2 = 0.$$

It is easily shown that this function satisfies the original equation provided r_1 and r_2 are greater than -1.

The case of equal roots presents more difficulties. Then the solution will be $u(x) = x^r \log x$ where r is a root of the two equations

$$a_0 r(r-1) + a_1 r + a_2 = 0,$$
$$a_0 r + a_0(r-1) + a_1 = 0.$$

To verify that $u(x)$ is indeed a solution consider the integral

$$I(x) = \int_0^x t^\lambda \log t\, (x-t)^\mu\, dt.$$

Making the transformation $t = xs$, we shall have

$$I(x) = \int_0^1 x^{\lambda+1} s^\lambda (\log x + \log s)\, x^\mu (1-s)^\mu\, ds =$$

$$x^{\lambda+\mu+1} \left\{ \log x \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)} + \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)} \left[\frac{\Gamma'(\lambda+1)}{\Gamma(\lambda+1)} - \frac{\Gamma'(\lambda+\mu+2)}{\Gamma(\lambda+\mu+2)}\right] \right\}$$

$$= x^{\lambda+\mu+1} \{\log x\, B(\lambda+1, \mu+1) + B(\lambda+1, \mu+1) [\psi(\lambda+1) - \psi(\lambda+\mu+2)]\}.$$

Substituting $u(x) = x^r \log x$ in the original equation and noting that $A_1 B(r+3,\, 1-\nu) + A_2 B(r+2,\, 2-\nu) + A_3 B(r+1,\, 3-\nu) = 0$, we get for the left hand side the expression

$$\begin{aligned} &A_1 B(r+3,\, 1-\nu)\,[\psi(r+3) - \psi(r+4-\nu)] \\ &\qquad + A_2 B(r+2,\, 2-\nu)\,[\psi(r+2) - \psi(r+4-\nu)] \\ &\qquad\qquad\qquad A_3 B(r+1,\, 3-\nu)\,[\psi(r+1) - \psi(r+4-\nu)], \end{aligned}$$

which we must show is identically zero.

If we make use of the fact that $\psi(r+2) = \psi(r+1) + 1/(r+1)$, $\psi(r+3) = \psi(r+1) + 1/(r+1) + 1/(r+2)$, it will be seen that the above expression reduces to

$$A_1(r+2) + A_1(r+1) + A_2(\nu - 1).$$

This is readily seen to be equivalent to $a_0 r + a_0(r-1) + a_1$, which is equal to zero.

6. *Sbrana's Integral Equation.* Another type of equation which yields easily to the calculus of fractional operators is that of F. Sbrana who, however, used other methods in his discussion.*

Sbrana's equation is of the form

$$f(x) = \int_0^x u(t)\, K(x-t)\, dt, \tag{28}$$

where
$$K(z) = \int_0^z \frac{e^{cu}}{u^a} \frac{e^{c_1(z-u)}}{(z-u)^a}\, du, \qquad 0 < a < 1.$$

If we let $z = v - t$, and $u = x - t$, we shall have

$$K(x-t) = \int_t^x \frac{e^{c(v-t)}}{(v-t)^a} \frac{e^{c_1(x-v)}}{(x-v)^a}\, dv.$$

If this is substituted in equation (28) and the order of integration inverted by Dirichlet's formula, we shall have

$$e^{c_1 x} \int_0^x \frac{e^{(c-c_1)v}}{(x-v)^a}\, dv \int_0^v \frac{e^{-ct} u(t)}{(v-t)^a}\, dt = f(x).$$

This equation can be written in the form

$${}_0D_x^{a-1} \{ e^{(c-c_1)x}\, {}_0D_x^{a-1} [e^{-cx} u(x)] \} = \frac{f(x)\, e^{-c_1 x}}{\Gamma^2(1-a)} \,.$$

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* "Sopra certi equazioni integrali di Volterra, risolubili con procedimenti finiti," *Atti dei Lincei*, Vol. 32 (1) (1923), pp. 156-157.

By successive integrations we obtain the solution

$$u(x) = \frac{e^{cx}}{\Gamma^2(1-a)}\, {}_0D_x{}^{1-a}\,\{e^{(c_1-c)x}\, {}_0D_x{}^{1-a}\,[f(x)\,e^{-c_1x}]\}.$$

As a special case the kernel can be a Bessel's function since

$$J_0(z) = 1/\pi \int_0^z \frac{e^{-i(z-2u)}}{\sqrt{u}\ \sqrt{z-u}}\ du.^*$$

It is easily proved that similar treatment can be made if the kernel is of the general form

$$K(z) = \int_0^z \frac{e^{c_n(z-u_{n-1})}}{(z-u_{n-1})^a}\ du_{n-1} \int_0^{u_{n-1}} \frac{e^{c_{n-1}(u_{n-1}-u_{n-2})}}{(u_{n-1}-u_{n-2})^a}\ du_{n-2} \cdots \int_0^{u_1} \frac{e^{cu}\,e^{c_1(u_1-u)}}{u^a\,(u_1-u)^{a_1}}\,du.$$

7. *Applications in Electrical Theory—Heaviside's Calculus.* The following applications to the flow of electricity in a cable are considered both by Heaviside and Carson although the methods which they employ do not explicitly make use of Riemann operators.

Consider the case of the flow of electricity in a non-inductive cable with distributed resistance R and capacity C per unit length subject to an impressed voltage $V_0(t)$ at the point $x = 0$.

The differential equations of the cable are †

$$RI = -(\partial/\partial x)\,V,$$
$$C\,(\partial/\partial t)\,V = -(\partial/\partial x)\,I,$$

where x is the distance measured along the cable from a fixed point $(x = 0)$, I is the current at the point x, and V the corresponding potential.

Using a method that is familiar in the theory of linear partial differential equations,‡ we can replace $\partial/\partial t$ by z and by elimination of I obtain the equation

$$z\,R\,C\,V = \partial^2V/\partial x^2.$$

Symbolically we shall then have

$$V(x,t) = e^{-ax}\,V_1(t) + e^{ax}\,V_2(t),$$

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* Riemann-Weber, *Partiellen Differentialgleichungen,* (Braunschweig), (1900), Vol. 1, § 73; (1919), § 75.

† Starling, *Electricity and Magnetism* (1916), p. 449; Jeans, *Mathematical Theory of Electricity and Magnetism* (1915), p. 333.

‡ Forsyth, *Differential Equations,* 5th ed. (1921), pp. 522-23.

where $a = \sqrt{z R C}$ and $V_1(t)$ and $V_2(t)$ are arbitrary functions.

By differentiation we get

$$I(x, t) = \sqrt{z C/R}\ (e^{-ax} V_1 - e^{ax} V_2).$$

Assuming that the cable is infinitely long so that the reflected wave is absent we may set $V_2 \equiv 0$ and have as the symbolical solution of our problem

$$I = \sqrt{z C/R}\ e^{-ax} V_1(t),$$

where $V_1(t)$ is to be determined from the boundary condition. Since this asserts that $V(x, t) \mid_{x=0} = V_0(t)$, it is clear that $V_1(t) \equiv V_0(t)$. Hence the complete solution of the problem is found in the expansion:

$$\begin{aligned} I(x, t) &= \sqrt{z C/R}\ e^{-ax} V_0(t), \\ &= [\sqrt{C/R}\ \{z^{\frac{1}{2}} + \frac{(Ax)^2}{2!} z^{3/2} + \frac{(Ax)^4}{4!} z^{5/2} + \cdots\} \\ &\quad - \sqrt{C/R}\ \{z A x + \frac{(Ax)^3}{3!} z^2 + \frac{(Ax)^5}{5!} z^3 + \cdots\}]\ V_0(t), \end{aligned}$$

where for brevity we have written $A = \sqrt{R C}$. Therefore we get

$$\begin{aligned} x, t) = \sqrt{C/R\pi}\ &\left\{ \frac{d}{dt} + \frac{(Ax)^2}{2!} \frac{d^2}{dt^2} + \frac{(Ax)^4}{4!} \frac{d^3}{dt^3} + \cdots \right\} \int_0^t \frac{V_0(s)}{(t-s)^{\frac{1}{2}}}\, ds \\ -\sqrt{C/R}\ &\{(Ax) \frac{d}{dt} + \frac{(Ax)^3}{3!} \frac{d^2}{dt^2} + \frac{(Ax)^5}{5!} \frac{d^3}{dt^3} + \cdots\}\ V_0(t). \end{aligned}$$

If $V_0(t) = 1$, that is to say if a unit E. M. F. is impressed at $x = 0$, we get the well-known solution,*

$$I(x, t) = \sqrt{C/R\pi t}\ e^{-\frac{RCx^2}{4t}}.$$

If $V_0(t) = \sin \omega t$, that is to say if an alternating E. M. F. is impressed on the circuit, we have at $x = 0$,

$$\begin{aligned} I(0, t) &= \sqrt{\frac{C}{R\pi}}\ \frac{d}{dt} \int_0^t \frac{\sin \omega s}{(t-s)^{\frac{1}{2}}}\, ds, \\ &= \sqrt{\frac{C}{R\pi}}\ \omega \int_0^t \frac{\cos \omega s}{(t-s)^{\frac{1}{2}}}\, ds. \end{aligned}$$

In establishing the convergence of the series representing $I(x, t)$ in the general case for real values of t greater than 0, it will be sufficient to assume that $V_0(t)$ is analytic in the neighborhood of $t = 0$. Let us first consider $I(x, t)$ as a function of x. If we replace $V_0(s)$ by a series $\sum_{n=0}^{\infty} a_n s^n$ and make the transformation $y = s/(t - s)$, we have

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* Jeans, *loc. cit.*, p. 334.

$$A(t) = \int_0^t \frac{V_0(s)}{(t - s)^{\frac{1}{2}}} ds = \int_0^\infty \sum_{n=0}^{\infty} t^{n+\frac{1}{2}} a_n \frac{y^n}{(1 + y)^{n+\frac{1}{2}}} dy,$$

$$= \sqrt{t} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1)\sqrt{\pi}}{\Gamma(n + 3/2)} a_n t^n,$$

$$= \sqrt{t}\, \phi(t),$$

where $\phi(t)$ is an analytic function.

Hence for values of $t \neq 0$, and for n sufficiently large, we have the inequality

$$\left| \frac{d^n}{dt^n}\{\sqrt{t}\, \phi(t)\} \right| < k^n n!, \tag{29}$$

where k is a constant independent of n. From this it follows, making use of Stirling's approximation for $n!$, that

$$\lim_{n=\infty} \left[\frac{|A x|^{2n-2}}{(2n - 2)!} \left| \frac{d^n}{dt^n}\{\sqrt{t}\, \phi(t)\} \right| \right]^{1/n} = 0.$$

Similarly the nth root of the general term of the second series leads to the same limit and the convergence of $I(x, t)$ regarded as a function of x is established. Regarding it as a function of t, we may use

$$\sum_{n=1}^{\infty} \frac{|A x|^{2n-2}}{(2n + 2)!} n! k^n$$

as a dominating series for the first series and

$$\sum_{n=1}^{\infty} \frac{|A x|^{2n-1}}{(2n - 1)!} n! k^n$$

as a dominating series for the second.

The case of a cable with distributed resistance R and capacity C with distributed leakage G per unit length leads similarly to the differential equations:

$$R I = -\, \partial/\partial x\, V,$$

$$(Cz + G) V = -\, \partial/\partial x\, I.$$

If the cable is infinitely long we shall then have

$$V = e^{-\sqrt{R(Cz+G)}\,x}\ V_0(t) =$$

$$\{1 + \frac{x^2(RB)}{2\,!} + \frac{x^4(RB)^2}{4\,!} + \frac{x^6(RB)^3}{6\,!} + \cdots\} V_0(t)$$

$$-\{x\,(RB)^{\frac{1}{2}} + \frac{x^3(RB)^{3/2}}{3\,!} + \frac{x^5(RB)^{5/2}}{5\,!} + \cdots\} V_0(t),$$

where we have used the abbreviation $B = Cz + G$.

Making use of the expansion of the operator $(d/dx + k)^n$ given in the second section and writing $\lambda = G/C$, this result may be put into the form,

$$V = e^{-\lambda t}\,[1 + \frac{x^2(RC)}{2\,!}\,z + \frac{x^4(RC)^2}{4\,!}\,z^2 + \cdots]\,\{e^{\lambda t}\,V_0(t)\}$$

$$- \frac{e^{-\lambda t}}{\sqrt{\pi}}\,[x\,(RC)^{\frac{1}{2}} z + \frac{x^3(RC)^{3/2}}{3\,!}\,z^2 + \cdots]\int_0^t \frac{e^{-\lambda s}\,V_0(s)}{\sqrt{t-s}}\,ds. \qquad (30)$$

The solution for the case where unit E. M. F. is impressed at $x = 0$ can be obtained with considerable ease by this formula. We first recall that:

$$\int_0^t \frac{e^{-\lambda s}}{\sqrt{t-s}}\,ds = \int_{-\infty}^t \frac{e^{\lambda s}}{\sqrt{t-s}}\,ds - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}\,ds,$$

$$= \sqrt{\pi/\lambda}\ e^{\lambda t} - \int_{-\infty}^0 \frac{e^{\lambda s}}{\sqrt{t-s}}\,ds.$$

Substituting this value for the integral in (30) and taking the indicated derivatives we get

$$V = e^{-x\sqrt{RG}} + \int_{-\infty}^0 \Big[\frac{1}{2}\,\frac{x(RC)^{\frac{1}{2}}}{(t-s)^{3/2}} - \frac{1\cdot 3}{2\cdot 2}\,\frac{x^3(RC)^{3/2}}{3\,!\,(t-s)^{5/2}}$$

$$+ \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\,\frac{x^5(RC)^{5/2}}{5\,!(t-s)^{7/2}} - \cdots\Big]\,\frac{e^{\lambda(s-t)}}{\sqrt{\pi}}\,dt. \qquad (31)$$

Making the transformation $t - s = \xi$, (31) reduces to

$$V = e^{-x\sqrt{RG}} + \frac{\sqrt{RC}}{2\sqrt{\pi}}\int_t^\infty \frac{x}{\xi^{3/2}}\Big[\ 1 - \frac{(RC)x^2}{4\xi}$$

$$+ \frac{(RC)^2x^4}{2\,!(4\xi)^2} - \frac{(RC)^3x^6}{3\,!(4\xi)^3} + \cdots\,]\,e^{-\lambda\xi}\,d\xi$$

$$= e^{-x\sqrt{RG}} + \frac{\sqrt{RC}}{2\sqrt{\pi}}\int_t^\infty \frac{x}{\xi^{3/2}}\,e^{-(RCx^2/4\xi)}\,e^{-\lambda\xi}\,d\xi.$$

That this, together with the value of $I(x, t)$ derived from it, forms a solution of the original differential equations can be verified by direct substitution. The boundary condition is obviously satisfied.

The general case of the inductive cable can be treated similarly by operational methods, but since it does not involve the use of fractional operators the discussion will be omitted.

A third problem employing fractional derivatives is that of determining the voltage at a cable terminal when an E. M. F., $f(t)$, is impressed on a long cable of distributed resistance R and capacity C per unit length through a condenser of capacity C_0. We have just seen that the current entering a cable whose terminal voltage is V is $\sqrt{C/R}\, z\, V$ and the current flowing into the condenser will be $C_0 z \{f(t) - V\}$ since $f(t) - V$ is the voltage across the condenser.

Equating these two values and dividing by z we obtain the following equation for the determination of V:

$$(C/R\ z^{-\frac{1}{2}} + C_0)\ V = C_0 f(t).$$

The solution is easily found since this is a fractional equation of order $-\frac{1}{2}$. From equation (18) we obtain

$$V(t) = f(0)\ e^{(c/RC_0^2)\, t} + \int_0^t \{f'(s) - \sqrt{\frac{C}{R\, C_0^2}}\ {}_0D_s^{\frac{1}{2}} f(s)\}\ e^{(c/RC_0^2)\ (t-s)}\, ds.$$

If $f(x) = 1$, this reduces to

$$V(t) = e^{(c/RC_0^2)\, t} - e^{(c/RC_0^2)\, t} \sqrt{\frac{C}{RC_0^2\, \pi}} \int_0^t \frac{e^{-(c/RC_0^2)\ s}}{\sqrt{s}}\ ds.$$

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Classification of Quadrics in Hyperbolic Space.

BY JAMES PIERPONT.

§ 1. *Introduction.*

In this paper,[*] I wish to present an extremely elementary, though somewhat lengthy method, of classifying quadric surfaces in H_3. Instead of using the usual projective methods of Cayley and Klein, I make use of the more direct methods of Riemann as I have developed in three earlier papers.[†] A brief resumé of this method now follows.

Let ξ, η, ζ be three real numbers, we say they define a point, whose coördinates are these numbers. How the coördinates are chosen is not specified. The metric is defined by

$$ds = \frac{4R^2 d\sigma}{\lambda} \qquad \lambda > 0, \qquad R > 0$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2, \quad \lambda = 4R^2 - r^2$

$$\mu = 4R^2 + r^2, \quad d\sigma = + \sqrt{d\xi^2 + d\eta^2 + d\zeta^2}.$$

H-straights [‡] are defined by

$$\delta \int ds = 0.$$

To perform the variation it is convenient to introduce a new set of coördinates

$$x_1 = \frac{4R^2\xi}{\lambda} \quad x_2 = \frac{4R^2\eta}{\lambda} \quad x_3 = \frac{4R^2\zeta}{\lambda} \qquad x_4 = R\,\mu/\lambda$$

Let us also set

$$[ab] = a_1b_1 + \cdots + a_4b_4 \qquad \{ab\} = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4.$$

We find the coördinates $x_1 \cdots x_4$ satisfy the relation $\{x^2\} = -R^2$.

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[*] Cf. P. Barbarin, "Études de Géométrie Analytique Non-Euclidienne," *Mém. Couronnés, Acad. Roy. Bruxelles*, Vol. 60 (1900-1); J. L. Coolidge, *Trans. Amer. Math. Soc.*, Vol. 4 (1903), p. 161; J. T. Bromwich, *Ibid.*, Vol. 6 (1905), p. 275.

[†] *The Amer. Math. Monthly*, Vol. 30, p. 425; Vol. 31, p. 26 (1923-4); *Proceedings of the Mathematical Congress*, Toronto (1924); *Bull. Amer. Math. Soc.*, 1926. The last two have not yet appeared.

[‡] For H-read hyperbolic, for e- read euclidean.

A relation $[ax]=0$ defines a H-plane; H-straights are the intersection of H-planes and conversely.

In order to have a model of our H-space we will regard ξ, η, ζ as rectangular coördinates of a point in e-geometry as in ordinary analytic geometry. Our H-space in this model is represented by the interior points of the e-sphere $\lambda=0$. H-planes in the model are e-spheres cutting $\lambda=0$ orthogonally, and H-straights are e-circles also cutting $\lambda=0$ orthogonally.

Besides the points so far considered it is necessary to consider the points (ξ, η, ζ) for which $\lambda=0$. We call these *ideal* and define their x coördinates by

$$x_1=\xi, \quad x_2=\eta, \quad x_3=\zeta, \quad x_4=2R;$$

hence for ideal points $\{x^2\}=0$. We must also introduce certain points called imaginary; their coördinates have the form

$$x_1=t_1\cdot i, \quad x_2=t_2\cdot i, \quad x_3=t_4 i, \quad t_4=t_4 i,$$

where $t_1 \cdots t_4$ are real numbers such that $\{t^2\}=R^2$. Hence imaaginary points also satisfy the relation $\{x^2\}=-R^2$. It is understood that when we speak of points we mean in general non-ideal points.

If a is a point real or imaginary, the locus of the points x such that $\{ax\}=0$, is called the absolute polar of a. We say x is conjugate to a. If a is imaginary its polar is real; if a is real its polar is imaginary. All H-straights perpendicular to a real plane pass through its pole.

The four planes $x_1=0 \cdots x_4=0$ form a tetrahedron, the τ tetrahedron. Each face $x_k=0$ is the polar of the opposite vertex which we call A_k.

Let $a\cdot b$ denote the H-straight joining the points a, b. If x is a point on this straight we denote the length of the segment a, x in the sense a to x by (a, x) etc. Then

$$x_k=\frac{\sinh (x,b)/R}{\sinh (a,b)/R}\cdot a_k+\frac{\sinh (a,x)/R}{\sinh (a,b)/R}\cdot b_k$$

$$=ma_k+nb_k \qquad k=1\cdots 4.$$

The cross ratio of four points (a, b, x, x') is $n/m : n'/m'$; it is harmonic when $=-1$. It is important to observe that a bisects the segment x, x', when (a, b, x, x') is harmonic and b is conjugate to a.

The linear transformation

$$x_i = \sum_j a_{ij} x'_j \qquad i, j = 1 \cdots 4, \qquad a_{ij} \text{ real}$$

defines an H-displacement or a motion if $ds = ds'$ and $\{x^2\} = \{x'^2\}$. In particular any e-rotation about A_4 in our model is an H-displacement. The transformation

$$x_1 = x_1', \quad x_2 = x_2', \quad x_3 = x_3' \cosh\theta - x'_4 \sinh\theta,$$
$$x_4 = -x_3' \sinh\theta + x_4' \cosh\theta$$

defines a rotation about the edge $A_1 \cdot A_2$ of the τ tetrahedron.

Finally the angle θ between two curves meeting at a point $(\xi\eta\zeta)$ is given by

$$\cos\theta = \frac{16R^4}{\lambda^2}\left(\frac{d\xi}{ds}\,\frac{d\xi'}{ds'} + \frac{d\eta}{ds}\,\frac{d\eta'}{ds'} + \frac{d\zeta}{ds}\,\frac{d\zeta'}{ds'}\right)$$
$$= \left[\frac{dx}{ds}\,\frac{dx'}{ds'}\right]$$

Hence this angle in H-measure is the same as the angle in e-measure between the corresponding curves in the model.

§ 2. *Classification.*

Any quadratic relation between $x_1 \cdots x_4$ defines a quadric surface. Here we consider only surfaces whose equations are homogeneous, viz.:

$$F = \sum a_{ij} x_i x_j = 0, \qquad a_{ij} = a_{ji}, \qquad i, j = 1 \cdots 4.$$

The a's are real numbers and the determinant $| a_{ij} | \neq 0$. We call the quadric $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$ the absolute, it remains invariant under a displacement.

If $a = (a_1 \cdots a_4)$ is a point we call

$$a_1 F_1(x) + \cdots + a_4 F_4(x) = 0, \qquad F_i = \tfrac{1}{2}\,\frac{\partial F}{\partial x_i}$$

the polar of a relative to F. Thus the plane $\{ax\} = 0$ which we called the absolute polar of a, is the polar of a relative to the absolute.

Suppose now the point g is such that its polar relative to F coincides with its absolute polar. If g is real, every chord of F through g is bisected by g; that is g is a center of F. If g is an imaginary point, its polar is a plane of symmetry, i. e. all chords perpendicular to this plane are bisected by it. Such g points must satisfy the equations:

$$\begin{aligned} (a_{11} - k) g_1 + a_{12} g_2 + a_{13} g_3 + a_{14} g_4 &= 0 \\ a_{21} g_1 + (a_{22} - k) g_2 + a_{23} g_3 + a_{24} g_4 &= 0 \\ a_{31} g_1 + a_{32} g_2 + (a_{33} - k) + a_{34} g_4 &= 0 \\ a_{41} g_1 + a_{41} g_2 + a_{43} g_3 + (a_{44} + k) g_4 &= 0. \end{aligned} \tag{1}$$

We call these the g-equations. For these to hold it is necessary that k satisfies the equation

$$H = \begin{vmatrix} a_{11} - k & \cdots & a_{14} \\ \cdot & \cdots & \cdot \\ a_{41} & \cdots & a_{44} + k \end{vmatrix} = 0 \tag{2}$$

which we call the H-equation; its roots we denote by a, b, c, d.

If we subject the surface F to a displacement the new surface will have an equation

$$\bar{F} = \sum a'_{ij} x_i x_j = 0$$

whose H-equation will have the same roots as (2). By this displacement the surface F, the point g and its polar will behave as a rigid body. If g is real, imaginary or ideal, it remains such after displacement. This being so, we have only to consider the various possibilities that may arise and in each such case move the surface relative to its center, plane or axis of symmetry in order that its equation takes on a simple form.* The following theorems readily proved except the last, simplify the work:

THEOREM 1. *To two different roots of the H-equation do not correspond the same g.*

THEOREM 2. *If g_a, g_b correspond to two different roots a, b of the H-equation then* $\{g_a, g_b\} = 0$.

THEOREM 3. *Moreover both g_a, g_b cannot be ideal.*

THEOREM 4. *If g_a is real, no g_b can be ideal.*

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* Dr. P. D. Schwartz in a paper given at the May meeting of the American Mathematical Society has shown how conics in H_2 may be classified by this method. Cf. *Bull. Amer. Math. Soc.*, Vol. 32 (1926), p. 315.

THEOREM 5. *The H-equation has at least two real roots.*[*]

I find now the following types of quadric surfaces, the numbers on the right are the types given by Bromwich.

No g necessarily ideal. Roots of H all real.

$\epsilon^2 = 1.$ η, arbitrary.

Roots unequal.	$(k-a)(k-b)(k-c)(k-d) = 0$	
(1) g_a real.	$bx_1^2 + cx_2^2 + dx_3^2 - ax_4^2 = 0$	I
One pair of equal roots	$(k-a)(k-b)(k-c)^2 = 0$	
(2) g_c real.	$ax_1^2 + bx_2^2 + c(x_3^2 - x_4^2) = 0$	Ib2
(3) g_c imag., g_a real.	$c(x_1^2 + x_3^2) + bx_2^2 - ax_4^2 = 0$	Ib1
Two pairs of equal roots	$(k-a)^2(k-b)^2 = 0$	
(4) g_a real.	$b(x_1^2 + x_3^2) + a(x_2^2 - x_4^2) = 0$ or $b(x_1^2 + x_2^2) + a(x_3^2 - x_4^2) = 0$	Id
(5) g_a, g_b imag.	$a(x_1^2 - x_4^2) + b(x_2^2 + x_3^2) = 0$	
Triple Root	$(k-a)^3(k-b) = 0.$	
(6) g_b real.	$a(x_1^2 + x_2^2 + x_3^2) - bx_4^2 = 0$	Ic1
(7) g_a real.	$bx_1^2 + a(x_2^2 + x_3^2 - x_4^2) = 0$	Ic2
(8) g_a, g_b imag.	$ax_1^2 + bx_2^2 + 2ax_3^2 + 2\epsilon a x_3 x_4 = 0$	IIIb
Quadruple Root	$(k-a)^4 = 0$	
(9) g_a real	$a(x_1^2 + x_2^2 + x_3^2 - x_4^2) = 0$	Ie
(10) g_a necessarily imag.	$a(x_1^2 + x_2^2 + 2x_3^2 + 2\epsilon x_3 x_4) = 0$	IIIc
(11) g_a imag. or ideal.	$a(x_1^2 + x_2^2) + 2ax_3^2 + \eta(x_2x_3 - x_2x_4) - 2ax_3x_4 = 0$	IVb

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[*] T. J. Bromwich, *Quadratic Forms.* Nr. 3 of the *Cambridge Tracts in Mathematics,* Cambridge, 1906, p. 69. The proof as given by Bromwich rests on the theory of elementary divisors. A very simple proof of a generalization of this theorem which I have found will be given in another paper.

A pair of conjugate imaginary roots.

Two real unequal roots $(k-a)(k-b)(k-a-i\beta)\ (k-a+i\beta)=0,\ \beta\neq0$

(12) g_a, g_b imag. $ax_1^2+bx_2^2+2ax_3^2+2\epsilon\sqrt{a^2+\beta^2}\cdot x_3x_4=0$ IIa

(13) $a=b$. g_a imag. $a(x_1^2+x_2^2)+2ax_3^2+2\epsilon\sqrt{a^2+\beta^2}\cdot x_3x_4=0$ IIb

One g necessarily ideal. Roots all real.

One pair of equal roots $(k-a)(k-b)(k-c)^2=0$

(14) g_c ideal, g_a, g_b imag. $ax_1^2+bx_2^2+2cx_3^2+2\epsilon cx_3x_4=0$ IIIa

Two pairs of equal roots $(k-a)^2(k-b)^2=0$

(15) g_a imag., g_b ideal. $a(x_1^2+x_2^2)+2bx_3^2+2\epsilon bx_3x_4=0$ IIId

Triple root. $(k-a)^3(k-b)=0$

(16) g_a ideal, g_b imag.

$$bx_1^2+ax_2^2+2ax_3^2+\eta(x_2x_3-x_2x_4)-2ax_3x_4=0 \qquad \text{IVa}$$

We note that Bromwich has no case corresponding to (5).

We now illustrate on a few cases how the reduction is effected.

Type 6. We move F so that g_a coincides with $A_4=(0,\ 0,\ 0,\ R)$. The g-equations give

$$a_{41}=a_{42}=a_{43}=0, \qquad a_{44}=-a. \tag{3}$$

We rotate F about A_4 till g_b coincides with $A_1=(iR,0,0,0)$. Then the new coefficients satisfy (3) and in addition the g-equations give

$$a_{11}=b, \quad a_{21}=a_{31}=a_{41}=0.$$ F now has the form

$$bx_1^2+a_{22}x_2^2+a_{33}x_3^2-ax_4^2+2a_{23}x_2x_3=0.$$

It will not confuse the reader if we omit from the a_{ij} here and in the following the accents. We now rotate F about the edge $A_1\cdot A_4$ of the τ tetrahedron through the angle θ determined by

$$\tan\theta=\frac{a_{23}}{a_{33}-a_{22}}$$

when $a_{22}\neq a_{33}$, in the latter case thru $\theta=45°$. Then F has the form

$$bx_1^2+a_{22}x_2^2+a_{33}x_3^2-ax_4^2=0,$$

whose H-equation is $(k-a)(k-b)(k-a_{22})(k-a_{33})=0$. By hypothesis $H=(k-a)^2(k-b)^2$; hence $a_{22}=a$, $a_{33}=b$ or $a_{22}=b$, $a_{33}=a$. We thus get type 6.

Type 7. We move F so that g_a becomes $A_1=(iR, 0, 0, 0)$. The g-equations give $a_{21}=a_{31}=a_{41}=0$, $a_{11}=a$. We rotate F about A_1 till g_b coincides with $A_2=(0, iR, 0, 0)$. Then $a_{12}=a_{32}=a_{42}=0$. $a_{22}=b$. It is easy to show that $|2a_{34}|<|a_{33}+a_{44}|$. We can therefore rotate F about the edge $A_1 \cdot A_2$ so that $a_{34}=0$ if it is not already 0. F now has the form

$$ax_1^2+bx_2^2+a_{33}x_3^2+a_{44}x_4^2=0,$$

whose H-equation is $(k-a)(k-b)(k-a_{33})(k+a_{44})=0$. Hence either

$$(1^\circ) \quad a_{33}=a, \quad a_{44}=-b; \quad \text{or} \quad (2^\circ) \quad a_{33}=b, \quad a_{44}=-a.$$

The g-equations in case 1° show that g_b may be taken real, which is contrary to hypothesis. Hence only 2° exists and this is type 7.

Type 10. We move F so that g_a coincides with $A_1=(iR, 0, 0, 0)$. The g-equations give $a_{11}=a$, $a_{21}=a_{31}=a_{41}=0$. The plane $x_1=0$ cuts F in the conic Φ whose g-equations are obtained by deleting the first row and column of (1). The H-equation of this system is $H_\Phi=(k-a)^3=0$.

Suppose now that the g-equations of Φ admitted a solution for $k=a$, to which corresponds a real non-ideal point $\gamma=(g_2, g_3, g_4)$, then $g'=(0, g_2, g_3, g_4)$ is a solution of the g-equations of F for $k=a$, and F belongs to type 9. We will suppose therefore that the g-equations of Φ admit only an imaginary γ, then g' is imaginary. We rotate F about A_1 till g' coincides with $A_2=(0, iR, 0, 0)$, then the g-equations give $a_{12}=a_{32}=a_{42}=0$, $a_{22}=0$ and F has the form

$$ax_1^2+ax_2^2+\alpha x_3^2+\beta x_4^2+2\gamma x_3x_4=0$$

whose H-equation is

$$(k-a)^2\{k^2-(\alpha-\beta)k+\gamma^2-\alpha\beta\}=(k-a)^4=0.$$

Comparing coefficients gives $\alpha-\beta=2a$, $\gamma-\alpha\beta=a^2$; hence $\alpha=a+\epsilon\gamma$, $\beta=\epsilon\gamma-a$, $\epsilon^2=1$, and F has the form

$$a(x_1^2+x_2^2)+(\alpha+\epsilon\gamma)x_3^2-(a-\epsilon\gamma)x_4^2+2\gamma x_3x_4=0.$$

This surface cuts the edge $x_1 = 0$, $x_2 = 0$ in two points given by

$$(a + \epsilon\gamma)x_3^2 - (a - \epsilon\gamma)x_4^2 + 2\gamma x_3 x_4 = 0,$$

one of which is $(0, 0, 2\epsilon R, 2R)$. If the other is P, we rotate F about $A_1 \cdot A_2$ so that P coincides with $A_4 = (0, 0, 0, R)$ when $a = \epsilon\gamma$. F now has the type (10)

As further illustration of our method let us investigate the cases which arise when

$$H = (k - a)^4 = 0 \qquad \textit{and } g \textit{ is necessarily ideal.} \tag{C}$$

We begin by rotating F about A_4 so that g_a coincides with $L = (0, 0, 2R, 2R)$. The g-equations give now $a_{14} = -a_{13}$, $a_{24} = -a_{23}$, $a_{34} = a - a_{33}$, $a_{44} = a_{33} - 2a$. The corresponding H-equation is

$$(k - a)^2 \begin{vmatrix} a_{11} - k & a_{12} \\ a_{12} & a_{22} - k \end{vmatrix} = (k - a)^4 = 0.$$

Comparing coefficients gives $a_{11} = a_{22} = a$, $a_{33} = \eta$, $a_{13} = \beta$, $a_{23} = \gamma$, hence F has the form

$$a(x_1^2 + x_2^2) + \eta x_3^2 + (\eta - 2a)x_4^2 + 2\beta(x_1x_3 - x_1x_4)$$
$$+ 2\gamma(x_2x_3 - x_2x_4) + 2(a - \eta)x_3x_4 = 0 \tag{4}$$

This cuts the edge $x_1 = x_2 = 0$ in L and in another point determined by

$$\eta x_3^2 + (\eta - 2a)x_4^2 + 2(a - \eta)x_3x_4 = 0.$$

Suppose $(C, 1)$ that $P \neq M = (0, 0, -2R, 2R)$. We rotate F about $A_1 \cdot A_2$ so that P coincides with $A_4 = (0, 0, 0, R)$. These coördinates set in (4) give $\eta = 2a$, and F reduces to the form

$$a(x_1^2 + x_2^2) + 2ax_3^2 + 2\beta(x_1x_3 - x_1x_4) + 2\gamma(x_2x_3 - x_2x_4) - 2ax_3x_4 = 0$$

whose first two g-equations for $k = a$ give

$$\beta(g_3 - g_4) = 0, \qquad \gamma(g_3 - g_4) = 0. \tag{5}$$

Suppose $(C, 1, a)$ that β or γ is $\neq 0$, then $g_3 = g_4$. The last two g-equations give

$$\beta g_1 + \gamma g_2 = 0. \tag{6}$$

Thus the g-equations for $k = a$ admit the solution

$$g_a = (\iota\rho\gamma, -\iota\rho\beta, 0, 0) \qquad \rho = \frac{R}{\sqrt{\beta^2 + \gamma^2}}$$

This is contrary to our hypothesis (C) that g_a is necessarily ideal; hence case $(C, 1, a)$ does not exist.

Suppose $(C, 1, b)$ *that* $\beta = \gamma = 0$. The g-equations for $k = a$ admit the solution $g_a = (iR, 0, 0, 0) = A_1$ which is also contrary to (C).

Suppose $(C, 2)$ *that* $P = M$. In this case a rotation about $A_1 \cdot A_2$ does not move M and the foregoing does not apply. We may proceed as follows. Setting the coördinates of M in (6) gives $\eta = a$ and F has the form

$$a(x_1^2 + x_2^2) + ax_3^2 - ax_4^2 + 2\beta(x_1x_3 - x_1x_4) + 2\gamma(x_2x_3 - x_2x_4) = 0$$

whose g-equations for $k = a$ give again (5), (6). Hence if β or γ is $\neq 0$ the g-equations are satisfied by $g_a = (\iota\rho\gamma, -\iota\rho\beta, 0, 0)$, ρ as above, i. e. by a non-ideal point contrary to (C). Finally if $\beta = \gamma = 0$, F reduces to type 9 whose g-equations are satisfied by any g, contrary to (C). Thus case (C) does not exist.

In a similar manner all possible cases may be treated.

YALE UNIVERSITY,
October, 1926.

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A Class of Functions Harmonic within the Sphere.*

By H. E. Bray and G. C. Evans.

1. *Introduction.* Consider a function $u(M)$ harmonic at all points M of the interior Σ of the unit sphere S with center O. Let $F(e)$ represent a completely additive function of point sets on the surface S and let P and M represent points on the unit sphere and on the concentric sphere S_r or radius r, respectively. Our purpose is to investigate the situation in which $u(M)$ is given in terms of a Stieltjes integral by the formula

$$\text{(I)} \qquad u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3}\, dF(e),$$

or in terms of the Poisson integral

$$\text{(II)} \qquad u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3}\, f(P)\, dP,$$

of which the former is a generalization.

On account of equation (11) below, and its consequences, we shall deal with limits which will generally be given in terms of functions of curves. For these a simple class of curves will be sufficient, namely, the class of regular curves on the sphere. A regular closed curve is simple and consists of a finite number of regular arcs. Each regular arc has a continuously turning tangent at every interior point; it has a continuously turning forward (backward) tangent at the initial (terminal) point. Since we are dealing with closed curves on a sphere it will be necessary to distinguish between the two regions which they bound, and a curve will be completely specified only by its association with one of these regions as an interior region. This may be done, for example, by assigning a sense to the curve and putting the interior region on the left. A regular closed curve w possesses evidently the property, that, given a point P, the superficial metric density, at P, of the points of the region interior to w—taken with respect to a family of concentric circles about P—is equal to the function

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* Report presented to the American Mathematical Society, April (1923). In the intervening time the authors have been able to simplify the treatment and reduce the conditions. See § 2.2 below.

$$q(P, w) = 1,\ 0,\ \psi/2\pi$$

according as P is inside, outside, or on w; ψ being the interior angle from the forward to the backward tangent at P. It is this property which is of chief importance for our purposes. The quantity $q(P, w)$ regarded as a function of the point P will be seen to be measurable in the Borel sense.

1.1. The most important of the regular curves is a *segment,* consisting of a curvilinear rectangle or polar cap, with reference to the usual angular coordinates on the sphere. We shall use the following notation:

N represents any fixed point of S whatever.

(r, θ, ϕ) represent spherico-polar coordinates of a point with regard to O, with N as north pole, θ as colatitude, ϕ as longitude.

s, or more explicitly $s(r;\ \theta_1, \theta_2;\ \phi_1, \phi_2)$ represents a figure on S_r called a segment. It is determined by the circles

$$\theta = \theta_1,\ \theta = \theta_2,\ \phi = \phi_1,\ \phi = \phi_2,$$

where

$$0 \leq \theta_1 < \theta_2 \leq \pi; \qquad 0 \leq \phi_1 < \phi_2 < 4\pi.$$

The figure is to be regarded as dividing the spherical surface into two regions, interior and exterior, mutually distinct; the interior points being defined as those whose coordinates (θ, ϕ) satisfy the following inequalities (a), (b), (c), (d), (e) according as the values of $\theta_1, \theta_2, \phi_1, \phi_2$, satisfy the respective relations (α), (β), (γ), (δ), (ϵ).

(a) $\theta_1 < \theta < \theta_2;\ \phi_1 < \phi < \phi_2$.	(α) $\theta_1 \neq 0,\ \theta_2 \neq \pi;\ \phi_2 - \phi_1 < 2\pi$.
(b) $\theta_1 < \theta < \theta_2$.	(β) $\theta_1 \neq 0,\ \theta_2 \neq \pi;\ \phi_2 - \phi_1 = 2\pi$.
(c) $\theta < \theta_2$.	(γ) $\theta_1 = 0,\ \theta_2 < \pi;\ \phi_2 - \phi_1 = 2\pi$.
(d) $\theta_1 < \theta$.	(δ) $\theta_1 > 0,\ \theta_2 = \pi;\ \phi_2 - \phi_1 = 2\pi$.
(e) $0 \leq \theta \leq \pi$.	(ϵ) $\theta_1 = 0,\ \theta_2 = \pi;\ \phi_2 - \phi_1 = 2\pi$.

These are the only cases considered in the definition of segments; (a) refers to a spherical rectangle, (b) to a zone, (c) and (d) to polar caps, and (e) to the whole sphere. It is to be noted, therefore, that a triangle or lune with a vertex at N or at its diametrically opposite point is not to be regarded as a segment. Of course, a point (θ, ϕ) on S is not to be regarded as distinct from the point $(\theta, \phi + 2\pi)$.

A *plurisegment* is a finite or denumerable set of segments no two of which have an interior point in common. If p_1 and p_2 are two plurisegments,

p_1 is said to be contained in p_2 if every segment of p_1 is contained in *a finite number* of segments of p_2. The two plurisegments are identical if each is contained in the other.*

1. 2. The basic instrument in the investigation to follow is the Stieltjes integral taken with respect to an additive bounded function of segments. The function $F(s)$ is an additive function of segments s if it is defined for every finite plurisegment and if when p_1 and p_2 are two finite plurisegments which have no interior point in common,

$$F(p_1) + F(p_2) = F(p_1 + p_2),$$

where $p_1 + p_2$ denotes the finite plurisegment consisting of the segments of p_1 and of p_2. The function $F(s)$ is said to be without point values if $F(s)$ becomes infinitely small with the diameter of s.

The positive variation function $P(s)$ for $F(s)$ is defined as the upper bound of $F(p)$ for all finite plurisegments p contained in s. This is again an additive function of segments and not negative; it is without point values if $F(s)$ is without point values.† Similarly, the function $N(s)$, defined as the analogous upper bound of $-F(p)$, is additive and of positive type, without point values if $F(s)$ is without point values. The same applies to the function $T(s)$, upper bound of $\Sigma \mid F(s_i) \mid$ where $p = \Sigma s_i$. In fact

$$F(s) = P(s) - N(s), \quad T(s) = P(s) + N(s).$$

The functions $N(s)$, $T(s)$ will be called respectively the negative and total variation functions of $F(s)$.

The extension of the definition of the function $P(s)$ is unique, if it is defined by means of the additive property for an infinite plurisegment $p = \Sigma s_i$. In fact, suppose $p = \Sigma s_i = \Sigma s'_i$ and $\Sigma P(s_i) \neq \Sigma P(s'_i)$, say

$$\sum P(s_i) = \sum P(s'_i) + 2\epsilon, \quad \epsilon > 0.$$

Then we can find a finite number n of segments s_i of p such that

$$\sum_1^n P(s_i) > \sum_1^\infty P(s'_i) + \epsilon.$$

But since $\sum_1^\infty s_i$ is identical with $\sum_1^\infty s'_i$, each s_i, and therefore $\sum_1^n s_i$, is contained in a finite number of segments s'_i. Hence, for some m,

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* A. J. Maria, "Functions of Plurisegments," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 448-471.

† The proof applies which is given in de la Vallée Poussin, *Intégrales de Lebesque*, Paris (1916), p. 99.

$$\sum_1^n P(s_i) \leq \sum_1^m P(s'_i);$$

which is in contradiction with the previous inequality.

It follows accordingly that the extension of the definition of $F(s)$, as an additive function, to infinite plurisegments p on S is unique:

$$F(p) = \sum_1^\infty F(s_i) \tag{1}$$

and the infinite series in question is absolutely convergent.

On the other hand, if $F(s)$ is a function of segments which is *completely additive,* that is, one for which $F(p)$ is uniquely defined, not infinite, for each plurisegment p on S by means of (1), it follows that $F(s)$ is a bounded additive function of segments, and further that

$$F(p_1 + p_2) = F(p_1) + F(p_2)$$

for infinite as well as finite plurisegments.*

In this résumé of the elementary properties of plurisegments all segments on S (with a given pole N) are considered. But it is important to notice that the theory also applies to certain subclasses of segments, e. g., to those of a *net.* We define a *net* H for a segment as follows. We form first a lattice H_n which is composed of a finite number of segments s'_{in} (with pole N), called meshes, each of diameter $< \delta_n$, and such that $s = \Sigma_i s'_{in}$. The system of lattices H_n corresponding to a set of numbers δ_n with $\lim \delta_n = 0$ constitutes the net H. In particular the net H may be defined for the whole sphere S, and we may thus deal with bounded additive functions $F(s')$ of segments formed from the meshes of the lattice.

1.21. The Riemann Stieltjes integrals.

$$\int_S h(P)\,dF(s_P) = \lim_{\delta=0} \sum_1^k h(P_i)F(s_i) \tag{2}$$

$$\int_S h(P)\,dF(s'_P) = \lim_{\delta=0} \sum_1^k h(P_i)F(s'_i) \tag{3}$$

where $$S = \sum_1^k s_i = \sum_1^k s'_i, \text{ diam } s_i < \delta, \text{ diam } s'_i < \delta,$$

and P_i is a point of s_i or s'_i respectively, are uniquely defined, just as in the case of the usual Stieltjes integral, the function $F(s_P)$ being additive and

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* A. J. Maria, *loc. cit.*, p. 449.

bounded as a function of segments on S (with pole N) and $F(s'_P)$ similarly as a function of segments of a net H.

The integrals (2) and (3) satisfy the properties (C), (A), (L), (M), of Daniell's S-integral.* Hence their definitions may be extended to any function $h(P)$ bounded on the sphere and measurable in the Borel sense, the fundamental interval in terms of which measure is defined being the segment; and if $h(P) = \lim_{k=\infty} h_k(P)$ where the $h_k(P)$ are bounded and have S-integrals, then the S-integral for $h(P)$ is the limit of that for $h_k(P)$. In particular, the function $q(P, w)$, already defined with reference to a regular curve w is measurable in the Borel sense, being obviously the limit of a sequence of continuous functions [see § 2].

If $F(s)$ is additive and bounded as a function of segments on S, the quantity

$$F_1(s) = \int_S q(P, s)\, dF(s_P) \tag{4}$$

is evidently also a bounded and additive function of all segments on S; this is true even if $F(s)$ is defined merely for the segments of a net H. If $F(s)$ satisfies the identity

$$F(s) = \int_S q(P, s)\, dF(s_P)$$

its possible discontinuities, in analogy to the case of a function of a single variable x, will be said to be *regular*.†

THEOREM. *Let $F(s')$ be bounded and additive on a net H and $F_1(s)$ be given by the definition*

$$F_1(s) = \int_S q(P, s)\, dF(s'_P). \tag{5}$$

Then if $h(P)$ is any bounded function, measurable Borel,

$$\int_S h(P)\, dF_1(s_P) = \int_S h(P)\, dF(s'_P) \tag{6}$$

and $F_1(s)$ has regular discontinuities.

To prove this, we observe first that if s is a segment of continuity for $F_1(s)$ then, clearly,

$$\int_S q(P, s)\, dF_1(s_P) = F_1(s) = \int_S q(P, s)\, dF(s'_P).$$

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* P. J. Daniell, "A General Form of Integral," *Annals of Mathematics*, Vol. 19 (1918), pp. 279-294.

† $F(s)$ is said to be continuous on s if it is continuous in the Volterra sense, as a function of curves. For a given $F(s)$ and pole N segments of discontinuity can be formed only from a certain denumerable infinity of meridians and parallels of latitude.

Let $h(P)$ be any continuous function. Consider a net on S all meshes of which are segments of continuity for $F(s)$, and let us choose in each mesh s_{in} of the lattice of order n a point P_{in} $(i = 1, 2, \cdots, k_n)$, where k_n is the number of meshes in the lattice of order n. Then it is clear that the function

$$h_n(P) = \sum_{i=1}^{k_n} h(P_{in})\, q(P, s_{in})$$

will furnish an approximation to $h(P)$ which, in fact, approaches $h(P)$ uniformly as n becomes infinite. And since

$$\int_S h_n(P)\, dF_1(s_P) = \int_S h_n(P)\, dF(s'_P)$$

we have, when n becomes infinite,

$$\int_S h(P)\, dF_1(s_P) = \int_S h(P)\, dF(s'_P).$$

But by the theory of the S-integral, the identity holds also for $h(P)$ bounded and measurable in the Borel sense.

In particular

$$\int_S q(P, s)\, dF_1(s_P) = \int_S q(P, s)\, dF(s'_P),$$

i. e.,

$$F_1(s) = \int_S q(P, s)\, dF_1(s_P).$$

THEOREM. *If $F(s)$ has regular discontinuities, its positive negative and total variation functions have regular discontinuities on any segment.*

Since $F(s) = P(s) - N(s)$, we have

$$F(s) = \int_S q(P, s)\, dF(s_P) = \int_S q(P, s)\, dP(s_P) - \int_S q(P, s)\, dN(s_P)$$

$$\begin{aligned} F(p) &= \int_S q(P, p)\, dP(s_P) - \int_S q(P, p)\, dN(s_P) \\ &\leq \int_S q(P, p)\, dP(s_P) \leq \int_S q(P, s)\, dP(s_P), \end{aligned}$$

for every p in s. Hence

$$P(s) \leq \int_S q(P, s)\, dP(s_P).$$

On the other hand if s_c represents the segment complementary to s with respect to S, we have similarly

$$P(s_c) \leq \int_S q(P, s_c)\, dP(s_P).$$

Hence by addition

$$\begin{aligned} P(S) &= P(s) + P(s_c) \\ &\leq \int_S [q(P, s) + q(P, s_c)]\, dP(s_P) = P(S) \end{aligned}$$

since $q(P,s) + q(P,s_c) \equiv 1$.

But this relation shows that only the equality sign is possible:

$$P(s) = \int_S q(P,s)\,dP(s_P)$$

and similarly

$$N(s) = \int_S q(P,s)\,dN(s_P)$$

and therefore

$$T(s) = \int_S q(P,s)\,dT(s_P).$$

1.22. If $F(s')$ is additive and bounded on a net H, the function

$$F_1(s) = \int_S q(P,s)\,dF(s'_P)$$

has been seen to be defined on all segments, with pole N, and to have regular discontinuities. To $F_1(s)$ there corresponds one and only one completely additive function of point sets $F(e)$ such that the equation

$$F_1(s) = F(e_s)$$

where e_s is the set of points interior to s, holds for every segment of continuity.* This function, in fact, is given for all sets e measurable Borel by the formula

$$F(e) = \int_S \phi(P,e)\,dF_1(s_P) = \int_S \phi(P,e)\,dF(s'_P) \tag{7}$$

where $\phi(P,e)$, the characteristic function for e, is defined by the relations

$$\begin{aligned} \phi(P,e) &= 1, \qquad P \text{ in } e, \\ &= 0, \qquad P \text{ in } Ce, \end{aligned} \tag{8}$$

and is therefore measurable Borel.

The Stieltjes integrals formed with respect to $F_1(s)$ and $F(e)$ are evidently the same for continuous integrands $h(P)$, since the approximating sums can be based on segments of continuity. Hence they are the same also for integrands $h(P)$ bounded and measurable Borel. Hence

$$\int_S h(P)\,dF(e_P) = \int_S h(P)\,dF_1(s_P) = \int_S h(P)\,dF(s'_P), \tag{9}$$

and in particular, for $h(P) = q(P,s)$

$$F_1(s) = \int_S q(P,s)\,dF(e_P). \tag{10}$$

We may speak of functions $F(e)$, $F_1(s)$, $F(s')$, etc. as corresponding if they satisfy (9) for all continuous functions (and therefore for all bounded functions, measurable Borel); given $F(s')$ there is therefore, by setting $h(P) = q(P, s)$ in (9), only one corresponding bounded additive function with regular discontinuities, and it is given by (5).

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* G. C. Evans, "Fundamental Points of Potential Theory," *Rice Institute Pamphlet*, Vol. 7 (1920), pp. 252-329. See p. 268.

1. 3. We shall say that a set of additive functions $F_k(s)$ [or $F_k(e)$ or $F_k(s')$] are of *uniformly limited variation* if their total variation functions are bounded in their set. We shall say that a sequence of bounded additive functions $F_k(s)$ *approaches* $F(s)$ *on the net* H, if $\lim\limits_{k=\infty} F_k(s_{in}) = F(s_{in})$ for every mesh s_{in} of the net.

It is enough to recall now the theorem of Helle-Bray:[*]

If the sequence $F_k(s)$, additive and of uniformly limited variation for all k, approaches $F(s)$, additive and bounded, on the net H of S, then

$$\lim_{k=\infty} \int_S h(P)\, dF_k(s_P) = \int_S h(P)\, dF(s_P), \tag{11}$$

where $h(P)$ is a continuous function of P. Moreover if $h_m(P)$ is continuous, and approaches $h(P)$ uniformly,

$$\lim_{\substack{k=\infty \\ m=\infty}} \int_S h_m(P)\, dF_k(s_P) = \int_S h(P)\, dF(s_P). \tag{12}$$

In particular, since H is a net for any plurisegment p composed of a finite number of meshes of the net, the theorem applies when S is replaced by p.

A similar theorem may evidently be stated in terms of additive functions of point sets, replacing $F_k(s)$ by $F_k(e)$, or $F(s)$ by $F(e)$, or both. If $\lim\limits_{k=\infty} F_k(s)$ exists for every segment s, the limiting function $F(s)$ will necessarily be bounded and additive. The same is true with regard to $\lim\limits_{k=\infty} F_k(e_s)$; the limit will always be a bounded additive function of segments s, but not necessarily a function of point sets.

2. *Necessary and sufficient conditions. Introductory Theorem. Given the function u, harmonic at every interior point of the unit sphere S, a necessary and sufficient condition that there exist on the surface of the sphere a bounded additive function of segments, $F(s)$, such that*

$$u(M) = \frac{1}{4\pi} \int_S \frac{1 - r^2}{\overline{MP}^3}\, dF(s_P) \tag{III}$$

is the following:

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[*] H. E. Bray, "Elementary Properties of the Stieltjes Integral," *Annals of Mathematics*, Vol. 20 (1919), pp. 177-186. See p. 180.

(a) *that there exist a sequence of values of* r, $r_1 < r_2 < r_3 \cdots$, *with* $\lim_{i=\infty} r_i = 1$, *such that for any segment* s, $F(r_i, s)$ *approaches a limiting value as* r_i *approaches* 1;

(b) *that* $F(r_i, s)$ *be of uniformly limited variation as a function of segments for all* i *where*

$$F(r, s) = \int_{\sigma(r,s)} u(M)\, dM,$$

and $\sigma(r, s)$ *represents the portion of* S_r *which is bounded by the projection from* O *of the segment* s *on* S.

The quantity $F(r, s)$ is a bounded additive function of segments s, for a given value of r. In order to prove the necessity of conditions (a) and (b) consider the quantity

$$p(P; r_i, s) = \frac{1}{4\pi} \int_{\sigma(r_i,s)} \frac{1 - r_i^2}{\overline{MP}^3}\, dM,$$

which is continuous in P for every r_i and is bounded as a function of P, s and r. Moreover, as i tends to infinity, it approaches a definite limit for any P. In fact, P may be surrounded by an arbitrarily small circle, and the portion of $\sigma(1, s)$ outside that circle neglected. If the portion of $\sigma(1, s)$ inside the small circle is denoted by σ' and α and θ denote the angles of colatitude and longitude respectively, referred to OP, we have

$$\lim_{i=\infty} p(P; r_i, s) = \lim_{i=\infty} \frac{1}{4\pi} \int_{\sigma'(r_i,s)} \frac{(1 - r_i)^2 r_i^2 \sin\alpha\, d\alpha\, d\theta}{(1 + r_i^2 - 2r_i \cos\alpha)^{3/2}}.$$

From this formula it follows immediately that

$$(13) \qquad \lim_{i=\infty} p(P; r_i, s) = q(P, s) = \begin{cases} 0 & (P \text{ outside } s) \\ 1 & (P \text{ inside } s) \\ \psi/2\pi & (P \text{ on } s), \end{cases}$$

where ψ denotes the angle from the forward to the backward tangent at any point of s.

But

$$\begin{aligned} F(r_i, s) &= \frac{1}{4\pi} \int_{\sigma(r_i,s)} dM \int_S \frac{1 - r_i^2}{\overline{MP}^3}\, dF(s_P) \\ &= \int_S p(P; r_i, s)\, dF(s_P) \end{aligned}$$

and since $p(P; r_i, s)$ is bounded, and $q(P, s)$ is measurable in the Borel sense on S, as a function of P

$$\lim_{i=\infty} F(r_i, s) = \int_S q(P, s)\, dF(s_P) = F_1(s)$$

where $F_1(s)$ is therefore the bounded additive function of segments with regular discontinuities, associated with $F(s)$ and identical with it except on segments for which the latter is discontinuous.

Condition (b) is established directly. In fact if the s_j are segments, we have, from the definition of the Stieltjes integral,

$$\begin{aligned}\sum_j |F(r_i, s_j)| &\leq \frac{1}{4\pi} \int dM \int_S \frac{1 - r_i^2}{\overline{MP}^3}\, dT(s_P) \\ &\leq T(S)\end{aligned}$$

where $T(s)$ is the total variation function of $F(s)$.

2. 1. To show that the conditions (a), (b) are sufficient, consider now an arbitrary point M_0, fixed, interior to the spherical region bounded by S, and express $u(M_0)$, u being harmonic inside S, by means of the Poisson integral over S_i, the sphere of radius r_i; sufficiently large to contain M_0 as an interior point. Thus

$$\begin{aligned}(14)\qquad u(M_0) &= \frac{1}{4\pi r_i} \int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, u(M)\, dM \\ &= \frac{1}{4\pi r_i} \int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, dF(r_i, s_M) \\ &= \lim_{r_i=1} \frac{1}{4\pi r_i} \int_{S_i} \frac{r_i^2 - r_0^2}{\overline{M_0M}^3}\, dF(r_i, s_M) \\ &= \frac{1}{4\pi} \int_S \frac{1 - r_0^2}{\overline{M_0P}^3}\, dF_1(s_P)\end{aligned}$$

by (12), since $F_1(s)$ is of limited variation.

Corollary. The conditions given are necessary and sufficient that $u(M)$ be given by (I), since the integrals (I) and (III) are identical when $F(s)$ and $F(e)$ are corresponding functions of segments and point sets respectively.

Corollary. For $u(M)$ given by (I) or (III) the conditions (a), (b) hold as r approaches 1 in any way whatever.

2. 2. *Removal of condition* (*a*). We shall now prove that condition (b) implies condition (a), from which it will follow that (b) is a necessary and sufficient condition in order that $u(M)$ be given by (III). With the notation already defined for s, we have

$$F(r, s) = F(r;\, \theta_1, \theta_2;\, \phi_1, \phi_2) = \int_{\sigma(r,s)} u(r, \theta, \phi)\, d\sigma$$

$$T(r, s) = T(r;\, \theta_1, \theta_2;\, \phi_1, \phi_2) = \int_{\sigma(r,s)} |u(r, \theta, \phi)|\, d\sigma < K.$$

$F(r,s)$, $T(r,s)$ are therefore additive functions of segments s and functions of the parameter r; of uniformly limited variation in s as r ranges over the sequence $r_1 < r_2 < r_3 \cdots$. We wish to prove the following auxiliary theorem:

THEOREM. *If $F(r_i, s)$ is of uniformly limited variation for all values of i, then there exists a subsequence of values of r, $\{r_i'\}$, of the sequence $\{r_i\}$ and there exist two sets $E'(\theta)$, $E''(\phi)$ dense in the respective intervals $(0,\pi)$, $(0,2\pi)$ such that*

$$\lim_{i=\infty} F(r'_i;\, \theta_1, \theta_2;\, \phi_1, \phi_2)$$

exists provided only that θ_1, θ_2, belong to $E'(\theta)$ and ϕ_1, ϕ_2 belong to $E''(\phi)$.

Suppose that $r' < r < 1$. Then, from Poisson's integral,

$$F(r', s) = \frac{1}{4\pi r}\int_{S_r} u(M)\,dM \int_{\sigma(r',s)} \frac{r^2 - r'^2}{\overline{MM'}^3}\, dM'$$

where M and M' represent points on the spheres of radii r, r' respectively. We can therefore write

$$F(r',s) = \int_{S_r} p(M;r,r';s)\,u(M)\,dM$$

where p is a positive function, since $r > r'$, and is given by the formula

$$p(M;\, r,\, r';\, s) = \int_{\sigma(r',s)} \frac{r^2 - r'^2}{4\pi r \cdot \overline{MM'}^3}\, dM'.$$

We will consider various positions for the point M with regard to the boundary s.

(i) Suppose that M is inside s and that its minimum angular distance from s is β. Then

$$p(M;r,r';s) > \frac{(r^2 - r'^2)r'^2}{4\pi r}\int_0^{2\pi} d\gamma \int_0^{\beta} \frac{\sin\alpha\, d\alpha}{(r^2 + r'^2 - 2rr'\cos\alpha)^{3/2}}$$
$$= \frac{r'(r+r')}{2r^2} - \frac{r'(r^2 - r'^2)}{2r^2(r^2 + r'^2 - 2rr'\cos\beta)^{1/2}}.$$

(ii) Suppose that M is outside s and that its minimum angular distance from the boundary s is β. Then

$$0 < p(M;r,r';s) < \frac{(r^2 - r'^2)r'^2}{4\pi r}\int_0^{2\pi} d\gamma \int_{\beta}^{\pi} \frac{\sin\alpha\, d\alpha}{(r^2 + r'^2 - 2rr'\cos\alpha)^{3/2}}$$
$$= \frac{(r^2 - r'^2)r'}{2r^2}\left[\frac{1}{(r^2 + r'^2 - 2rr'\cos\beta)^{1/2}} - \frac{1}{r+r'}\right]$$

(iii) Whatever may be the position of M with respect to s

$$0 < p(M; r, r'; s) \leq \frac{(r^2 - r'^2) r'}{2r^2} \left[\frac{1}{r - r'} - \frac{1}{r + r'} \right] = \frac{r'^2}{r^2} < 1.$$

We see from the above inequalities, that, given the increasing sequence $\{r_k\}$ it is possible to associate with any integer n a quantity δ_n with $\lim\limits_{n=\infty} \delta_n = 0$ such that if the point $M(\theta, \phi)$ is inside the segment

$$\bar{s}(\theta_1 + \delta_n, \ \theta_2 - \delta_n; \ \phi_1 + \delta_n, \ \phi_2 - \delta_n)$$

or outside the segment

$$\bar{\bar{s}}\,(\theta_1 - \delta_n, \ \theta_2 + \delta_n; \phi_1 - \delta_n, \ \phi_2 + \delta_n)$$

then, for all $j > i > n$

$$| \, p(M; \ r_j, \ r_i; \ s) - q(M, s) \, | < \delta_n,$$

where $q(M, s)$ represents the function which is equal to 1, 0, $\psi / 2\pi$ according as M is inside, outside, or on s.

Consequently,

$$\begin{aligned} & | \, F(r_i; \ \theta_1, \ \theta_2; \ \phi_1, \ \phi_2) - F(r_j; \ \theta_1, \ \theta_2; \ \phi_1, \ \phi_2) \, | \\ & = \left| \int_{S_j} \{ q(M, s) - p(M; \ r_j, \ r_i; \ s) \} u(M) \, dM \right| \\ & \leq \int_{S_j} | \, q(M, s) - p(M; \ r_j, \ r_i; \ s) \, | \cdot | \, u(M) \, | \, dM \\ & \leq \delta_n K + T(r_j; \ \theta_1 - \delta_n, \ \theta_1 + \delta_n; \ c, \ d) \\ & \qquad + T(r_j; \ \theta_2 - \delta_n, \ \theta_2 + \delta_n; \ c, \ d) \\ & \qquad + T(r_j; \ a, \ b; \ \phi_1 - \delta_n, \ \phi_1 + \delta_n) \\ & \qquad + T(r_j; \ a, \ b; \ \phi_2 - \delta_n, \ \phi_2 + \delta_n) \end{aligned}$$

where the numbers $\theta_1 \pm \delta_n$, $\theta_2 \pm \delta_n$ lie between a and b, and $\phi_1 \pm \delta_n$, $\phi_2 \pm \delta_n$ lie between c and d; $0 \leq a < b \leq \pi$, $0 \leq c < d \leq 2\pi$. We now show that, given $\epsilon > 0$, n can be chosen so large that the integral

$$\int_R | \, F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) \, | \, dR < \epsilon \qquad (n < i < j)$$

where R is the region $[0 \leq \theta_1 \leq \pi, 0 \leq \theta_2 \leq \pi; \ 0 \leq \phi_1 \leq 2\pi, 0 \leq \phi_2 \leq 2\pi]$.

We first choose a positive number η so small that for all values of i, j

$$\int_{R - R_\eta} | \, F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) \, | \, dR < \epsilon / 2 .$$

where R_η represents the region

$$[\eta \leqq \theta_1 \leqq \pi - \eta,\ \eta \leqq \theta_2 \leqq \pi - \eta;\quad \eta \leqq \phi_1 \leqq 2\pi - \eta,\ \eta \leqq \phi_2 \leqq 2\pi - \eta].$$

This can be done because F, by hypothesis, is bounded. The η being now fixed, we consider only values of n for which $\delta_n < \eta$. For such values of n, it follows, that if $j > i > n$ and if $(\theta_1, \theta_2; \phi_1, \phi_2)$ lies in R_η

$$\begin{aligned} &| F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) | \\ &< \delta_n K + T(r_j;\ \theta_1 - \delta_n,\ \theta_1 + \delta_n;\ 0,\ 2\pi) \\ &\qquad + T(r_j;\ \theta_2 - \delta_n,\ \theta_2 + \delta_n;\ 0,\ 2\pi) \\ &\qquad + T(r_j;\ 0,\ \pi;\ \phi_1 - \delta_n,\ \phi_1 + \delta_n) \\ &\qquad + T(r_j;\ 0,\ \pi;\ \phi_2 - \delta_n,\ \phi_2 + \delta_n). \end{aligned}$$

Consequently, we obtain, after integrating over R_η,

$$(15)\qquad \begin{aligned} &\int_{R\eta} | F(r_i; \theta_1, \theta_2; \phi_1, \phi_2) - F(r_j; \theta_1, \theta_2; \phi_1, \phi_2) | \, dR \\ &\leqq \delta_n K \cdot 4\pi^4 + 4\pi^3 \int_\eta^{\pi-\eta} T(r_j;\ \theta_1 - \delta_n,\ \theta_1 + \delta_n;\ 0,\ 2\pi) \, d\theta_1 + \cdots \\ &\leqq \delta_n K \cdot 4\pi^4 + 4\pi^3 \int_\eta^{\pi-\eta} \{ T(r_j; 0, \theta_1 + \delta_n; 0, 2\pi) \\ &\qquad\qquad - T(r_j; 0, \theta_1 - \delta_n; 0, 2\pi) \} \, d\theta_1 + \cdots \end{aligned}$$

where the terms omitted in the last expression are similar to the integral term written down. By changing the variable of integration this term can be written in the form

$$\begin{aligned} &4\pi^3 \left\{ \int_{\eta+\delta_n}^{\pi-\eta+\delta_n} - \int_{\eta-\delta_n}^{\pi-\eta-\delta_n} T(r_j; 0, \theta; 0, 2\pi) \, d\theta \right\} \\ &= 4\pi^3 \left\{ \int_{\pi-\eta-\delta_n}^{\pi-\eta+\delta_n} - \int_{\eta-\delta_n}^{\eta+\delta_n} T(r_j; 0, \theta; 0, 2\pi) \, d\theta \right\} \\ &< 4\pi^3 \int_{\pi-\eta-\delta_n}^{\pi-\eta+\delta_n} T(r_j; 0, \theta; 0, 2\pi) \, d\theta \\ &< 4\pi^3 K \cdot 2\delta_i \end{aligned}$$

since $T(r_j;\ 0,\ \theta;\ 0,\ 2\pi)$ is a non-decreasing function of θ, less than K. By treating the other terms of (15) in a similar manner we arrive at the result

$$\int_{R\eta} | F(r_i;\theta_1,\theta_2;\phi_1,\phi_2) - F(r_j;\theta_1,\theta_2;\phi_1,\phi_2) | \, dR$$

$$< \delta_n K\,[4\pi^4 + 8\pi^3 + 8\pi^3 + 4\pi^3 + 4\pi^3] = \delta_n \cdot 4\pi^3 K(\pi + 6) \qquad (j > i > n).$$

We now choose n so large as to make this expression less than $\epsilon/2$. From this it follows that the integral extended over the whole of R will be less than ϵ for a sufficiently large value of n; that is to say, *the sequence of functions*

$$F(r_i;\theta_1,\theta_2;\phi_1,\phi_2)$$

converges in the mean (of order 1) *in the region* R.

We can now state the following auxiliary results which are implied by convergence in the mean, of order 1:

(A) *There exists a subsequence* $\{\bar{r}_i\}$ *of* $\{r_i\}$ *and a function* $\Phi(\theta_1,\theta_2;\phi_1,\phi_2)$ *defined at nearly every point* $(\theta_1,\theta_2;\phi_1,\phi_2)$ *of* R, *such that*

$$\lim_{i=\infty} F(\bar{r}_i;\theta_1,\theta_2;\phi_1,\phi_2) = \Phi(\theta_1,\theta_2;\phi_1,\phi_2)$$

except for a set of points, in R, *of measure zero.*

(B) *There exists a subsequence* $\{r'_i\}$ *of* $\{\bar{r}_i\}$ *and a function of* θ *denoted by* $\Phi(0,\theta;0,2\pi)$, *defined at nearly every point* θ *of the interval* $0 \leqq \theta \leqq \pi$, *such that*

$$\lim_{i=\infty} F(r'_i;0,\theta;0,2\pi) = \Phi(0,\theta;0,2\pi),$$

except for a possible set of values of θ *of linear measure zero.*

The proof of (B) is similar to that already given, of (A), but it is less complicated and may therefore be omitted. But (A) is valid, of course, if r takes on the values in the subsequence $\{r'_i\}$ of $\{\bar{r}_i\}$; hence there exists a single sequence $\{r'_i\}$ for which both (A) and (B) are valid.

Let $E_1(\theta)$ represent the set of values of θ, of measure π, for which $\lim F(r'_i;0,\theta;0,2\pi) = \Phi(0,\theta;0,2\pi)$. It follows from (A) that there exists at least one number a in $E_1(\theta)$ and a certain subset $E_2(\theta)$ of $E_1(\theta)$ of measure π, depending on a, such that the set of points (ϕ_1,ϕ_2) for which $\lim F(r'_i;a,\theta_2;\phi_1,\phi_2)$ does not exist is of superficial measure zero, if θ_2 is a number of the set $E_2(\theta)$. And since

$$F(r'_i;\theta_1,\theta_2;\phi_1,\phi_2) = F(r'_i;a,\theta_2;\phi_1,\phi_2) - F(r'_i;a,\theta_1;\phi_1,\phi_2),$$

it follows that if θ_1, θ_2 are any two numbers in $E_2(\theta)$, then there can be at most a set of points (ϕ_1,ϕ_2) of two-dimensional measure zero for which $\lim F(r'_i;\theta_1,\theta_2;\phi_1,\phi_1)$ does not exist.

Let us now choose a denumerable subset $E'(\theta)$ of $E_2(\theta)$ dense in the interval $(0, 2\pi)$ and let us exclude all points (ϕ_1, ϕ_2) for which $\lim F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2)$ does not exist; allowing θ_1, θ_2 to take on every pair of numbers in $E'(\theta)$. We thus exclude a denumerable aggregate of sets of points $\phi_1, \phi_2)$ each of measure zero. The points (ϕ_1, ϕ_2) remaining constitute a set $E_1(\phi_1, \phi_2)$ of measure $4\pi^2$ in the region $0 \leq \phi_1 \leq 2\pi$; $0 \leq \phi_2 \leq 2\pi$. If now θ_1, θ_2 are in $E'(\theta)$ and if (ϕ_1, ϕ_2) is in $E_1(\phi_1, \phi_2)$ then $\lim F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2)$ exists.

There exists a number b, $0 < b < 2\pi$, and a set $E''(\phi)$ such that meas. $E''(\phi) = 2\pi$ and such that $\lim F(r_i'; \theta_1, \theta_2; b, \phi_2)$ exists if ϕ_2 is any number in $E''(\phi)$. And, since

$$F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2) = F(r'_i; \theta_1, \theta_2; b, \phi_2) - F(r'_i; \theta_1, \theta_2; b, \phi_1),$$

it follows that if θ_1, θ_2 are in $E'(\theta)$ and if ϕ_1, ϕ_2 are in $E''(\phi)$

$$\lim_{i=\infty} F(r'_i; \theta_1, \theta_2; \phi_1, \phi_2) = \Phi(\theta_1, \theta_2; \phi_1, \phi_2) = \Phi(s')$$

exists. The auxiliary theorem is thus proved.

We are now enabled to complete the proof of the statement that condition (b) implies condition (a), that is, that $\lim F(r'_i, s)$ exists for every segment s.

First, we have

$$\begin{aligned} F(r'_i, s) &= \lim_{j=\infty} \int_{S_j} p(M; r'_j, r'_i; s)\, dF(r_j', s'_M) \\ &= \int_S p(P; 1, r'_i; s)\, d\Phi(s'_P) \end{aligned}$$

where the integrals are formed with respect to any net H of segments s' determined exclusively by the circles $\theta = \text{const.}$ in $E'(\theta)$, $\phi = \text{const.}$ in $E''(\phi)$. For $p(M; r_j, r_i; s)$ approaches $p(P; 1, r_i; s)$ uniformly, and $F(r_j, s')$ approaches $\Phi(s')$ on every segment s' in H. Evidently $\Phi(s')$ is a bounded additive function of segments on the net H. Hence

$$\begin{aligned} \lim_{i=\infty} F(r'_i, s) &= \lim_{i=\infty} \int_S p(P; 1, r'_i; s)\, d\Phi(s'_P) \\ &= \int_S q(P, s)\, d\Phi(s'_P) \end{aligned}$$

by the definition of the generalized Stieltjes integral. For the function $q(P, s)$ is again the limit function of the bounded sequence of continuous functions of P, $p(P; 1, r'_i; s)$. Thus condition (a) is satisfied.

We are thus able to state the following theorem, which summarizes the results of these sections.

THEOREM. *A necessary and sufficient condition that* $u(M)$, *harmonic in* Σ, *be given by the formula* (I),

$$u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3} \, dF(e), \tag{I}$$

where $F(e)$ *is a completely additive function of point sets, or by* (III)

$$u(M) = \frac{1}{4\pi} \int_S \frac{1-r^2}{\overline{MP}^3} \, dF(s) \tag{III}$$

where $F(s)$ *is a bounded and additive function of segments is that*

$$\int_{S_r} | u(M) | \, dM \tag{b}$$

remain bounded as r *approaches* 1, *over a denumerable sequence of values, or, in fact, in any way.*

The function $F(r, s)$ *has a limit as* r *approaches* 1 *for every segment* s, *and*

$$\begin{aligned} \lim_{r=1} F(r, s) &= \lim_{r=1} \int_S p(P;\, 1,\, r;\, s) \, dF(s) \\ &= \int_S q(P, s) \, dF(s) = F_1(s) \end{aligned} \tag{16}$$

is identical with $F(s)$ *on all segments of continuity and has regular discontinuities; the function* $F(e)$ *is the corresponding completely additive function of point sets.*

In fact, the above relations are true if $F_1(s)$ and $F(e)$ are functions corresponding to $F(s)$ as described in § 1. 22. On the other hand, if $F_2(e)$ were a completely additive function of point sets, different from $F(e)$, i. e., not corresponding to $F(s)$, the corresponding bounded additive function of segments $F_2(s)$ with regular discontinuities would differ from $F_1(s)$ on some segment and relation (16) would fail.

2. 3. The class of functions given by (I) or (III) within the unit sphere may also be characterized in another way.

THEOREM. *A necessary and sufficient condition that* $u(M)$ *be given by* (I) *or* (III) *within the unit sphere is that it be the difference of two functions*

$$u(M) = u_1(M) - u_2(M)$$

which are harmonic and not negative within the unit sphere.

The condition is evidently necessary, since such functions $u_1(M)$, $u_2(M)$

may be obtained by substituting for $F(s)$ in (III) the positive and negative variation functions, respectively, of $F(s)$. In order to prove the sufficiency we note that

$$\int_{S_r} | u(M) | \, dM \leq \int_{S_r} u_1(M) dm + \int_{S_r} u_2(M) dM$$
$$= 4\pi r^2[u_1(O) + u_2(O)] < 4\pi[u_1(O) + u_2(O)].$$

Hence condition (b) is satisfied.

2.4. In § 1.21 the reader was allowed to assume that the pole for the systems of segments remained invariant, although the formulae there given provide convenient means for just such changes of coordinates. Rather than complete the demonstrations in this more general sense let us turn at once to the general class of regular curves on S.

We define a *regular net* G on S, for a regular curve w as a system of lattices G_n corresponding to positive numbers δ_n successively decreasing and approaching 0 as a limit. Each lattice represents a partition of w and its interior into a finite number of cells w_{in} each of which is a regular curve of diameter $< \delta_n$. Given a net G for a regular curve w it may obviously be extended throughout the complement w_c of w so as to become a net for S itself.

Let $F(s')$ be additive and bounded on a net H of segments, and define

$$F_1(w) = \int_S q(P, w) \, dF(s'_P) \tag{17}$$

for any regular curve w on S. Then $F_1(w)$ is a function of regular curves, additive and bounded on S. Moreover we can integrate by means of a Riemann sum with respect to $F_1(w)$ any function $h(P)$ if it is continuous on S, and extend the definition of the integral as an S-integral for any $h(P)$ bounded and measurable Borel. Take first $h(P)$ to be continuous.

Divide S into portions bounded by regular curves w_i of diameter $< \delta$. Write

$$h_\delta(P) = \textstyle\sum_i h(P_i) q(P, w_i),$$

P_i a point of w_i. Then

$$\textstyle\sum_i h(P_i) F_1(w_i) = \int_S h_\delta(P) \, dF(s'_P)$$

by definition of $F_1(w_i)$. But the limit of the right hand side is unique as δ approaches 0, since $\lim\limits_{\delta=0} h_\delta(P) = h(P)$ uniformly. Hence the Riemann-Stieltjes sum in the left-hand member has a unique limit; it also satisfies the postulates (C) (A) (L) (M) of the S-integrals.* It follows therefore that

$$\int_S h(P)\,dF_1(w_P) = \int_S h(P)\,dF(s'_P) \tag{18}$$

for all functions $h(P)$ measurable Borel and bounded.

In particular we may take $h(P) = q(P, w)$ so that

$$\int_S q(P, w)\,dF_1(w_P) = F_1(w) \tag{19}$$

and the discontinuities of $F_1(w)$ may be said to be *regular.*

Let $P_1(w)$ be the upper bound of $F_1(p_w)$ for all finite pluricells p_w composed of arbitrary regular curves w_i contained in w, and let $N_1(w)$, $T_1(w)$ be defined accordingly. Let $P_1(s')$, $N_1(s')$, $T_1(s')$ be the functions similarly formed on H. We have the following theorem.

THEOREM. *The function $P_1(w)$ is given by the formula*

$$P_1(w) = \int_S q(P, w)\,dP_1(s'_P) \tag{20}$$

and is a bounded additive function of regular curves with regular discontinuities. Corresponding relations hold for $N_1(w)$ and $T_1(w)$.

It is necessary merely to prove (20), since the rest of the theorem is an immediate consequence of that relation. Let G be a net of regular curves w'_{in} for both w and w_c and form $P'(w')$ and $N'(w')$ as upper bounds, of $F_1(p_{w'})$ and $-F_1(p_{w'})$ respectively, on G, for curves w' which are finite pluricells on the net. The analysis of §§ 1. 2, 1. 21 shows that these functions are bounded and additive for curves w' and satisfy the relations

$$\begin{aligned} F(w') &= P'(w') - N'(w') \\ P'(w') &= \int_S q(P, w')\,dP'(w'_P) \\ P'(w) + P'(w_c) &= P'(S) \leq P_1(w) + P_1(w_c). \end{aligned} \tag{21}$$

We have, however, from (19), if p is a finite plurisegment in H,

$$\begin{aligned} F_1(p) &= \int_S q(P, p)\,dF_1(w'_P) \\ &= \int_S q(P, p)\,dP'(w'_P) - \int_S q(P, p)\,dN'(w'_P) \\ &\leq \int_S q(P, p)\,dP'(w'_P) \leq \int_S q(P, s')\,dP'(w'_P) \end{aligned}$$

when the plurisegment p is contained in the segment s' of H. Hence

$$P_1(s') \leq \int_S q(P, s')\,dP'(w'_P);$$

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* P. J. Daniell, *loc. cit.*

also

$$P_1(s'_c) \leq \int_S q(P, s'_c)\, dP'(w'_P),$$

and therefore

$$P_1(S) = P_1(s') + P_1(s'_c) \leq \int_S [q(P,s') + q(P,s'_c)]\, dP'(w'_P)$$

$$(22) \qquad P_1(S) \leq P'(S).$$

On the other hand

$$\begin{aligned} F_1(p_w) &= \int_S q(P, p_w)\, dF_1(s'_P) \\ &= \int_S q(P, p_w)\, dP_1(s'_P) - \int_S q(P, p_w)\, dN_1(s'_P) \\ &\leq \int_S q(P, p_w)\, dP_1(s'_P) \leq \int_S q(P, w)\, dP_1(s'_P). \end{aligned}$$

Hence

$$P_1(w) \leq \int_S q(P, w)\, dP_1(s'_P),$$

$$P_1(w_c) \leq \int_S q(P, w_c)\, dP_1(s'_P).$$

Unless therefore (20) is satisfied we have

$$P_1(w) + P_1(w_c) < P_1(S).$$

But this equation is not compatible with (22) and the third of equations (21). Hence (20) must be valid.

Incidentally, *we have the fact that*

$$(23) \qquad P_1(w) = P'(w).$$

Since w is itself a pluricell of G.

2. 5. As an application of § 2. 4 we have the following theorem:

THEOREM. *If $u(M)$ is given by* (III), *and $F_1(w)$ is the additive and bounded function of regular curves w associated with $F(s)$ by the equation*

$$(24) \qquad F_1(w) = \int_S q(P, w)\, dF(s_P)$$

then

$$(25) \qquad \lim_{r=1} \left(F(r, w) = \int_{\sigma(r,w)} u(M)\, dM \right) = F_1(w) = P_1(w) - N_1(w)$$

$$\lim_{r=1} \left(T(r, w) = \int_{\sigma(r,w)} | u(M) |\, dM \right) = T_1(w) = P_1(w) + N_1(w)$$

where $\sigma(r, w)$ is the projection of the region bounded by w on the sphere of radius r.

In fact

$$F(r,w) = \lim_{r'=1} \int_{S_{r'}} p(M;\, r', r;\, w)\, dF(r', s)$$
$$= \int_S p(P;\, 1, r;\, w)\, dF(s),$$

by (12), as in § 2. 2. And therefore

$$\lim_{r=1} F(r,w) = \int_S q(P,w)\, dF(s) = F_1(w).$$

Since $p(P;\, 1, r;\, w)$ remains bounded as r tends to 1. Thus the first of the relations (25) is established.

For the second relation, we obtain directly from the Poisson integral the inequality

$$\int_{\sigma(r,w)} | u(M) |\, dM \leq \int_S p(P;\, 1, r;\, w)\, dP_1(s) + \int_S p(P;\, 1, r;\, w)\, dN_1(s),$$

and therefore

$$\lim_{r=1} \int_{\sigma(r,w)} | u(M) |\, dM \leq \int_S q(P,w)\, dP_1(s) + \int_S q(P,w)\, dN_1(s)$$

or

$$\lim_{r'=1} T(r,w) \leq P_1(w) + N_1(w).$$

On the other hand, since $P_1(w)$ by (23) is the upper bound $P'(w)$ of $F_1(p_w)$ for all finite pluricells of a net G for w, contained in w, we can find such a finite pluricell p_w such that

$$F_1(p_w) > P_1(w) - \epsilon/4, \qquad \epsilon \text{ arbitrary, positive};$$

and if we represent by p'_w the finite pluricell of G complementary to p_w with respect to w, we have

$$-F_1(p'_w) > N_1(w) - \epsilon/4.$$

We may now choose r near enough to 1 so that

$$F(r, p_w) > F_1(p_w) - \epsilon/4$$
$$-F(r, p'_w) > -F_1(p'_w) - \epsilon/4.$$

By combining these results, we obtain,

$$F(r, p_w) - F(r, p'_w) > P_1(w) + N_1(w) - \epsilon.$$

Hence

$$T(r,w) \geq F(r, p_w) - F(r, p'_w)$$
$$> P_1(w) + N_1(w) - \epsilon;$$

or

$$\lim_{r=1} T(r, w) \geq P_1(w) + N_1(w).$$

By comparing this inequality with the earlier one, the second of the relations (25) is verified.

3. *Behaviour of $u(M)$ in the neighborhood of the boundary.* An important theorem comes as a consequence of (I) or (III).

THEOREM. *Let A be a point of S at which $F(s)$ possesses a derivative $f(A)$,* and $M_1, M_2, \cdots$ be any sequence of points, with $\lim_{n=\infty} M_n = A$ such that, if r_i denotes OM_i and θ_i the angle between OM_i and OA, the quantity $\theta_i/(1-r_i)$ is bounded. Then, if $u(M)$ is given by* (III) *or* (I),

$$\lim_{i=\infty} u(M_i) = f(A).$$

If $F(s)$ has a unique derivative at A, the same is true of the corresponding function of segments with regular discontinuities. Hence we may assume that the discontinuities of $F(s)$ are regular.

We may write

$$F(s) = \sigma f(A) + g(s)$$

where $g(s)$ is a bounded additive function of segments with regular discontinuities, and has a unique derivative at A which is zero.

If a bounded additive function of segments $g(s)$ has a zero derivative at a point A, the same is true of its positive and negative variation functions. In fact, if p_n is such a regular family, we can choose p'_n in p_n so that

$$P(p_n) < F(p'_n) + 1/2^n \text{ (meas. } p_n),$$

and if we let p''_n be the complement of p'_n in p_n,

$$N(p_n) < -F(p''_n) + 1/2^n \text{ (meas. } p_n).$$

For every value of n, either p'_n or p''_n is of measure $\geq p_n/2$. Hence if we denote by p'_i those p'_n of measure $\geq p_n/2$, and by p''_j the p''_n corresponding to the remaining values of n, we shall have two regular families of plurisegments, of which the parameter of regularity is not more than twice that for p_n. Accordingly we have

$$\lim_{i=\infty} F(p'_i)/\text{meas. } p'_i = 0$$

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* The derivative is taken with regard to any regular family of plurisegments. See A. J. Maria, *loc. cit.*, p. 459.

and therefore

$$\lim_{i=\infty} \frac{P(p_i)}{\text{meas. } p_i} \leq \lim_{i=\infty} \frac{F(p'_i)}{\text{meas. } p_i} + \lim_{i=\infty} 1/2^n = 0,$$

where i ranges over this certain sequence of values.

Similarly for the other sequence of values,

$$\lim_{j=\infty} N(p_j)/\text{meas. } p_j = 0$$

and therefore

$$\lim_{j=\infty} \frac{P(p_j)}{\text{meas. } p_j} = \lim_{j=\infty} \frac{F(p_j) + N(p_j)}{\text{meas. } p_j} = 0.$$

Hence for the whole sequence of values of n

$$\lim_{n=\infty} P(p_n)/\text{meas. } p_n = 0.$$

Accordingly we may assume for simplicity that $g(s)$ is of positive type. Since the discontinuities of $F(s)$, $g(s)$ are regular (see § 1. 21) the definition of these functions may be extended, by means of § 2. 4, to all regular curves on S, and the integral in (I) or (III) written with respect to $F(w)$. Moreover since $g(s)$, of positive type, has a zero derivative at A on a regular family of segments for which A is an interior point, the same is true at A for any regular family whatever of regular curves w.

Let now a small circle of latitude, C, be drawn upon S with OA as axis, and let α' be its colatitude; and let α' be chosen so small that for any family of circles (c) within C and containing A—this would be a regular family having the number 4 as parameter of regularity with respect to circles of center A—the value of F can be expressed as

$$F(c) = [f(A) + \eta(c)]\sigma \tag{26}$$

where σ is the area of c, and $0 \leq \eta(c) < \epsilon$ a quantity which approaches 0 with α'. Otherwise there would be a sequence of c's constituting a regular family on which the derivative of g would not be 0.

Since M_i is to approach A we may assume that $\theta_i < \alpha'/2$. Thus, if we denote $\alpha' - \theta_i$ by α_i, a circle c_i of colatitude α_i about OM_i as axis will contain A in its interior and will touch C internally. Now we have

$$\lim_{j=\infty} u(M_i) = f(A) + \lim_{j=\infty} (1/4\pi) \int_S \frac{1 - r_i^2}{M_i P^3} \, dg(w)$$

$$= f(A) + \lim_{i=\infty} (1/4\pi) \int_S q(P, c_i) \; \frac{1 - r_i^2}{\overline{M_i P}^3} \, dg(w),$$

since the circle c_i ultimately tends to C, and the distance from the projection of M_i on S to the complement $S - c_i$ of c_i exceeds a positive number which is independent of i.

Write now, for circles c with OM_i as axis and α as colatitude,

$$g(c) = h_i(\alpha)$$

which is thus a function of α of limited variation, uniformly for all i. Moreover

$$h_i(\alpha) = \eta_i(\alpha) \cdot 2\pi(1 - \cos\alpha) \qquad \theta_i \leqq \alpha \leqq \alpha_i$$

$$h_i(\alpha) \leq \eta_i(\theta_i) \cdot 2\pi(1 - \cos\theta_i) \qquad 0 \leq \alpha \leq \theta_i,$$

in which $\eta_i(\theta_i)$ approaches zero as i becomes infinite. Now the integral may be evaluated with respect to any net on S. Hence by an integration by parts, we have for the limit to be considered one which does not exceed

$$\lim\, (1/4\pi) \int_0^{\alpha_i} \frac{1 - r_i^2}{1 + r_i^2 - 2r_i \cos\alpha} \; dh_i(\alpha)$$

$$= \lim\, (1/4\pi) \left[\frac{(1 - r_i^2) h_i(\alpha)}{(1 + r_i^2 - 2r_i \cos\alpha)^{3/2}} \right]_0^{\alpha_i}$$

$$+ \lim\, (3/4\pi) \int_0^{\alpha_i} \frac{(1 - r_i^2) r_i \sin\alpha \cdot h_i(\alpha)}{(1 + r_i^2 - 2r_i \cos\alpha)^{5/2}} \, d\alpha$$

of which the first term is zero; for at $\alpha = \alpha_i$ the denominator has a fixed lower bound, and at $\alpha = 0$

$$(1/4\pi) \left[\frac{(1 - r_i^2) h_i(\alpha)}{(1 + r_i^2 - 2r_i \cos\alpha)^{3/2}} \right]_0 \leq \eta_i(\theta_i)\,(1 + r_i) \; \frac{\sin^2(\theta_i/2)}{(1 - r)^2} \; .$$

The second term, when the integral is rewritten as $\int_0^{\theta_i} + \int_{\theta_i}^{\alpha_i}$ may be handled directly, and is

$$\leq \lim\, (1/4\pi) \left[\frac{\eta_i(\theta_i)\,(1 - \cos\theta_i)\,(1 - r_i^2)}{(1 + r_i^2 - 2r_i \cos\alpha)^{3/2}} \right]_{\alpha=0}^{\alpha=\theta_i}$$

$$+ (3/2)\, \epsilon \lim \int_0^{\alpha_i} \frac{(1 - r_i^2) r_i (1 - \cos\alpha) \sin\alpha}{(1 + r_i^2 - 2r_i \cos\alpha)^{5/2}} \, d\alpha,$$

where for simplicity the lower limit of integration in the second term has been replaced by 0. The first term has the limit 0, on account of the restriction on $\theta_i/(1-r_i)$ and the second may be valuated as

$$\epsilon \lim (1/4\pi) \left[\frac{(1-r_i)^3(1+r_i)}{(1+r_i^2-2r_i\cos a)^{3/2}} - \frac{3(1-r_i)(1+r_i)}{(1+r_i^2-2r_i\cos a)^{1/2}}\right]_{a=0}^{a=a_i}$$

$$= \epsilon$$

Hence

$$\left|\lim_{i=\infty} u(M_i) - f(A)\right| < \epsilon$$

and

$$\lim_{i=\infty} u(M_i) = f(A).$$

In other words $u(M)$ *approaches* $f(A)$ *not only as* M *approaches* A *along a radius, but also as* M *approaches* A *along any path which may be contained in a circular cone of vertex* A *and axis* OA *with vertical semi-angle* $< \pi/2$ *.

4. *Poisson's integral and the Dirichlet problem for summable boundary values.* The functions $u(M)$ of the class we have been considering are evidently not determined by their boundary values $f(A)$. In fact it is the $F(e)$ which is arbitrary. That is, if $\lim\limits_{r=1} F(r,s)$ is an arbitrary additive and bounded function of curves, with regular discontinuities, for regular curves on S, there is one and only one function $u(M)$ harmonic within S and subject to this boundary condition if $\int_{S_r} |u(M)|\, dM$ remains bounded. This is a *generalized* Dirichlet problem.

If however the lower net derivative † of $F(e)$, (which is $f(A)$ almost everywhere), does not anywhere have a positively infinite value (which it can have, in general on a point set of measure 0), or the upper net derivative a negatively infinite value, the $F(e)$ or $F(s)$ is absolutely continuous, the formula (I) or (III) degenerates into Poisson's Integral, and the function $u(M)$ is determined by assigning its boundary values, summable in the Lebesgue sense, almost everywhere on S.

THEOREM. *A necessary and sufficient condition in order that* $u(M)$, *harmonic in* Σ *be given by the Poisson integral*

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* A generalization of Fatou's well-known theorem for the circle; "Séries trigonométriques et séries de Taylor," *Acta Mathematica*, Vol. 35 (1906), pp. 335-400, see p. 345 and p. 357.

† The net derivative in the sense of de la Vallée Poussin, *loc. cit.*, p. 98.

$$\text{(II)}, \qquad u(M) = (1/4\pi)\int_S \frac{(1-r^2)f(P)}{\overline{MP}^3} \quad dP$$

in which $f(P)$ is summable in the Lebesgue sense, is

(c) *that the absolute continuity of the integral $F(r_i, s)$ be uniform for all i.*

The condition (c) implies (a). Hence if $u(M)$ satisfies (c), it is given by the formula (III). But

$$F(s) = \lim F(r_i, s)$$

being the limit of a sequence of functions whose absolute continuity is uniform, is itself absolutely continuous. Hence, in this case, the formula (III) reduces to the Poisson integral (II).

Conversely, if u is given by (II), u is the difference of two harmonic functions, u_1, u_2, which are also given by (II) in terms of two non-negative summable functions f_1, f_2, where $f = f_1 - f_2$; u_1 and u_2 are therefore non-negative and

$$\lim u_1(M) = f_1(A); \quad \lim u_2(M) = f_2(A)$$

at nearly every point A, with $\lim (M) = A$.

But the necessary condition that for a non-negative function

$$\int_{\sigma(r_i,s)} u_1(M)\,dM = \int_{\sigma(1,s)} f_1(P)\,dP,$$

is that the absolutely continuity of $\int u_1(M)\,dM$ be uniform.* The same is true of $u_2(M)$ and therefore the absolute continuity of $\int u(M)\,dM$ is uniform.

In precisely the same way as in the corresponding proposition in the case of the circle, the following corollary may be proved.

Corollary. The condition (c) *implies that the absolutely continuity of*

$$\int_{\sigma(r,s)} u(M)\,dM$$

is uniform over all r, $r < 1$.†

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* de la Vallée Poussin, "Sur l'intégrale de Lebesgue," *Transactions of the American Mathematical Society,* Vol. 16 (1915), pp. 435-501. See p. 447.

† G. C. Evans, "Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems," *American Mathematical Society, Colloquium Series,* (in press).

From this theorem we obtain easily the theorem, which is proved in the case of the circle by a different method, by M. Noaillon.*

THEOREM. *A necessary and sufficient condition that* $u(M)$ *be given by the Poisson integral* (II) *is*

$$(c') \qquad \lim_{i,j=\infty} \int_S | u_i(P) - u_j(P) | \, dP = 0$$

where $u_i(P)$ *represents the value of* u *at the point which is the projection of* P *from* O *on the sphere* S_i *of radius* r_i; *in other words the sequence of functions* $u_i(P)$ *is convergent in the mean of order* 1.

From the inequality

$$\int_e | u_i(P) | \, dP \leqq \int_e | u_j(P) | \, dP + \int_e | u_i(P) - u_j(P) | \, dP, \qquad i > j,$$

it is easily seen that the condition (c') implies (c). For if (c') is true it is possible, given ϵ to take a fixed j so large that the second term of the second member is less than $\epsilon/2$ for any set e on S; and then to choose meas. (e) so small that

$$\int_e | u_i(P) | \, dP < \epsilon/2 \qquad (i = 1, 2, 3, \cdots, j).$$

Thus, for every i, we shall have $\int_e | u_i(P) | \, dP < \epsilon$. Hence the sufficiency of (c') is established.

Conversely, if u is given by (II), $u_i(P)$ approaches $f(P)$ nearly everywhere on S, as we have seen. And the absolute continuity of $\int u_i(P) dP$ is uniform; the same is true of $\int | u_i(P) - f(P) | \, dP$. But

$$\int_S | u_i(P) - u_j(P) | \, dP \leqq \int_S | u_i(P) - f(P) | \, dP + \int_S | u_j(P) - f(P) | \, dP;$$

and since the second member of this inequality approaches zero as i and j become infinite, it follows that

$$\lim_{i,j=\infty} \int_S | u_i(P) - u_j(P) | \, dP = 0.$$

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* Noaillon, *Comptes Rendus*, Vol. 182 (1926), p. 1371.

We have now the following result:

THEOREM for the Dirichlet Problem. *Given $f(A)$ summable on S there is one and only one function of the class defined by condition* (c) *or of the class defined by* (c′), *harmonic in Σ, which takes on* (*in the sense of the theorem of* § 3) *the values $f(A)$ as boundary values almost exerywhere on S. It is given by the Poisson integral* (II).

If $f(A)$ is bounded on S, or bounded except on a point set of measure 0, $u(M)$ is bounded in Σ and vice versa. In this case condition (c) is satisfied. *Hence there is one and only one bounded function, harmonic in Σ which takes on $f(A)$ almost everywhere on S, where $f(A)$ is bounded and measurable in the Lebesgue sense.*

We have already shown that any function which satisfies (c) and is harmonic within S takes on boundary values almost everywhere on S.

The case just considered is a special case of that where $|\, u(M) \,| \leq \psi(A)$ and $\psi(A)$ is summable over S. In fact, under this inequality, the absolute continuity of $\int u(M)\, dM$ is uniform. Another special case of this same condition is that where $|\, \partial u/\partial r \,|$ is summable over Σ, a situation where the $\psi(A)$ is not necessarily a mere constant. For we have

$$\int_{r_0}^{r} dr \int_{\sigma(r,s)} \frac{1}{r^2}\, \frac{\partial u}{\partial r}\, d\sigma = \int_{\sigma(r,s)} u(r, M)\, dM - \int_{\sigma(r_0,s)} u(r_0, M)\, dM,$$

so that

$$\left|\, \int_{\sigma(r,s)} u(r, M)\, dM \,\right| \leq \int_{\sigma(r_0,s)} |\, u(r_0, M) \,|\, dM + (1/r_0{}^2) \int_{\Sigma_s} \left|\, \frac{\partial u}{\partial r} \,\right| d\tau.$$

the volume integral being extended over the conical region with vertex O and base $\sigma(1, s)$. But the right-hand member and $\int u(M)\, dM$ define additive functions of point sets on S_r. Hence

$$\int_{\sigma(r,s)} |\, u(r, M) \,|\, dM \leq \int_{\sigma(r_0,s)} |\, u(r_0, M) \,|\, dM + (1/r_0{}^2) \int_{\Sigma_s} \left|\, \frac{\partial u}{\partial r} \,\right| d\tau.$$

Moreover, since the right-hand member is absolutely continuous, and independent of r, it may be written in the form $\int_{\sigma(1,s)} \psi(P)\, dP$, whence, wherever $\psi(P)$ is the derivative of its own integral,

$$|\, u(r, M) \,| \leq (1/r_0{}^2)\, \psi(P)\,,$$

P being the projection of M from O on S. But since this condition holds almost everywhere, if we define $\psi(P) = +\infty$ where it is not the derivative of its own integral, the function $\psi(P)$ will still be summable and the inequality will hold throughout.

A further special case, of interest on account of its relation to classical treatments of the Dirichlet problem, is that where the integral

$$D(u) = \int_{\Sigma} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right\} dM,$$

extended over the interior of Σ, exists. In fact the existence of $D(u)$ implies that $(\partial u/\partial r)^2$ and therefore that $|\partial u/\partial r|$ is summable over Σ.

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Notes on Formal Modular Protomorphs.*

BY O. C. HAZLETT.

1. *Relation to the literature.* In a recent paper,† W. L. G. Williams has proved several theorems relating to formal modular protomorphs of a certain class of binary forms with respect to the general Galois field. He considers the binary form

$$f(a;x) = a_0x_1^q + \binom{q}{1} a_1x_1^{q-1}x_2 + \binom{q}{2} a_2x_1^{q-2}x_2^2 + \cdots + a_qx_2^q$$

where the a's are independent variables. It is well known that one set of algebraic protomorphs of $f(x,y)$ is $C_1, C_2, \cdots, C_q$ where $C_1 = a_0$, $C_2 = a_0a_2 - a_1^2$, and in general

$$C_{2m} = a_0a_{2m} - \binom{2m}{1} a_1a_{2m-1} + \binom{2m}{2} a_2a_{2m-2} \cdots$$

$$+ (-1)^{m-1} \binom{2m}{m-1} a_{m-1}a_{m+1} + \tfrac{1}{2}(-1)^m \binom{2m}{m} a_m^2$$

$$C_{2m+1} = (a\theta - 2a_1)\, C_{2m}$$

where $\quad \theta = a_1\, \partial/\partial a_0 + a_2\, \partial/\partial a_1 + \cdots + a_q\, \partial/\partial a_{q-1}.$

He then proves that the seminvariants C_i $(i = 1, 2, \cdots, q)$ and β form a fundamental system of protomorphs of the binary q-ic form, mod p, p being a prime such that $\binom{q}{j} \not\equiv 0$, mod p $(j = 1, 2, \cdots, q-1)$. Here $\beta \equiv a_1^p - a_0^{p-1}a_1$. Similarly, a fundamental system of protomorphs of a system of binary forms, mod p (with similar restrictions on p) is a fundamental system of algebraic protomorphs of this system together with one additional formal modular seminvariant, namely β formed for any one of the forms of the system. He proves a similar theorem for formal modular protomorphs with respect to any Galois field, $GF[p^n]$, of order p^n, with the same restriction on p.

Unfortunately, however, his proofs are such that they hold only for a form or system of forms for which no binomial coefficient is congruent to zero modulo p and hence one wonders if the theorems are true for the general form or system of forms. Although any integer $< p$ is admissible as a value of q by his theorem, yet for values of $p < 20$, a rather large number of values of $q \leqq 20$ are not admissible, as shown by the following table, in which $A =$ admissible values of q, $I =$ inadmissible values of q:

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* Read to the American Mathematical Society, Sept. 1926.

† "Fundamental Systems of Formal Modular Protomorphs of Binary Forms," *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 183-197.

$p = 2$, $I = 2, 4, 5, 6, 8 - 14, 16 - 20$, $A = 3, 7, 15$;
$p = 3$, $I = 3, 4, 6, 7, 9 - 16, 18 - 20$, $A = 2, 5, 8, 17$;
$p = 5$, $I = 5 - 8, 10 - 13, 15 - 18, 20$, $A = 2 - 4, 9, 14, 19$;
$p = 7$, $I = 7 - 12, 14 - 19$, $A = 2 - 6, 13, 20$;
$p = 11$, $I = 11 - 20$, $A = 2 - 10$;
$p = 13$, $I = 13 - 20$, $A = 2 - 12$;
$p = 17$, $I = 17 - 20$, $A = 2 - 16$;
$p = 19$, $I = 19, 20$, $A = 2 - 18$.

Hence, it may be of interest to observe that his theorems quoted above hold for a form or system of forms provided merely that we exclude forms whose order is divisible by p.

2. *Protomorphs for the algebraic case.* Now the theorem about algebraic seminvariants of a binary form $f(a; x)$ of order q asserts that any algebraic seminvariant S is expressible in the form $a_0^{\mu} F(a_0, B_2, B_3, \cdots, B_q)$ where F is a polynomial in its arguments and μ is an integer, positive, negative or zero. Here $B_2, B_3, \cdots, B_q$ is any set of $q - 1$ rational integral seminvariants of f which are such that B_i is of weight i and actually involves a_i.* In classical invariant theory it is usually customary † to use as these B's a special set of seminvariants $C_2, C_3, \cdots, C_q$ in which each is of the lowest possible degree. Those of even weights are of the second degree and those of odd weights are of the third degree. Sometimes, however, one uses the coefficients of f in the special form where the coefficient of the next to the highest power of x_1 is zero. If we do not use binomial coefficients (in this paper we shall not use binomial coefficients), then the transformation which carries $f(a; x) \equiv \sum_i a_i x_1^{q-i} x_2^i$ into this special form is $x_1 = x_1' - a_1 x_2'/q a_0$, $x_2 = x_2'$ and the resulting coefficients are $A_0 = a_0$, $A_1 = 0$, $A_i = a_i + \phi_i$ $(i > 1)$ where each ϕ_i is a polynomial in the a_j $(j < 1)$ divided by a power of a_0. Moreover, the coefficients of each ϕ_i are integers divided by a power

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* Elliott, *Algebra of Quantics*, first edition, p. 213.

† *Ibid.*, pp. 214-215.

of q and thus the transformation is admissible for the modular case if q is not congruent to zero in the field. Also, any algebraic seminvariant S of a system of forms $f_i(a^{(i)}; x)$ $(i = 1, 2, \cdots, k)$ is such that $a_0^{(1)\mu_1} a_0^{(2)\mu_2} \cdots a_0^{(k)\mu_k} S$ is a polynomial in seminvariants of each of the forms f_i taken separately and in the leaders $a_0^{(1)} a_1^{(2)} - a_1^{(1)} a_0^{(2)}$, etc. of the Jacobians of one of the forms f_1 and the rest.*

3. *Modular protomorphs as algebraic protomorphs.* This last theorem is of interest in connection with a theorem about the formal modular invariants of a binary form $f(a; x)$. First, however, we need

LEMMA I. *Any formal modular seminvariant of a form or system of forms with respect to the Galois field, $GF[p^n]$, of order p^n is the sum of a finite number of seminvariants which are pseudo-isobaric.*

For it was proved elsewhere † that any seminvariant is annihilated, modulo p, by Ω' and its p'th, p^2' th, $\cdots$, p^{n-1}' th, symbolic powers where

$$\Omega' = \Omega + (1/[n]!)\Omega^{[n]} + \cdots + (1/[kn]!)\Omega^{[kn]} + \cdots$$

in which $[in] = p^{in}$, Ω is the first Aronhold operator of classical invariant theory and $\Omega^{[in]}$ denotes the $[in]$'th symbolic power of Ω. The converse is also true. Now if Ω operates on any polynomial in the a's, it decreases the weight by 1 and, in general, $\Omega^{[kn]}$ decreases the weight by p^{kn}. Moreover, Ω replaces an isobaric function by an isobaric function and a pseudo-isobaric function by a pseudo-isobaric one. Hence, if Ω' annihilates ΣS_i modulo p, where each S_i is pseudo-isobaric and no two S_i have weights which are congruent, modulo $p^n - 1$, then Ω' must annihilate each S_i, modulo p. Hence each S_i is a seminvariant.

But elsewhere ‡ it was proved that any isobaric formal modular invariant C of a system, S, of binary forms with respect to the Galois field, $GF[p^n]$, of order p^n is congruent, modulo p, to an algebraic invariant of S. In a later paper § it was proved that either C was congruent to a rational algebraic invariant of S or that a suitable power of C is congruent to such a covariant. Since[*] a pseudo-isobaric covariant C of a binary system S is such that a suitable power of C (or C multiplied by a suitable power of L) is an isobaric covariant of an enlarged system, S', then these results apply also to pseudo-isobaric invariants of S. But these theorems, although stated only for invariants under the total linear group on the x's, yet the same proof holds without change for the invariants of S under any linear group with coefficients in $GF[p^n]$ and hence, in particular, for seminvariants. Thus we have

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* *Ibid.*, p. 218.

† Hazlett, "Annihilators of Modular Invariants and Covariants," *Annals of Mathematics*, Series 2, Vol. 23 (1922), p. 210.

‡ Hazlett, "A Symbolic Theory of Formal Modular Invariants," *Transactions of the American Mathematical Society*, Vol. 24 (1922), p. 300. Referred to as S. T.

§ Hazlett, *Formal Modular Covariants as Algebraic Invariants.* (As yet unpublished) § 5. Referred to as A. I.

THEOREM I. *If* $f(a; x)$ *is any binary form of order* q *and if* S *is any formal modular seminvariant of* f *for the Galois field,* $GF[p^n]$, *of order* p^n, *then a suitable power of* S *is congruent in the field to a polynomial in the algebraic seminvariants of* $f_0 = f(a; x)$, $f_1 = f(a^{p^n}; x)$, $\cdots$, $f_q = f(a^{q^n}; x)$ *and in the invariants of the* a*'s under the total linear group on the* a*'s divided by a power of* L. *Similarly for a system of forms.*

This is equivalent to saying that any such seminvariant S, when multiplied by a suitable power of L is congruent to a polynomial in the algebraic seminvariants of the system f_0, f_1, $\cdots$, f_{q+1}. For those formal modular seminvariants which are congruent to rational integral algebraic seminvariants of these $q+2$ forms, a complete system of protomorphs consists of the seminvariants a_0, A_2, A_3, $\cdots$, A_q of f_0 and also the corresponding seminvariants $a_0^{p^n}$, $A_2^{p^n}$, $\cdots$ of the other f_i together with the leaders of certain covariants, viz., the Jacobians $J_1 \equiv a_0 a_1^{p^n} - a_0^{p^n} a_1$, $J_2 \equiv a_0 a_1^{p^{2n}} - a_0^{p^{2n}} a_1$, etc. of f_0 and each of the other forms. But the seminvariants $a_0^{p^n}$, $A_2^{p^n}$, etc. are positive integral powers of a_0, A_2, etc. and it is readily seen that

$$J_2 = (a_0 J_1^{p^n} + a_0^{p^{2n}} J_1) \,/\, a_0^{p^n},$$
$$J_3 = (a_0 J_1^{p^{2n}} + a_0^{p^{3n}} J_1) \,/\, a_0^{p^{2n}},$$

and, in general, that

$$J_l \equiv (a_0 J_1^{p^{(l-1)n}} + a_0^{p^{ln}} J_{l-1}) \,/\, a_0^{p^{(l-1)n}}.$$

Thus every J_l is expressible as a polynomial in J_1 and a_0 divided by a power of a_0. Thus we have the following theorem for those formal modular seminvariants of f which are congruent to rational integral algebraic seminvariants of f_0, f_1, $\cdots$ f_{q+1}.

THEOREM II. *Let* $f(a; x)$ *be any binary form of order* q *and consider*

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[*] S. T., pp. 301-303; A. I., § 5.

its formal modular seminvariants with respect to the Galois field, $GF[p^n]$, *of order* p^n. *If* q *is not divisible by* p, *then a complete set of formal modular protomorphs consists of a complete set of algebraic protomorphs of* f, *together with* $a_0 a_1^{p^n} - a_0^{p^n} a_1$.

4. *General case.* To prove this theorem for the general case, we can proceed precisely as in the article by W. L. G. Williams cited above, using the A_i instead of the C_i. Or we can proceed as in the proof of the corresponding theorem for algebraic seminvariants.

In the second of these proofs, we note that since the formula for A_i can be solved for a_i in terms of A_i and the a_j $(j < i)$, we can readily express any formal modular seminvariant, S, as a polynomial, S_1, in the A_i and a_0, a_1 divided by a power of a_0. Now S_1 can involve a_1 only through the function $a_1^{p^n} - a_0^{p^n-1} a_1$. For, since S_1 is identically equal to S, it is annihilated modulo p by Ω' and thus, if

$$S_1 = a_0^{-\mu} F(a_0, a_1, A_2, \cdots, A_q)$$

where F is a polynomial in the arguments, then

$$\Omega' S_1 = a_0^{-\mu} \left[q\delta F + \sum_{ij} C_i a_i \frac{\partial F}{\partial A_j} \frac{\partial A_j}{\partial a_{i+1}} \right]$$

where each C_i is a numerical coefficient, where

$$\delta = a_0 \frac{\partial}{\partial a_1} + a_0^{[n]} \frac{\partial}{a_1^{[n]}} + \cdots + a_0^{[(m+1)n]} \frac{\partial}{\partial a_1^{[(m+1)n]}}$$

and where i, j range over the values 1, 2, 3, $\cdots$, q. Hence

$$\Omega' S_1 = q\, a_0^{-\mu}\, \delta F \equiv 0 \qquad (\text{mod } p).$$

In this connection, we need

LEMMA 2. *Let* F *be a polynomial in* a_0, a_1, *and variables* A_2, A_3, *etc. in which* a_0 *and* a_1 *are the first two coefficients of the binary form* f *of order* q, *but in which* A_2, A_3, *etc. may be any variables which may assume any set of values, independent of the values assumed by* a_0 *and* a_1. *Then* F *is annihilated by the differential operator* δ *given above if and only if* F *is expressible as a polynomial in* a_0, *the* A_i *and* $a_1^{p^n} - a_0^{p^n-1} a_1$ *divided by a positive integral power of* a_0.

Let $\quad t_0 = a_0^{l_0} a_1^{K_0} \phi_0(A_2, \cdots, A_q)$

where $\quad K_0 = k_0 + k_0' p^n + k_0'' p^{2n} + \cdots$

be the term of F which is of lowest degree in a_0 and let k be a non-negative integer less than p^n. The result of operating on t with $a_0 \dfrac{\partial}{\partial a_1}$ is congruent to

$$(1) \qquad k_0\, a_0^{l_0+1}\, a_1^{K_1}\, \phi_0$$

where $\qquad K_1 = (k_0 - 1) + k'_0 p^n + k_0'' p^{2n} + \cdots$

and, accordingly, since F is annihilated modulo p by δ, the F must contain a term t_1 which is such that, when t_1 is operated on by $a_0^{p^n} \dfrac{\partial}{\partial a_1^{p^{2n}}}$, or by $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^n}}$ etc. it produces precisely (1) above. Hence (1) would have to come from a term whose degree in a_0 is at most $l_0 - (p^n - 1)$, which is contrary to the definitnon of t_0. Hence $k_0 = 0$.

But the result of operating on t_0 with $a_0^{p^n} \dfrac{\partial}{\partial a_1^{p^n}}$ is

$$(2) \qquad k_0'\, a_0^{l_0+p^n}\, a_1^{K_2}\phi_0$$

where

$$K_2 = (k_0' - 1) p^n + k_0'' p^{2n} + \cdots.$$

Since F is annihilated by δ, F must contain a term t_2 which is such that, when t_2 is operated upon by $a_0 \dfrac{\partial}{\partial a_1}$ or $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^{2n}}}$ or etc., it provides (2). But, since t_0 is the term of lowest degree in a_0, none of these alternatives is possible except the first, and thus

$$t_2 = - k_0'\, a_0^{l_0+p^n-1}\, a_1^{K_3}\phi_0$$

where

$$K_3 = 1 + (k_0' - 1) p^n + k_0'' p^{2n} + \cdots.$$

$I \neq k_0' \neq 0$, this means that the term of F which is of lowest degree in a_0 is accounted for by

$$t_0 + (1/k_0') t_2 = (a_1^{p^n} - a_0^{p^n-1}\, a_1)\, a_0^{l_0}\, a_1^{K_2}\phi_0;$$

whereas, if $k_0' = 0$, we proceed to consider the result of operating on t_0 with $a_0^{p^{2n}} \dfrac{\partial}{\partial a_1^{p^{2n}}}$,

$$\text{Now } \delta[t_0 + (1/k_0') t_2] = [\delta(a_1^{p^n} - a_0^{p^n-1} a_1)]\,[a_0^{l_0} a_1^{K_2}\phi_0] \\ + [a_1^{p^n} - a_0^{p^n-1} a_1)]\, \delta\, [a_0^{l_0} a_1^{K_2}\phi_0]$$

But $\delta(a_1^{p^n} - a_0^{p^n-1}a_1) = 0$ and

$$\delta(a_0^{l_0}\, a_1^{K_2}\phi_0)$$
$$= [(k_0' - 1)a_0^{l_0+p^n}a_1^{K_2-p^n} + k_0''a_0^{l_0+p^{2n}}a_1^{K_2-p^{2n}}$$
$$+ k_0'''a_0^{l_0+p^{3n}}a^{K_2-p^{3n}} + \cdots]\, \phi_0 .$$

Thus δ operating on $t_0 + (1/k_0')t_2$ gives rise only to terms whose degree in a_0 is higher than that of t_0 and not less than that of t_2. Moreover, any such term is of the form

$$t_3 = C\, a_0^{l_1} + a_1^{K_1'}\phi_0$$

where $l_1 = l_0 + p^{in}$ and $K_1' = k_1'p^n + k_1''p^{2n} + \cdots + k_1^{(m)}p^{mn}$, or is of the form $a_0^{p^n-1}a_1t_3$. Here i and each of the k' is a non-negative integer subject to the restrictions that $1 \leqq i \leqq m$ and $k_1^{(j)} \leqq k_0^{(j)}$, and C is a constant. Note that t_3 (or $a_0^{p^n-1}a_1t_3$) arises from $t_0 + 1/k_0'\ t_2$ only by operating with $a_0^{[i]}\ \dfrac{\partial}{\partial a_1^{[i]}}$.

Since F is annihilated by δ, then the totality F_1 of terms of F not accounted for by $t_0 + (1/k_0')t_2$ must be annihilated by δ. Now proceed with F_1 as we did with F. Since the degree of F_1 in a_0 is higher than the degree of F in a_0, then at the end of a finite number of steps we arrive at a pair τ of terms of the same general type as $t_0 + (1/k_0')t_2$ of which the term of the highest degree in a_0 is of degree 1 in a_1. By the foregoing argument, this and the pairs of terms obtained by the preceding argument exhaust all terms of F except that conceivably it may not account for precisely all the coefficient of the term in the highest power of a_0. But if there were a term in the highest power of a_0—say t_5—which is not accounted for by the pair τ, then when we operate on t_5 by δ we should get a term, s_5, independent of a_1. Now the only way that s_5 can be cancelled is by $-s_5$ arising from differentiating a term t_6 of F. By the foregoing argument, t_6 must be of higher degree than t_5 in a_0. Since t_5 is a term of F which is of the lowest degree in a_1, this is impossible and thus the last pair τ precisely exhausts all remaining terms of S. Combining results, we see that the totality of terms of F which contain a_1 as factor is divisible by $a_1^{p^n} - a_0^{p^n-1}a_1$.

Moreover, since

$$(3)\quad \delta(a_1^{p^n} - a_0^{p^n-1}a_1)Q = Q\delta\,(a_1^{p^n} - a_0^{p^n-1}) + (a_1^{p^n} - a_0^{p^n-1}a_1)\delta Q\,,$$
$$= (a_1^{p^n} - a_0^{p^n-1}a_1)\delta Q\,,$$

it follows that the quotient, Q, is annihilated by δ. By induction, F is a polynomial in a_0, A_2, A_3, etc. and $a_1^{p^n} - a_0^{p^n-1}a_1$, and the lemma is proved.

Hence we at once have Theorem II for a single binary form. But since the protomorphs A_i are all expressible as rational functions of the S_j where the S_j form a complete set of algebraic protomorphs of f, we have

COROLLARY I. *Under the hypothesis of the theorem, a complete set of formal modular protomorphs of f is any complete set of algebraic protomorphs and* $a_1^{p^n} - a_0^{p^n-1}a_1$.

Similarly, we have

THEOREM III. *Let* $f(a;x), g(b;x), h(c;x), \cdots$ *be any system of binary forms and consider the formal modular seminvariants of this system with respect to the Galois field,* $GF[p^n]$, *of order* p^n. *If the order of no one of the ground forms is divisible by p, then a complete set of formal modular protomorphs is any complete set of algebraic protomorphs of the system together with* $a_0a_1p^{p^n} - a_0^{p^n}a_1$.

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Irreducible Continuous Curves.[*]

BY HARRY MERRILL GEHMAN.

A point set M is said to be *an irreducible continuous curve about a point set A*, if M is a continuous curve and contains A, but contains no proper subset which is a continuous curve and contains A. In general, a point set M is said to be *irreducible* with respect to a property X, if M has property X, but contains no proper subset which has property X.

In the present paper, we shall first show that a set which is both a continuous curve and an irreducible continuum about a point set A,[†] is also an irreducible continuous curve about A. Since every continuum is an irreducible continuum about itself, it follows that if the set A is properly selected, *every* continuous curve is an irreducible continuous curve about a set A. We are therefore principally concerned in this paper with some properties of the set A for a given continuous curve (Theorems 2, 3, and 4), and with the possibility of constructing a continuous curve which is irreducible about a given set A (Theorems 5 and 6). Some rather important results concerning continuous curves are obtained incidentally (Corollaries 4B, 4C, and 4D).

THEOREM 1. *A point set which is both a continuous curve and an irreducible continuum about a point set A is also an irreducible continuous curve about A. Conversely, an irreducible continuous curve about a point set A is an irreducible continuum about A.*

Proof. If M is an irreducible continuum about a set A, then no proper subcontinuum of M contains A, whether the subcontinuum be a continuous curve or not. Therefore if M is also a continuous curve, it is an irreducible continuous curve about A.

Suppose the converse were not true, and there existed a set M which is an irreducible continuous curve about a set A, but is not an irreducible continuum about A. Then M contains a proper subcontinuum N containing A. Let P be a point of $M - N$, and let C be a circle about P whose interior contains no points of N. If X denotes the points of M in the interior of C, the set K which is the maximal connected subset of $M - X$ that contains N, is a continuous curve.* That is, M contains a proper subset K which is a continuous curve and contains A, which is impossible since we assumed that M was an irreducible continuous curve about A. Therefore, M is an irreducible continuum about A.

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[*] Presented to the American Mathematical Society, Feb. 27, 1926.

[†] This generalization of the notion of an irreducible continuum between two points is due to W. A. Wilson, "On the oscillation of a continuum at a point," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 429-440. See also H. M. Gehman, "Concerning irreducibly connected sets and irreducible continua," *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), pp. 544-547. We shall refer to this paper hereafter as I. C. S.

THEOREM 2. *Every continuous curve is an irreducible continuous curve about the set consisting of its non-cut-points.*†

Proof. In Theorem 4 of our paper I. C. S., we have proved that every bounded continuum M is an irreducible continuum about the set consisting of its non-cut-points. Therefore if M is a continuous curve, it follows from Theorem 1 that M is an irreducible continuous curve about the set consisting of its non-cut-points.

THEOREM 3. *A necessary and sufficient condition that a continuous curve M be an irreducible continuous curve about one of its subsets A, is that A'* ‡ *contain all the non-cut-points of M.*

Proof. Any continuum containing A will contain A'. Therefore, if A is any subset of M which is such that A' contains all the non-cut-points of M, then the only continuous curve lying in M that can contain A is M itself, by Theorem 2. Therefore the condition is sufficient.

The condition is necessary, for suppose M is an irreducible continuous curve about a set A, but some point P which is a non-cut-point of M, is not contained in A'. We shall show that this leads to a contradiction. There are two cases to be considered, according as P is an end-point of M, or a point of a simple closed curve in M.

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* H. M. Gehman, "Some relations between a continuous curve and its subsets," to appear in the *Annals of Mathematics*. See especially Theorem 6. We shall refer to this paper hereafter as S. R.

† If M is a connected point set, and P is a point of M, then if $M - P$ is not connected, P is said to be a *cut-point* of M; if $M - P$ is connected, P is said to be a *non-cut-point* of M. See R. L. Moore, "Concerning the cut-points of continuous curves, etc.", *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106.

‡ If A is any point set, A' denotes the set consisting of A and all other points which are limit points of A.

Case 1. Suppose P is an end-point * of M. Let ϵ be a positive number which is less than the distance from P to a point of A'. Then by Menger's definition, there exists a point Q which is such that M can be expressed as the sum of two continuous curves † having only Q in common, one of which contains P but no point of A', and the other contains A' but not P. In that case, a proper subset of M is a continuous curve and contains A, which is contrary to our supposition that M is an irreducible continuous curve about A.

Case 2. Suppose P is a point of a simple closed curve J of M. Let C_1 be a circle about P as center, excluding all points of A', and let C_2 be a circle about P as center whose radius is less than that of C_1. Let QR be an arc of J lying entirely within C_2, and let N be the maximal connected subset of M in C_2 plus its interior, that contains QR. As we have shown before, N is a continuous curve, and therefore $M - N$ contains only a finite number of maximal connected subsets containing points of A, because only a finite number of these subsets can be of diameter greater than the difference in radii of C_1 and C_2.‡ Let these be $D_1, \cdots, D_k$. By Theorem 7 of our paper S. R., the set D_i' is a continuous curve. Each of these continuous curves has a point in common with N, and every such common point is a point of C_2. Let P_i be a point common to D_i' and N, and for each point P_i select a definite arc P_iQ_i in N, such that Q_i is the only point that the arc has in common with QR. The set $D_1' + D_2' + \cdots + D_k' + P_1Q_1 + P_2Q_2 + \cdots + P_kQ_k + QR$ is a continuous curve containing A, and must therefore be identical with M. In that case, the points of M in the interior of C_2 lie on a finite set of arcs, and we can select two points X and Y of QR such that there are no points of $Q_1 + Q_2 + \cdots + Q_k$ between them. The set $(M - XY)'$ is a continuous curve (by Theorem 7 of S. R.) which contains A and is a proper

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* We are using *end-point* here in the sense defined by R. L. Wilder, "Concerning continuous curves," *Fundamenta Mathematicae*, Vol. 7 (1925), pp. 340-377. For the equivalence of Wilder's definition with a modification of one due to Menger, see Theorem 2 of our forthcoming paper entitled, "Concerning end-points of continuous curves and other continua." The modification of Menger's definition may be stated thus: A point P of a continuous curve M is said to be an end-point of M, if given any positive number ϵ, there exists a simple closed curve of diameter less than ϵ, enclosing P, and having only one point in common with M. See Karl Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1925), pp. 277-306. The theorem that an end-point of a continuous curve M is a non-cut-point of M which belongs to no simple closed curve in M is due to W. L. Ayres.

† This follows from Theorem 6 of our paper S. R.

‡ W. L. Ayres, "Concerning the arcs and domains of a continuous curve," (abstract), *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 37.

subset of M. But this is contrary to our supposition concerning M. Therefore the condition is necessary.

Definition. If a set M is an irreducible continuous curve about a closed set A, but not about any proper closed subset of A, then A is said to be a *basic closed set* about which M is an irreducible continuous curve.

From this definition and Theorem 3, we have the following corollary.

COROLLARY 3A. The only basic closed set about which a continuous curve is an irreducible continuous curve, is the set consisting of all the non-cut-points of the continuous curve and all points which are limit points of non-cut-points.

COROLLARY 3B. If a set M is an irreducible continuous curve about a closed set A, then M is irreducibly connected about A.

Proof. By Theorem 3, if M is an irreducible continuous curve about a closed set A, then A contains all the non-cut-points of M. By Theorem 2 of our paper I. C. S., the set M is irreducibly connected about any set that contains all the non-cut-points of M, and therefore M is irreducibly connected about A.

THEOREM 4. *If a set M is an irreducible continuous curve about a set A, then every subcontinuum K of M is connected im kleinen at every point not in A', and if K contains every point of a maximal connected subset B of A', then K is also connected im kleinen at every point of B.*

Proof. Suppose K were not connected *im kleinen* at a point P of $M - A'$. Then we can construct a circle C about P excluding all points of A'. Within C there exists the state of affairs described in R. L. Moore's characterization of continua which are not continuous curves,* and by the argument given in the proof of Theorem 1 of our paper S. R., it follows that M contains two arcs $P'P''$, $Q'Q''$ which have no points in common, are interior to C, and are such that each of them has points in common with each of the sets M_i for i equal to or greater than some fixed number n, each of the sets M_i being an arc. Therefore the set $P'P'' + Q'Q'' + M_n + M_{n+1}$ contains a simple closed curve J which is interior to C. Every point of a simple closed curve in M is either a non-cut-point of M, or a limit point of non-cut-points, † and by Theorem 3, all such points must be points of A'. Therefore A' contains J. But this is impossible, as all points of A' are exterior to C, and J is interior to C. Therefore K is connected *im kleinen* at every point of $M - A'$.

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* R. L. Moore, "Report on continuous curves from the viewpoint of analysis situs," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 289-302.

† R. L. Moore, "Concerning the cut-points of continuous curves, etc.", *loc. cit.*, Theorem B*.

Suppose K contains every point of a maximal connected subset B of A', but is not connected *im kleinen* at a point P of B. Then there exists in K the state of affairs described in the preceding paragraph, where the continuum of condensation W (or $\bar{M}$, in Moore's notation) at points of which K is not connected *im kleinen*, contains P. Since we have shown that K is connected *im kleinen* at every point not in A', it follows that W is a subset of A'. Since W contains P, the continuum W is a subset of B.

If on the arc $P'P''$ [*] we select any point X different from P'', there is a point P_i of one of the sets $M_i (i > n)$ on the arc XP''. The set $P'P_i + M_n + M_i + Q'Q''$ contains a simple closed curve containing the arc $P'P_i$. As we have shown above, such an arc lies in A', and since X was an arbitrary point of $P'P''$, it follows that the entire arc $P'P''$ lies in A'. Since the arc $P'P''$ has the point P'' in common with W, the arc $P'P''$ is also a subset of B and therefore of K. But this contradicts part (2) of Moore's characterization. Therefore K is connected *im kleinen* at every point of B.

COROLLARY 4A. If M is an irreducible continuous curve about a set A, every maximal connected subset of A' is a continuous curve, and not more than a finite number of maximal connected subsets of A' are of diameter greater than any given positive number.

Proof. That every maximal connected subset of A' is a continuous curve, follows directly from Theorem 4.

Suppose that an infinite number of these continuous curves were of diameter greater than some positive number ϵ. Then let us construct in each of these, an arc of diameter greater than ϵ, thus obtaining an infinite collection of arcs, such that no arc in A' joins points of any two of them. From this collection let us select a sequence of arcs approaching a limiting set W.[†] Since M is connected *im kleinen* at every point of W, there exists in M an arc $P'P''$ which has only P'' in common with W, but which has points in common with infinitely many of the arcs of the given sequence. Then, as was pointed out in the proof of Theorem 4, the arc $P'P''$ belongs to A'. But this contradicts our supposition that the arcs were constructed so that no arc in A' joins points of any two of them.

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[*] The point P'' is the only point which this arc has in common with W.

[†] Compare pp. 40-41 of H. M. Gehman, "Concerning the subsets of a plane continuous curve," *Annals of Mathematics*, Vol. 27 (1925), pp. 29-46.

COROLLARY 4B. If H denotes the set of all non-cut-points of a continuous curve, then every maximal connected subset of H' is a continuous curve, and not more than a finite number of maximal connected subsets of H' are of diameter greater than any given positive number.

Proof. The continuous curve is irreducible about H' by Theorem 3, and then the conclusion follows by Corollary 4a.

The following example shows that if K denotes a maximal connected subset of the non-cut-points of a continuous curve M, it is not necessarily true that K' is a continuous curve.

Let M_1 denote the lines $y=0$ and $y=1/2^n$ $(n=0, 1, 2, \cdots)$ between $x=0$ and $x=1$. Let M_2 denote the lines $x=k/2^n$ $(n=0, 1, 2, \cdots$, and $k=0, 1, \cdots, 2^n)$ between $y=0$ and $y=1/2^n$. Let M_3 denote the lines $y=5/2^{n+3}$ and $y=7/2^{n+3} \cdot (n=0, 1, 2, \cdots)$ between $x=k/2^n$ and $x=(k/2^n)+(1/2^{n+3})$, for $k=1, 2, \cdots, 2^n$. The set $M=M_1+M_2+M_3$ is a continuous curve, and every point of $M-M_3$ is a non-cut-point of M, while every point of M_3 is a cut-point of M except the points whose x-coordinates are $(k/2^n)+(1/2^{n+3})$ for the values of n and k given above. If K is the maximal connected subset of the non-cut-points of M, that contains the point $(1, 1)$, the set K' evidently contains the points $(1, 0)$ and $(1, 1/2^n)$, for $n=0, 1, 2, \cdots$. But no two of these points can be joined by an arc in K' unless the arc has points in common with the line $x=0$. Therefore K' is not connected *im kleinen* at the point $(1, 0)$, and is therefore not a continuous curve. On the other hand, the maximal connected subset of H' that contains the point $(1, 1)$; is the set M_1+M_2 which is a continuous curve.

COROLLARY 4C. If M is an irreducible continuous curve about a set A, and if every subcontinuum of A' is a continuous curve, then every subcontinuum of M is a continuous curve.

Proof. If K is any subcontinuum of M, then K is connected *im kleinen* at every point of $M-A'$, by Theorem 4. If K is not connected *im kleinen* at a point of A', there exists in K the state of affairs described in the proof of Theorem 4. As we pointed out there, each of the arcs M_i (for i greater than n) contains a subarc A_iB_i which has only A_i in common with $P'P''$ and only B_i in common with $Q'Q''$. Each of these arcs lies in a simple closed curve in M, and is therefore contained in A'. Since $P'P''$ is also in A', the infinite collection of arcs A_iB_i lies in one maximal connected subset B of A'. But in that case, some subcontinuum of B is not a continuous curve,* which is contrary to hypothesis.

COROLLARY 4D. A necessary and sufficient condition that every subcontinuum of a continuous curve M be a continuous curve, is that every subcontinuum of H' be a continuous curve, where H denotes the set of all non-cut-points of M.

Proof. The necessity of the condition is obvious. The sufficiency of the condition follows from Theorem 3 and Corollary 4c.

THEOREM 5. *If A is a subset of a continuous curve S, then a necessary and sufficient condition that S contain an irreducible continuous curve about A, is that every maximal connected subset of A' be a continuous curve and that not more than a finite number of the maximal connected subsets of A' be of diameter greater than any given positive number.*

Proof. The condition is necessary, for if S contains a continuous curve M which is irreducible about a set A, then A' satisfies the given condition by Corollary 4a.

To show that the condition is sufficient, we shall indicate how to construct in S a continuous curve M which is irreducible about a given set A satisfying the condition. In our paper, *Concerning acyclic continuous curves*,† we give a method (in the proof of Theorem 4) for constructing an acyclic continuous curve about a certain subset of a continuous curve. This method bears a close relation to the proof of Theorem 1 of our paper, *On extending a continuous* $(1-1)$ *correspondence of two plane continuous curves to a correspondence of their planes*.‡ If we modify this method so that it bears a similar relation to Theorem 2 of the paper just mentioned, we obtain a method for constructing in S a continuous curve M containing A. Since the method is such that every point of $M-A'$ is a cut-point of M, it follows from Theorem 3 that the continuous curve M is irreducible about A.

THEOREM 6. *A necessary and sufficient condition that a bounded con-*

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* H. M. Gehman, "Concerning the subsets of a plane continuous curve," *loc. cit.*, Theorem V.

† To appear in the *Transactions of the American Mathematical Society*. An acyclic continuous curve is one which contains no simple closed curve.

‡ *Transactions of the American Mathematical Society*, Vol. 28 (1926), pp. 252-265.

tinuum S contain an irreducible continuous curve about every subset of S is that every continuum of S be a continuous curve.

Proof. The condition is sufficient, because if A is any subset of S, then S contains a set M which is an irreducible continuum about A.* If every subcontinuum of S is a continuous curve, then M is a continuous curve, and therefore is an irreducible continuous curve about A, by Theorem 1.

Also, the condition is sufficient, because if every subcontinuum of S is a continuous curve, every subset A of S will be such that A' satisfies the conditions of Theorem 5,† and therefore S will contain an irreducible continuous curve about A.

The necessity of the condition will follow as a corollary of Theorem 7.

THEOREM 7. *If a bounded continuum S contains an irreducible continuous curve about every closed disconnected subset A', then every subcontinuum of S is a continuous curve.*

Proof. Let K be any proper subcontinuum of S, and let P be a point of $M - K$. The set $A' = K + P$ is closed and disconnected, and therefore S contains an irreducible continuous curve about A', by hypothesis. Since the set K is a maximal connected subset of A', it is a continuous curve, by Corollary 4a. Therefore every proper subcontinuum of S is a continuous curve. It remains to be shown that S itself is a continuous curve.

If S is an irreducible continuum about a set A' consisting of two points P and Q, then by Theorem 1, S itself is the only subcontinuum of S which can be an irreducible continuous curve about the set A'. Therefore S is a continuous curve.

If S is not an irreducible continuum about any set A' consisting of two points P and Q, then S is decomposable, and by Theorem 2 of our paper S. R., the continuum S is a continuous curve, since we have proved above that every proper subcontinuum of S is a continuous curve.

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* W. A. Wilson, *loc. cit.*, p. 433.

† H. M. Gehman, "Concerning the subsets of a plane continuous curve," *loc. cit.*, Theorem V.

The Plane Quintic With Five Cusps.

By Marguerite Lehr.

The quintic with five cusps (characterized by the Plücker numbers $m = n = \kappa = i = 5;\ \nu = \tau = 0;\ p = 1$) has been considered by del Pezzo,* Field † and Basset.‡ Field gives a general descriptive account of the appearance of the curve under various conditions on the coefficients in its equation; Basset mentions it as the limiting case under quintics with five nodes; but neither paper gives a detailed study of the curve. In such a study the del Pezzo work is fundamental. Starting with the fact that a quintic with five cusps may be obtained by quadratic transformation from a quartic with two cusps, in- and circumscribed to the triangle of reference, he proves that the quintic is uniquely determined by its five cusps. Considering the two cusps of the quartic as the two intersections of the line $\sum\limits_{xyz} x = 0$ with the conic $\sum\limits_{\substack{xyz\\abc}} ayz = 0$ circumscribed to the triangle of reference ABC, he shows that the quartic is uniquely determined by the conditions for contact with $x = 0$, $y = 0$, $z = 0$, respectively, and passage through A, B and C. By means of a quadratic transformation, he obtains the equation of the quintic with cusps at $A\,B\,C$ and at K_1K_2, the intersections of the line $\Sigma\, \alpha x = 0$ with the conic $\Sigma\, yz = 0$, in the form: §

$$(1)\qquad 16xyz(\Sigma\, \alpha x)^2 - 4(\Sigma\, \alpha x)(\Sigma\, yz)(\Sigma\, \alpha yz) - (\Sigma\, yz)^2 \Sigma\, (b - c)^2 x = 0$$

where $\alpha = -a + b + c;\ \beta = a - b + c;\ \gamma = a + b - c$. All the forms involved are symmetric in x, y, z and a, b, c. Write

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* P. del Pezzo, "Equazione d'una curva del quinto ordine dotata di cinque cuspidi," *Rendiconti dell' Academia delle Scienze Fisiche e Matematiche* (*di Napoli*), Serie 2; Volume 3, (1889), pp. 46-49.

† Peter Field, "On the Form of a Plane Quintic with Five Cusps," *Transactions of the American Mathematical Society*, Volume 7, (1906), pp. 26-32.

‡ A. B. Basset, "On Quinquenodal and Sexnodal Quintics," *Quarterly Journal of Mathematics*, Volume 37, pp. 199-214; "On Quintic Curves with Four Cusps," *Rendiconti del Circolo Matematico di Palermo*, Volume 26 (1908), pp. 332-335.

§ Slight inaccuracies in some of the forms in the del Pezzo article have been corrected here.

$$\sum (b-c)^2 = l$$
$$\sum ax = k$$
$$\sum yz = \phi$$
$$\sum ayz = \psi.$$

The equation then assumes the form:

$$(2) \qquad 16xyzk^2 - 4k\phi\psi - \phi^2 l = 0.$$

Del Pezzo gives also the cuspidal tangents at A, B, C, the typical form being the equation of the cuspidal tangent at A:

$$(3) \qquad y(\gamma + a) = z(\beta + a).$$

This completes the account of results obtained in his paper.

In the present investigation, which takes as starting point the del Pezzo work, it is proved that if five points in the plane are given as cusps on a quintic, the curve thus uniquely determined is unipartite, with the five cusps and five inflexions occurring alternately. Moreover, it is shown that from the five given points the cuspidal tangents, the inflexions, the inflexional tangents, as well as a series of ordinary points on the quintic, may be obtained by linear constructions.

The method of treatment is based on a slight modification of the form of (2). Replace ψ and yz by:

$$\psi \equiv ayz + \beta zx + \gamma xy = a\phi - 2xh$$

where $h \equiv (c-a)y + (b-a)z$

$$yz = \phi - x(y+z) \equiv \phi - xg.$$

Then the equation for the quintic Q may be written:

$$(4) \qquad \phi^2(l + 4ak) - 8xk\phi(h + 2k) + 16x^2k^2g = 0.$$

Here the importance of the system of conics depending on ϕ and xk is suggested at once. Consequently, setting apart the cusp A, consider the pencil of conics through $B\,C\,K_1K_2$.

The Parametric Treatment of the Quintic. Five given points are to be made cusps on a quintic, which is then uniquely determined. Take as $B\,A\,C$, triangle of reference, any set of three cusps adjacent in that order on the conic ϕ determined by the five points.* The two remaining points determine the line $k \equiv \Sigma\, ax = 0$. No restriction is imposed on the curve either by such a choice of $A\,B\,C$, or by writing ϕ as $\Sigma\, yz = 0$.

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* For the analytical work, it is not necessary that BAC be adjacent on ϕ; for the interpretation, however, such a choice is convenient since it gives a, b, c all positive.

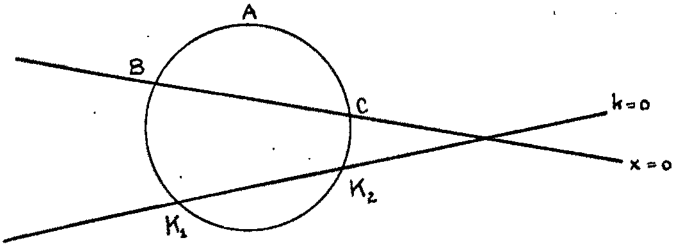

FIG. 1.

The equation of the quintic is then equation (4), where a, b, c are all positive. The cusps $K_1\,K_2$ are given by the lines AK_1, AK_2:

$$by^2 + ayz + cz^2 = 0.$$

The reality of K_1K_2 depends on $a^2 - 4bc$; call this σ^2.

Any conic of the pencil $S_2(\lambda)$

$$\lambda\phi - 4xk = 0 \tag{5}$$

has ten intersections with the quintic, of which eight are fixed at the cusps $B\,C\,K_1K_2$; the two remaining variable intersections are cut out on the conics by the singly-infinite quadratic system of lines $S_1(\lambda^2)$:

$$(l + 4ak) - 2\lambda(h + 2k) + \lambda^2 g = 0 \tag{6}$$

obtained by direct substitution of (5) in (4). To each λ-conic with a given λ-value there corresponds a λ-line, which intersects the conic in two points on Q. The lines (6) envelope a conic Θ:

$$(h + 2k)^2 - g(l + 4ak) = 0$$

which reduces to

$$(b - c)^2\phi - 4axk = 0 \tag{7}$$

or

$$\delta^2\phi - 4axk = 0 \qquad \text{where } \delta = b - c,$$

and is therefore a λ-conic,* corresponding to the value $\lambda = (b - c)^2/a$. The λ-line corresponding to this value is:

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* If $B\,C\,K_1K_2$ are ordinary nodes, the conics through them have eight fixed points of intersection with the Q in question and two variable intersections given by a system of lines; this point of view I hope to develop at some other time. The specialization in this case is the passage of the envelope of the lines through $B\,C\,K_1K_2$, which are therefore cusps.

(8a) $$l + 4ak - 2(\delta^2/a)(h + 2k) + (\delta^4/a^2)g = 0$$

or

(8b) $$x(4a\alpha - 3\delta^2) + y(\alpha + b - \delta^2/a)^2 + z(\alpha + c - \delta^2/a)^2 = 0.$$

The two points which this line j cuts out on Θ are points on Q; j is tangent to Θ, being a λ-line, and is therefore tangent to Q.

For all points on Θ

$$\frac{l + 4ak}{h + 2k} = \frac{h + 2k}{g} = \rho.$$

The points of contact of λ-lines (6) are given by the value $\rho = \lambda$. In particular, for the point of contact of the line j

(9) $$\frac{l + 4ak}{h + 2k} = \frac{h + 2k}{g} = \frac{\delta^2}{a}.$$

The line j is a special line in the system $S_1(\lambda^2)$; it may be expected to have a special relation to Q. In (4) write

$$l + 4ak = 2(\delta^2/a)(h + 2k) - (\delta^4/a^2)g.$$

The resulting equation

$$[(\delta^2/a)\phi - 4xk]\,\{2\phi(h + 2k) - g[(\delta^2/a)\phi + 4xk]\} = 0$$

gives intersections of Q with the line j. The quadratic factor is accounted for—the line j is tangent to Q on Θ. The cubic cuts out the three remaining points on the line. This cubic meets Θ, i. e. $(\delta^2/a)\phi - 4xk = 0$ on ϕ (at $B\,C\,K_1K_2$) and on the line

$$a(h + 2k) - \delta^2 g = 0;$$

that is, the cubic passes through the point on Θ given by:

$$\frac{h + 2k}{g} = \frac{\delta^2}{a}.$$

This point, however, is the point of contact of j with Θ. The cubic, therefore, cuts the line j at its point of contact with Θ; the line j has three intersections with Q at that point. All singularities are accounted for at the cusps; therefore the line j is an inflexional tangent of Q, and its point of contact with Θ is an inflexion on Q. The cusp A, then, is associated with one of the inflexions by the system of conics through the remaining four cusps. The equation of j, more properly called j_A, is:

(10a) $$a^2x(4a\alpha - 3\delta^2) + y(\sigma^2 + ac)^2 + z(\sigma^2 + ab)^2 = 0.$$

The corresponding inflexion I_A has coordinates

$$x:y:z=\frac{\delta}{a}:-\frac{\sigma^2+a^2+2ac}{\sigma^2+ac}:\frac{\sigma^2+a^2+2ab}{\sigma^2+ab}.$$

In the same way, each cusp is connected with an inflexion by the system of conics through the remaining four cusps; e. g. j_B is given by:

$$x(\sigma^2+bc)^2+b^2y\left[4b\beta-3(c-a)^2\right]+z(\sigma^2+ab)^2=0 \tag{10b}$$

with the corresponding inflexion I_B

$$x:y:z=\frac{\sigma^2+b^2+2bc}{\sigma^2+bc}:\frac{c-a}{b}:-\frac{\sigma^2+b^2+2ab}{\sigma^2+ab}$$

The equation of j_{K_1} is most easily obtained by transforming the triangle of reference to $A\,K_1K_2$, so that the conic $\Sigma\,yz=0$ becomes the conic $\Sigma\,y'z'=0$. The points K_1K_2 are so named that K_1 is adjacent to B on ϕ; the coordinates are

$$\begin{aligned} K_1 &: x:y:z=-2bc:c(\gamma+\sigma):b(\beta-\sigma)\\ K_2 &: x:y:z=-2bc:c(\gamma-\sigma):b(\beta+\sigma). \end{aligned}$$

The equations of transformation are:

$$\begin{array}{lll} & \sigma x'=ax+by+cz & \text{or } k\\ (11) & 2ay'=(\gamma+\sigma)y+(\sigma-\beta)z & \text{or } AK_2\\ & 2az'=(-\gamma+\sigma)y+(\beta+\sigma)z & \text{or } AK_1 \end{array}$$

The new $a'\ b'\ c'$ are the coefficients in the equation of BC referred to the new triangle. This is:

$$-4\sigma^2x'+(\gamma-\sigma)(\beta+\sigma)y'+(\gamma+\sigma)(\beta-\sigma)z'=0.$$

So the required values are

$$\begin{array}{lll} & a'=-4\sigma^2 & \alpha'=-4a\alpha\\ (12) & b'=(\beta+\sigma)(\gamma-\sigma) & \beta'=-4\sigma(\sigma+\delta)\\ & c'=(\beta-\sigma)(\gamma+\sigma) & \gamma'=4\sigma(-\sigma+\delta). \end{array}$$

The equation of j_{K_1} is now given by (10b), with $x'\ y'\ z'$ and $a'\ b'\ c'$ written for $x\,y\,z$ and $a\,b\,c$ respectively,

$$x'(\sigma'^2+b'c')^2+b'^2y'\left[4b'\beta'-3(c'-a')^2\right]+z'(\sigma'^2+a'b')^2=0$$

with equations (11) and (12) as equations of the transformation. The corresponding inflexion I_{K_2} is obtained from I_B by a similar substitution. Then j_{K_2} and I_{K_2} are obtained from j_{K_1} and I_{K_1} by changing the sign of σ.

The five inflexions and their inflexional tangents have been found by means of the systems of conics through four cusps. The cuspidal tangents may be expressed in terms of the same systems. If A_1 denote the conjugate of A with respect to $S_2(\lambda)$, i. e. the common point of the polars of A with respect to the conics of $S_2(\lambda)$, then t_A, the cuspidal tangent at A, is the polar of A_1 with respect to ϕ: for:—The polar of A with respect to ϕ is g; the polar with respect to xk is $ax + k$; the common point A_1 has coordinates

$$x : y : z = \delta : -2a : 2a$$

The polar of A_1 with respect to ϕ is:

$$y(2a + \delta) + z(-2a + \delta) = 0\,;$$

that is:

$$y(\gamma + \alpha) - z(\beta + \alpha) = 0$$

which is t_A.

Application of this method gives the equation of the cuspidal tangents at K_1 and K_2. The conjugate of K_1 with respect to the related system of conics through $A\,B\,C\,K_2$ is the common point of its polars with respect to any two conics of the system;

e. g.

$$\text{i)}\quad z[2ax + (\gamma + \sigma)y] = 0$$
$$\text{ii)}\quad y[2ax + (\beta + \sigma)z] = 0.$$

The polar of K_1 with respect to i) is:

$$ax \cdot 2b(\beta - \sigma) + by(\beta - \sigma)(\gamma + \sigma) - cz \cdot 2\sigma(\gamma + \sigma) = 0.$$

The polar with respect to ii) is:

$$ax \cdot 2c(\gamma + \sigma) - by \cdot 2\sigma(\beta - \sigma) + cz(\gamma + \sigma)\cdot(\beta - \sigma) = 0.$$

The common point has coordinates

$$x : y : z = \frac{(\gamma + \sigma)(\beta - \sigma) + 4\sigma^2}{-2a} : \frac{b(\beta - \sigma) + \sigma(\alpha - \sigma)}{b} : \frac{c(\gamma + \sigma) - \sigma(\alpha + \sigma)}{c}$$

The polar of this with respect to ϕ is:

$$2abc \sum ax - \sigma \sum a\alpha(b - c)x - \sigma^2 \sum a(b + c)x = 0.$$

This is the cuspidal tangent to Q at K_1. The cuspidal tangent at K_2, obtained by changing the sign of σ, is:

$$2abc \sum ax + \sigma \sum a\alpha(b - c)x - \sigma^2 \sum a(b + c)x = 0.$$

The results obtained may be restated:

The quintic is traced out by pairs of points obtained as the intersections of the conics of the singly-infinite linear system $S_2(\lambda)$ with the corresponding lines of the singly-infinite quadratic system $S_1(\lambda^2)$, all such lines being tangent to the conic Θ of the system $S_2(\lambda)$. For each value of λ there are determined:

1) a λ-conic of the pencil $\lambda\phi - 4axk = 0$;
2) a λ-line tangent to Θ;
3) therefore, two points (common to 1 and 2) on Q;
4) a λ-point (of contact of 2) on Θ.

Each point on Q has its λ-value and has a correspondent with respect to A, i. e. the second point with that same λ-value. Similarly it has a μ-value and a correspondent with respect to B, i. e. determined by means of the conics $\mu\phi - 4yk = 0$ through ACK_1K_2. For a given point $x'y'z'$,

$$\lambda = \frac{4x'k'}{\phi}\,; \qquad \mu = \frac{4y'k'}{\phi}$$

therefore

$$\lambda : \mu = x' : y'$$

A similar correspondence is set up for each of the five cusps. All investigations will be made with respect to the cusp A and then extended to $B\,C\,K_1K_2$.

The curve may be expressed parametrically by means of direct solution of $S_2(\lambda)$ with $S_1(\lambda^2)$. The lines through C thus obtained are given by the equation:

$$(13) \quad x^2[-\lambda(\beta + a)^2 - 4a(a - c)^2 + 4c\delta^2] + y^2[-\lambda(\lambda - a - b)^2] \\ + 2xy[\lambda^2(\beta + a) - \lambda(2aa - 3a\delta + b\delta) - 2\delta(a^2 - bc)] = 0.$$

These are pairs of lines joining C to a varying pair of corresponding points P_1P_2, common to a λ-conic and its λ-line. Important properties of the curve may be obtained from this system of lines without proceeding to the complete parametric expression. For the value $\lambda = \delta^2/a$, i. e. at the inflexion I_A, the two lines coincide, for the λ-line given by this value is tangent to its conic; thus I_A is its own correspondent. Similarly, for $\lambda = \infty$, the λ-line $g = 0$ is tangent to its conic ϕ, and the cusp A is its own correspondent. As λ changes continuously from ∞, the two lines CP_1, CP_2, move from coincidence at A, the point P_1 tracing out one branch from the cusp, the point P_2 tracing out the other branch. If this circuit is to be closed, the lines must again come to coincidence; if another circuit exists, another position of coincidence of P_1P_2 is necessary as a transition between imaginary and real values. In other words, the number of values of λ which give self-corresponding points determines how many circuits the curve has.

Such values of λ are given by the discriminant of the quadratic (13), set equal to zero. This discriminant is:

$$\begin{aligned}\Delta \equiv & [\lambda^2(\beta + a) - \lambda(2a\alpha - 3a\delta + b\delta) - 2\delta(\alpha^2 - bc)]^2 \\ & + [-\lambda(\beta + a)^2 - 4a(\alpha - c)^2 + 4c\delta^2]\,[\lambda(\lambda - \alpha - b)^2].\end{aligned}$$

Two known zeroes of this expression are $\lambda = (b - c)^2/a$, $\lambda = \infty$; the reduction is therefore simplified. The reduced form of Δ is:

$$(14) \qquad \Delta = 4(\delta^2 - a\lambda)\,[\lambda^2(\alpha^2 - 3bc) - \lambda\alpha(2\alpha^2 - 5bc) + (\alpha^2 - bc)^2].$$

The quadratic factor set equal to zero gives other λ-values which determine self-corresponding points on Q. The reality of such values depends on the discriminant of the quadratic in question, i. e. on

$$\alpha^2(2\alpha^2 - 5bc)^2 - 4(\alpha^2 - 3bc)(\alpha^2 - bc)^2$$

which reduces to

$$(15) \qquad -3b^2c^2(\alpha^2 - 4bc).$$

Consideration of the relative values of α, a, b, c will determine the sign of this discriminant. With $ABCK_1K_2$ placed as described, a, b, c are all positive. The points K_1K_2 are given as the intersections of k wth ϕ, i. e. by the lines AK_1, AK_2:

$$(16) \qquad by^2 + \alpha yz + cz^2 = 0.$$

If K_1K_2 are to be real,*

$$(17) \qquad \alpha^2 > 4bc.$$

Application of (17) to the discriminant (15) shows that the result is negative; the two λ-values given by the quadratic equation are imaginary. Hence there are only two values for λ giving self-corresponding points; there is therefore one circuit only.

The solution of equation (13) may be written in the form:

$$(18) \qquad Px = (T \pm R)y$$

where

$$\begin{aligned} P &= -\lambda(\beta + a)^2 - 4a(\alpha - c)^2 + 4c\delta^2 \\ T &= -\lambda^2(\beta + a) + \lambda(2a\alpha - 3a\delta + b\delta) + 2\delta(\alpha^2 - bc) \\ \Delta^{1/2} \equiv R &= + \{4(\delta^2 - a\lambda)\,[\lambda^2(\alpha^2 - 3bc) - \alpha\lambda(2\alpha^2 - 5bc) + (\alpha^2 - bc)^2]\}^{1/2}.\end{aligned}$$

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* The present considerations are limited, in the interests of directness of development, to the case of five real cusps.

This with the value for $x : z$ obtained from $S_1(\lambda^2)$ gives a complete parametric representation of the quintic.

$$x : y : z = (\lambda - \alpha - c)^2(T \pm R) : P(\lambda - \alpha - c)^2 : (T \pm R)(4a\lambda - 4a\alpha - \delta^2) - P(\lambda - \alpha - b)^2. \tag{19}$$

The general investigation of properties of the curve derived from equation (18) is greatly facilitated if the λ-values of special elements of interest are found first by direct use of $S_2(\lambda)$, $S_1(\lambda^2)$.

Conic ϕ: corresponds to the value $\lambda = \infty$; its λ-line is $g = 0$, which cuts out A, self-corresponding. The λ-point on Θ, i. e. the contact point of g with Θ is A_1, the conjugate point of A with respect to the pencil $S_2(\lambda)$; for A_1 is

$$x : y : z = \delta : -2a : 2a$$

which lies on Θ and on g; but g is a λ-line and therefore tangent to Θ. Its point of contact is then A_1. So Θ is determined as the conic of the pencil $S_2(\lambda)$ through A_1.

Conic Θ: corresponds to $\lambda = (b - c)^2/a$; its λ-line is j_A (the tangent at the inflexion I_A); the λ-point of contact on Θ is I_A, self-corresponding on Q.

Conic $xk = 0$: corresponds to $\lambda = 0$; its λ-line is $l + 4ak = 0$, which cuts out on x and on k respectively a point of Q, the fifth point on each, since both x and k have already four points of intersection with Q, viz., x two each at B and C, k two each at K_1 and K_2. These are the conics of importance; the other line pairs in the pencil are given by $\lambda = 2(\alpha - \sigma)$ and $\lambda = 2(\alpha + \sigma)$, but as only one line pair is needed, xk is selected for simplicity.

Naturally the distribution of the five cusps and five inflexions on the single circuit invites discussion. For this purpose, the λ-values which give $B\,C\,K_1K_2$ are necessary. For a given point on Q, in general, direct substitution in $S_2(\lambda)$ determines the proper λ-value and therefore fixes the λ-line. For the points BCK_1K_2 which are fixed for the pencil $S_2(\lambda)$, however, this process fails to determine λ; these are points of special interest on Q. Here the system $S_1(\lambda^2)$ may be used to advantage, for though in general two λ-lines pass through a point, these points, being on Θ, have only one λ-line passing through them, i. e. the tangent to Θ at the particular point in question. For B, then,

$$x(-4a\lambda + 4a\alpha + \delta^2) + y(\lambda - \alpha - b)^2 + z(\lambda - \alpha - c)^2 = 0$$

must pass through $(0, 1, 0)$; the λ-value for B is:

$$\lambda = \alpha + b.$$

The λ-conic is

$$(\alpha + b)\phi - 4xk = 0$$

which has as tangent at B

$$x(\gamma + b) = z(\alpha + b)$$

that is, the cuspidal tangent to Q at B, t_B. Thus B and its A-correspondent are given by the conic of the pencil tangent at B to t_B, and the tangent to Θ at B.

For C, symmetrically,

$$\lambda = \alpha + c$$

the λ-conic is tangent to t_C at C
the λ-line is tangent to Θ at C.

To obtain the λ-values corresponding to K_1K_2, another method is more easily handled. It would apply as well as B and C. Since K_1 is given on Θ by a λ-value (i. e. as the point of contact of a λ-line),

$$(h + 2k)/g = \lambda$$

must pass through K_1, the coordinates of which are known. The λ-value obtained is:

$$\lambda = (a\alpha - \beta\gamma - \sigma\delta)/a$$

The λ for K_2 (obtained by changing the sign of σ) is:

$$\lambda = (a\alpha - \beta\gamma + \sigma\delta)/a.$$

For $ABCK_1K_2$ designated as described, α is negative; β, γ are positive; δ, which is less than $b + c$ and therefore less than a, may be assumed positive, for since only the order BAC on ϕ is of importance, BC may be so named that $b > c$. The special case $b = c$ will be discussed at the end of this section. Now $a\alpha - \beta\gamma + \sigma\delta$ is negative, for $(\beta\gamma - a\alpha)^2 - \sigma^2\delta^2$ reduces to

$$4a^4 - 4a^3(b + c) + a^2(b + c)^2 + 3a^2\delta^2 + 2a(b + c)\delta^2.$$

Since $4a^4 - 4a^3(b + c)$ is $-4a^3\alpha$, which is positive, and all the other terms are positive,

$$|\beta\gamma - a\alpha| > |\sigma\delta|$$

so λ_{K_2} is negative. The λ for K_1 is negative and greater than λ_{K_2} in absolute value.

For points on Q in the neighborhood of A, x is positive and k is positive. Since Q is unipartite, consisting of one infinite branch, it lies entirely outside ϕ, (it has ten intersections with ϕ fixed at the cusps and so cannot cross ϕ).

Consequently, for points on Q in the neighborhood of A, ϕ is negative. Now

$$\lambda\phi = 4xk$$

therefore λ for such points is negative.

As λ varies from ∞ at A through negative values, the points of intersection of the λ-conic and λ-line trace out the two branches from the cusp A. The λ-values for both K_1 and K_2 are negative. Further, with $B\,C$ so named that $b > c$, λ_C is negative, for

$$a^2 > 4bc > 4c^2; \qquad |\,a\,| > 2c$$

so $\lambda = a + c$ is negative. The λ for B

$$\lambda_B = a + b = (\delta^2 - \sigma^2 - ac)/a < \delta^2/a$$

may be negative, or positive less than δ^2/a. On each branch from A, therefore, λ passes through negative values, zero, positive values to $\lambda = (b - c)^2/a$ the order being $\lambda_A\,\lambda_{K_1}\,\lambda_{K_2}\,\lambda_C\,\lambda_B\,\lambda_{I_A}$, the two branches connecting at I_A to complete the single circuit. This apparently does not determine on which branch the cusps lie; it may be that all the cusps lie on one branch between A and I_A, and their correspondents lie on the other. The sequence of λ values does show that whatever the order of $B\,B'\,C\,C'$ may be, there exists an arc BC containing no other cusp, which does or does not contain I_A according as $B\,C$ are on the same or opposite branches; therefore B, C are adjacent cusps on Q. But $B\,A\,C$ are any three cusps adjacent in that order on the conic ϕ; $B\,C$ are any two cusps not adjacent on ϕ; the theorem may therefore be stated:

Any two cusps not adjacent on the conic determined by the five cusps are adjacent on the quintic determined by the five cusps.

Since the quintic is unipartite, and there are only two cusps not adjacent to B on ϕ, viz. C and K_2, the order of cusps on Q is $C\,B\,K_2$. Repetition of this argument for every pair of non-adjacent cusps gives the order on Q:—$B\,K_2\,A\,K_1\,C$. Any given λ value, however, gives two points on Q, e. g. B and its A-correspondent B', which are known to be on opposite branches. Hence consideration of the sequence of λ-values determines the following order on Q:

$$B\,C'\,K_2\,K'_1\,A\,K_1\,K'_2\,C\,B'$$

where the primed letters denote A-correspondents of the cusps. But the λ for I_A is greater than λ for B and B'; I_A lies between B and B'. Consequently $B\,C$ are separated by $A\,I_A$ in such a way that on the arc $B\,I_A\,C$ there are no other cusps.* Since $B\ C$ are any two cusps non-adjacent on ϕ, and A is then the cusp adjacent to both, the same process may be repeated, determining the relative position of the other four inflexions. The resulting order on Q is:

$$A\ (C)\ K_2\ (K_1)\ B\ (A)\ C\ (K_2)\ K_1\ (B)\ A$$

where the bracketed letters denote inflexions associated with the given cusps.

Comparison of the order on Q with the order on ϕ gives the two theorems:

Any two cusps adjacent on Q are not adjacent on the conic ϕ determined by the five cusps.

Any two cusps not adjacent on Q are adjacent on ϕ.

The assumption has been made that b is different from c. The special relations $a = b$ or $a = c$ are precluded by the fact that a is negative, so that $a > b + c$. Consequently, $b = c$ is the only specialization which affects the foregoing argument. The vanishing of $b - c$ does not interfere with the existence of the equations developed, though the system $S_1(\lambda^2)$, while still involving λ^2, degenerates into a system of lines through a point; that is, equation (6) becomes

$$g(\lambda + a - 3b)^2 + 4ax(\lambda - a + 2b) = 0.$$

Only in the geometrical interpretation is the specialization noticeable. In the preceding development, the following changes appear:

The λ-value for Θ is zero, so Θ becomes the line pair $xk = 0$. The double point of the line pair lies on the Q and is therefore the inflexion, since Θ cuts out I_A on Q. The inflexional tangent j_A is the line $l + 4ak$ with $b = c$, given by $\lambda = 0$ in equation (8). The coordinates for I_A reduce to $x : y : z = 0 : -1 : 1$, that is, the common point of $x = 0$, $ax + b(y + z) = 0$. But A_1, the conjugate of A with respect to $S_2(\lambda)$, with coordinates $x : y : z = \delta : -2a : 2a$ becomes $(0, -1, 1)$. In this case, therefore, A_1 is the inflexion I_A, and the tangent to ϕ viz. $g = 0$, which is the line AA_1, passes through the point (xk). The cuspidal tangent at A, t_A, becomes $y - z = 0$. The tangents t_B and t_C become respectively

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* Analytical proof may be given that $B\ C$ are separated by $A\ I_A$. In equation (18), call $Px = (T - R)y$ solution I and $Px = (T + R)y$ solution II. Points common to solution I and the λ-conics trace out one branch of Q from A; points of solution II trace the other branch; the two solutions come again to coincidence at I_A. Substitution of coordinates (0 1 0), B, gives $T - R = 0$; $\lambda = a + b$ gives $T = 2b(\sigma^2 + ac)$, which is positive; therefore $T - R$ is the desired coefficient, and B lies on branch I. The R thus evaluated (for B) is $2b(\sigma^2 + ac)$.

For C, $\lambda = a + c$; $R = 2c(\sigma^2 + ab)$. The desired line is tangent to the λ-conic at C and is therefore of the form $y(a + c) = x(\beta + c)$. Now $\lambda = a + c$ gives $P = (\beta + c)(\delta^2 - 4ac)$, $T + R = (a + c)(\delta^2 - 4ac)$. These coefficients have the desired form, so C lies on branch II. Then $B\ C$ are separated by $A\ I_A$.

$$x(a+b) = z(-a+3b)$$
$$x(a+b) = y(-a+3b).$$

and hence meet on $y - z = 0$, that is, on t_A. The equations for t_{K_1} and t_{K_2} reduce to

$$2ax(ab-\sigma^2) + y[2ab^2 - \sigma(a-b) - \sigma^2(a+b)] + z[2ab^2 + \sigma(a-b) - \sigma^2(a+b)] = 0$$
$$2ax(ab-\sigma^2) + y[2ab^2 + \sigma(a-b) - \sigma^2(a+b)] + z[2ab^2 - \sigma(a-b) - \sigma^2(a+b)] = 0$$

and so meet on t_A. The two points of intersection on t_A

$$(20) \qquad x : y : z = -a+3b : a+b : a+b$$
$$x : y : z = 2ab^2 - \sigma^2(a+b) : a(\sigma^2 - ab) : a(\sigma^2 - ab)$$

are distinct if no further condition is imposed.

Conversely, if a pair of cuspidal tangents t_B, t_C meet on t_A, the condition for concurrence gives

$$(\beta + c)(\alpha + b) = (\gamma + b)(\alpha + c)$$

therefore $\alpha = 0$ or $b = c$. With five real cusps $\alpha = 0$ is impossible, so $b = c$; then t_{K_1} and t_{K_2} also meet on t_A, and I_A is the point common to $x = 0$, $k = 0$. If I_A is known to be the point (xk), then $b = c$, and the cuspidal tangents meet as stated. Any one of the three conditions brings about the other two.

If now t_B t_C and t_{K_1} t_{K_2} meet as paired on t_A, and (e. g.) t_A t_{K_1} and t_C t_{K_2} meet on t_B, in other words, if all five cuspidal tangents are concurrent, then $ABCK_1K_2$ may be projected into the vertices of a regular pentagon; for I_A is the point (xk), I_B is the point (AK_1, CK_2). Project the line joining these two inflexions to infinity, and the conic ϕ into a circle. Then g, x, k are parallels, so the arcs AB, BK_1, AC, CK_2 are all equal. Similarly, the tangent to ϕ at B and its parallels give the arcs BA, AC, K_1K_2, K_1B all equal, so $ABCK_1K_2$ is a regular pentagon and the original five cusps were such a set as could be projected into its vertices. The five inflexions in this symmetrical case are at infinity.

Such concurrence of the cuspidal tangents may be obtained from the case $b = c$, $\alpha < 0$, by imposition of another condition on a, b, such as coincidence of the two points (20) on $y - z = 0$. Proportionality of the respective x and y coordinates gives (for $b = c$, $\sigma^2 = a(a - 4b)$)

$$a(-a+3b)(a-5b) = [2b^2 - (a-4b)(a+b)](a+b).$$

The resulting condition is:

$$a = b(2 \pm \sqrt{5}); \quad \text{so} \quad a = b(2+\sqrt{5}) \text{ since } a > 2b.$$

This is a high degree of specialization and results in much simplification of the framework of related lines and points (see Fig. 2); as one result, the linear constructions about to be given are not available.

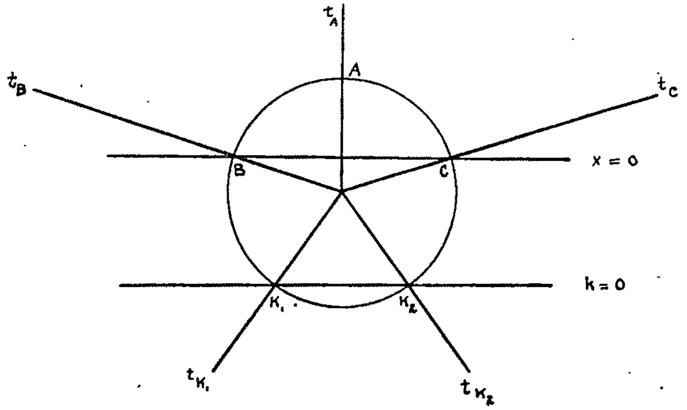

FIG. 2.

The Appearance of the Quintic Curve. The curve is unipartite, composed of one infinite branch. It has five cusps and five inflexions occurring alternately, each cusp associated with the inflexion symmetrically placed with respect to it and the remaining four cusps. The parametric treatment has given this information. A line having only one real intersection with the curve certainly exists, for the line joining two cusps adjacent on ϕ meets the curve in one other point, P. Rotate this line about P through a very small angle so that its intersections with the branches from the cusps become imaginary; since the curve lies entirely outside ϕ, such rotation about an external point is always possible. The line in its new position has only one real intersection with Q. Projection of this line to infinity gives the type of the curve with only one real asymptote. Figures 3 and 4, obtained by quadratic transformation of a quartic with two cusps, in- and circumscribed to the triangle of reference ABC, show this form and also the form with five real asymptotes.

The inflexional tangent j_A at I_A divides ϕ into two arcs, on one of which A lies, while $B\,C\,K_1K_2$ all lie on the other arc. The equation of j_A is (10a). Let j_A cut ϕ in points $J_1\,J_2$; the lines AJ_1, AJ_2 are then:

$$y^2\left[\frac{\sigma^2+ac}{a}\right]^2+yz\left[3\delta^2-4a\alpha+\frac{(\sigma^2+ac)^2}{a^2}+\frac{(\sigma^2+ab)^2}{a^2}\right]$$
$$+z^2\left[\frac{\sigma^2+ab}{a}\right]^2=0.$$

The discriminant here is:

$$\left[3\delta^2 - 4a\alpha + \frac{(\sigma^2+ac)^2}{a^2} + \frac{(\sigma^2+ab)^2}{a^2}\right]^2 - 4\,\frac{(\sigma^2+ab)^2(\sigma^2+ac)^2}{a^4}$$

$$= (3\delta^2-4a\alpha)^2 + 2(3\delta^2-4a\alpha)\left[\frac{(\sigma^2+ac)^2}{a^2}+\frac{(\sigma^2+ab)^2}{a^2}\right]$$

$$+\left[\frac{(\sigma^2+ac)^2}{a^2}-\frac{(\sigma^2+ab)^2}{a^2}\right]^2.$$

The terms are all positive; the roots are therefore real. The coefficients of the quadratic are all positive; the lines AJ_1, AJ_2 therefore lie outside the triangle, and J_1J_2 separate A from BCK_1K_2 on ϕ.

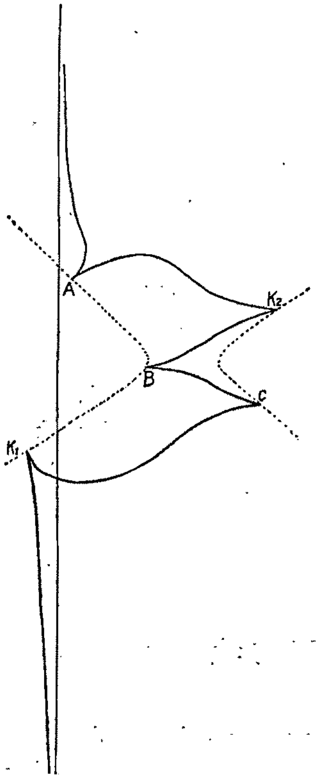

FIG. 3.

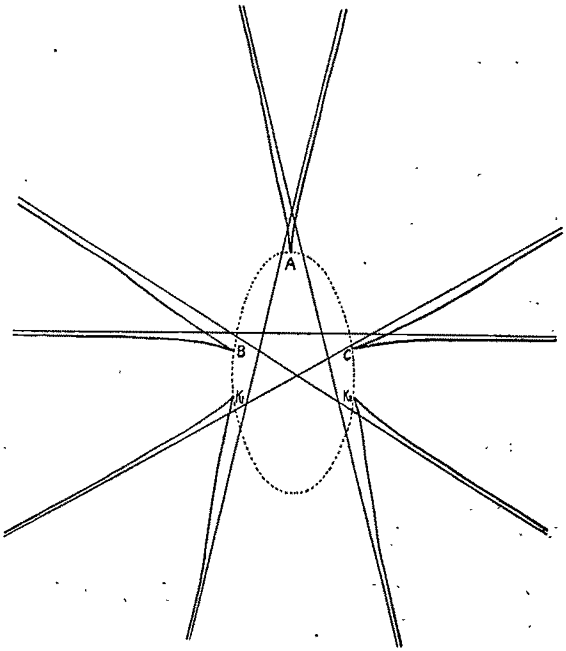

FIG. 4.

The curve lies in such a way that tangents to ϕ at the cusps have each three real intersections with Q elsewhere. For: $g=0$ has intersections with Q given by $y^2=0$ and the cubic

$$y^3\delta(a+3\alpha)+y^2x(9\delta^2+4a\alpha)+yx^2(24a\delta)+16a^2x^3=0.$$

The substitution $4ax/y = w - 2\delta$ gives

$$w^3 + w(4a\alpha - 3\delta^2) + 2\delta^2(a^2 - \sigma^2) = 0.$$

The derived equation has two real roots w_1, w_2; the cubic expression evaluated at $-\infty$, w_1, w_2, $+\infty$ exhibits three changes of sign; three real roots exist; therefore $g = 0$ has three real intersections with Q not at A.

Since the Plücker numbers for the quintic with five cusps, or pentoid, as it may be called, are $m = n = 5$; $\kappa = i = 5$, the reciprocal curve is another pentoid. For this curve, corresponding theorems may be stated. The conic of the cusps, ϕ, becomes χ the conic of the five inflexional tangents, $j_A\, j_B\, j_{K_1}\, j_{K_2}\, j_C$, with points of contact on χ in the order given. The order of points of contact on Q is then (A') (K_2) (B) (C) (K_1). From the point of contact of j_A with χ, three real tangents may be drawn. Through the cusp A there pass two tangents to χ which cut off an arc of χ to which $j_B\, j_C\, j_{K_1}\, j_{K_2}$ are tangents and an arc to which j_A alone is tangent. Since any pentoid has an equation of the type of equation (4), these properties, which relate only to order on the curve and reality of elements, characterize the original pentoid also.

Constructions. 1). The cuspidal tangent to Q at A is the polar of A_1 (the conjugate of A with respect to $S_2(\lambda)$), with respect to ϕ. Now A_1 on g may be constructed linearly; (AA_1 are harmonic with respect to the points (xk) on g); then a linear construction for t_A follows. Similarly, t_B is the polar of B_1 with respect to ϕ, (where B_1 is the conjugate of B with respect to the pencil of conics through ACK_1K_2). So the five cuspidal tangents are obtained by linear constructions from the five given cusps.

2). The five inflexional tangents may be constructed linearly, with the help of certain intermediate lines.

$p \equiv \Sigma x = 0$ is the Pascal line for ABC inscribed in ϕ; it may therefore be constructed linearly.

$h = (c - a)y + (b - a)z$ is then known: for p meets k in a point P on the line AP:

$$(b - a)y + (c - a)z = 0;$$

then h is the harmonic conjugate of AP with respect to $y \pm z = 0$;

$$[(b - a)y + (c - a)z] + [(c - a)y + (b - a)z] = (y + z) \cdot (\alpha - a)$$
$$[(b - a)y + (c - a)z] - [(c - a)y + (b - a)z] = (y - z)\delta.$$

Now h and k are known; $h+2k$ is the line through their common point and through A_1; $h+2k$ may be constructed.

The pencil of lines joining A_1 to the λ-points on Θ

$$\lambda g-(h+2k)=0$$

is projective with the pencil of polars of A with respect to $S_2(\lambda)$, viz.,

$$\lambda t_A-4\delta(by+cz)=0.$$

In these pencils the rays correspond as follows:

(A_1 pencil)		(A pencil)
$g(\lambda=\infty)$	corresponds to	t_A
$h+2k(\lambda=0)$	corresponds to	$by+cz=0$
$A_1B\ \ (\lambda=a+b)$	corresponds to	polar of A_1w.r. to conic of $S_2(\lambda)$ tangent at B to t_B.

All three lines in the A_1 pencil are constructed; t_A, t_B are drawn; $by+cz=0$ is the line joining A to the common point of $x=0$, $k=0$; the polar of A_1 with respect to a conic through four given points, with a given tangent at one of them, may be constructed linearly. Therefore: three pairs of corresponding rays in the $A\ A_1$ pencils are drawn and the correspondent to any other ray may be constructed. Since g as a ray of the A pencil corresponds to $\lambda=\delta^2/a$,

$$\begin{aligned}(\delta^2/a)t_A-4\delta(by+cz) &= (\delta/a)\{y[\delta(2a+\delta)-4ab]+z[-\delta(2a-\delta)-4ac]\}\\ &= (\delta/a)(y+z)(\sigma^2-a^2),\end{aligned}$$

its corresponding ray (which may be linearly constructed), determines I_A on Θ; j_A, tangent to a conic given by five points, at a known point on it, is then given by linear construction. For B, a similar process is performed, with the system $S_2(\mu)$ of conics through ACK_1K_2 as basis, and the conic of $S_2(\mu)$ through B_1 in place of Θ. Thus all five inflexions and the inflexional tangents are obtained by linear construction from the five given cusps.*

3). The intermediate lines necessary in this construction give other points on the quintic.

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* Any of the given quantities a, b, c, σ may be irrational, but the parameters and coordinates of all the points in question are built up from these quantities by rational operations.

The fifth point of intersection on x, k respectively (given by the conic of $S_2(\lambda)$ corresponding to $\lambda = 0$), on the line $l + 4ak = 0$ may be constructed, for $l + 4ak = 0$ is the tangent to $\circledast$ at the point given by $h + 2k$ (through A_1); a linear construction gives $l + 4ak$. The fifth point on all lines joining two cusps may be obtained by repetition of this process at B, C, K_1, K_2. There are ten such lines, therefore ten points are given.

The A-correspondent to B lies on the conic of $S_2(\lambda)$ tangent at B to t_B, and on the line tangent to $\circledast$ at B. This line and its intersection with the conic in question may be constructed linearly; B' is therefore found. Four such points are given for B (correspondents with respect to $A\,C\,K_1\,K_2$). Similarly, four correspondents for each cusp may be found—twenty points in all. The series of points obtained by the linear constructions in (3) may be extended in this way indefinitely.

BRYN MAWR, PA., 1925.

The Convergence of General Means and the Invariance of Form of Certain Frequency Functions.

By Edward L. Dodd.

Introduction. The problem of choosing for given statistical data the most suitable mean or average offers many difficulties. Perhaps for some time, the intuition or the judgment of the investigator must be relied upon chiefly. However, it seems possible to work out to some extent the relations between frequency functions and means so that eventually the choice may be more guided.

To illustrate, suppose that a distribution presents itself that can be fitted admirably by the function or curve

$$y = \frac{x^{-2/3}}{3(2\pi)^{1/2}} e^{-(x^{1/3}-4)^2/2}. \tag{1}$$

discussed by Rietz.* In obtaining (1) the assumption made—that diameters are normally distributed corresponding to the volumes under consideration—suggests that a very suitable mean for (1) is

$$M = [(1/n) \sum m_i^{1/3}]^3, \qquad (i = 1, 2, \cdots, n). \tag{2}$$

Moreover, when for independent measurements m_i, the probability that $x < m_i < x + dx$ is $y\ dx$, as given by (1), then the probability that $x < M < x + dx$ is $v\ dx$, where

$$v = \frac{n^{1/2}\, x^{-2/3}}{3(2\pi)^{1/2}} e^{-n(x^{1/3}-4)^2/2}. \tag{3}$$

And this expression not only maintains the essential form of (1), but it shows that with increasing n there is a probability bordering on certainty that M will converge to 64, a value that the underlying assumption suggests as the "normal value" or "true value" of m.

This mean given by (2) is one of a general class of means † that can be written in the form

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* "Frequency Distributions Obtained by Certain Transformations of Normally Distributed Variates," *Annals of Mathematics*, Ser. 2, Vol. 23, (1922), pp. 292-300—see p. 295. Primes used by Rietz have been dropped.

† For a more general mean involving weights, see Dodd, "Functions of Measurements under General Laws of Error," *Skandinavisk Aktuarietidskrift*, Vol. 5 (1922), pp. 133-158. See p. 141.

$$M = G[(1/n) \sum_1^n f_i(m_i)], \tag{4}$$

where each $f_i(u)$ is a continuous function of u, univariant over the interval in which the m_i appear, $v = 1/n \Sigma f_i(u) = F(u)$, $u = G(v)$. Setting $f_i(u) = u$, $\log u$, $1/u$, u^2, $u^{1/3}$, we obtain the arithmetic mean, the geometric mean, the harmonic mean, the root-mean-square, and the cube of the mean cube-root in (2), respectively.

The purpose of this paper is to exhibit certain relations that connect arbitrary frequency functions with the general mean M in (4), relative to the convergence of M to a value a, which may be taken as the true value, and relative also to the permanence or essential invariance of the form of the frequency function, as illustrated by (1) and (3).

1. *Theorems on General Means and Frequency Functions.*

THEOREM I. *Let the probability that a measurement m_i will take on a value less than u be $\Phi(u)$, where $\Phi(\alpha) = 0$ and $\Phi(\beta) = 1$. And suppose that in the interval from α to β, finite or infinite, $\Phi(u)$ is an increasing function, and—except possibly for a finite number of values of u in any finite interval—has a positive derivative $\Phi'(u)$. Let $\Psi(v)$ be a second arbitrary cumulative frequency function, defined in the interval from α' to β'* with *corresponding restrictions, and such that*

$$\Psi(a) = \Phi(a) \tag{5}$$

$$\int_{\alpha'}^{\beta'} v\, \Psi'(v)\, dv = a, \qquad \int_{\alpha'}^{\beta'} v^2\, \Psi'(v)\, dv < K \text{ (finite)}. \tag{6}$$

Then the continuous increasing function

$$v = f(u) \tag{7}$$

with inverse $u = g(v)$, constructed[*] *by setting*

$$\Psi(v) = \Phi(u), \tag{8}$$

is such that if for n independent measurements m_i,

$$M = g[(1/n) \sum f(m_i)], \tag{9}$$

and if ϵ and η are arbitrarily small positive numbers, there is a probability greater than $1 - \eta$, when n is sufficiently large, that

$$| M - a | < \epsilon. \tag{10}$$

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[*] Dodd, "The Frequency Law of a Function of One Variable," *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 27-31. See p. 30.

Proof. Equation (8) may be written in the differential form

$$\Psi'(v)\,dv = \Phi'(u)\,du, \tag{11}$$

and each member may be regarded as the probability that both the following equivalent inequalities will be satisfied:

$$u < m_i < u + du, \qquad v < f(m_i) < v + dv, \tag{12}$$

noting that $f(u)$ is an increasing function.

Then, with $\epsilon' > 0$, $\eta > 0$, it is possible to take n large enough—on account of (6) and the Tchebycheff Theorem—so that there will be a probability greater than $1 - \eta$ that

$$a - \epsilon' < (1/n) \sum_1^n f(m_i) < a + \epsilon'. \tag{13}$$

But, from (5), (7), and (8), it follows that

$$f(a) = a, \qquad g(a) = a. \tag{14}$$

And thus (13) may be written, on account of (9);

$$a - \epsilon_1 = g(a - \epsilon') < M < g(a + \epsilon') = a + \epsilon_2, \tag{15}$$

where ϵ_1 and ϵ_2 approach zero with ϵ', since $g(v)$ is a continuous function. This establishes (10).

THEOREM II. *Let the probability that a measurement m_i will take on a value less than u be $\Phi_i(u)$, where in the interval $\alpha_i \leqq u \leqq \beta_i$, each $\Phi_i(u)$ is under the restrictions of Theorem I. Suppose, furthermore, that the measurements are independent, and that there exist constants a, δ, and k, so that if $a - \delta \leqq u \leqq a + \delta$,*

$$\Phi_i'(u) > k > 0, \qquad (i = 1, 2, \cdots, n). \tag{16}$$

Then there exist continuous increasing functions $f_i(u)$, defined in (α_i, β_i), so that if we set

$$v = F(u) = (1/n) \sum_1^n f_i(u), \qquad u = G(v), \tag{17}$$

$$M = G[(1/n) \sum f_i(m_i)], \tag{18}$$

and if we take two positive numbers ϵ and η small at pleasure, there is a probability greater than $1 - \eta$, when n is sufficiently large, that

$$|M - a| < \epsilon. \tag{19}$$

Proof. We can select in an infinite number of ways functions $\Psi_i(v)$, so that if $a - \delta \leqq v \leqq a + \delta$,

$$(20) \qquad \Psi_i'(v) < k, \qquad (i = 1, 2, \cdots, n).$$

functions conforming, moreover, to the requirements of Theorem I, in particular, (5), (6), and (8); since these merely require that the area under Ψ_i' up to the point v shall be equal to that under $\Phi'(u)$ up to the point u—where u and v reach a simultaneously—and that the first moment or center of gravity for Ψ_i' shall be equal to a, and the second moment shall be finite. Thus we can write (13) and (14) with f replaced by f_i, g by G. Then, from (16), (17), and (20), it follows that when $a - \delta \leqq v \leqq a + \delta$, $G'(v) < 1$. Hence, a fortiori, (13) is valid if the middle member is replaced by M; and this leads to (19).

THEOREM III. *Let the probability that a measurement m_i will take on a value less than u be $\Phi_i(u)$, where in the interval from α_i to β_i, $\Phi_i(u)$ is under the restrictions stated in Theorem II. Suppose, furthermore, that*

$$(21) \qquad \Phi_i(a) = \tfrac{1}{2}, \qquad (i = 1, 2, \cdots, n).$$

Take

$$(22) \qquad w = \Theta(y) = (1/\pi^{1/2}) \int_{-\infty}^{y} e^{-t^2} dt, \qquad y = \Theta^{-1}(w),$$

$$(23) \qquad f_i(u) = a + \Theta^{-1}[\Phi_i(u)], \qquad v = f(u) = (1/n) \sum_{1}^{n} f_i(u),$$

$$(24) \qquad u = g(v), \qquad M = g[(1/n) \sum f_i(m_i)].$$

Then the probability that

$$(25) \qquad u_1 < M < u_2,$$

when $a - \delta \leqq u_1 \leqq u_2 \leqq a + \delta$, is given by

$$(26) \qquad P(u_1, u_2) = \Theta\{n^{1/2}[f(u_2) - a]\} - \Theta\{n^{1/2}[f(u_1) - a]\},$$

whether n be large or small. Moreover, if ϵ and η are arbitrarily small positive numbers, there is a probability greater than $1 - \eta$ when n is sufficiently large, that

$$(27) \qquad | M - a | < \epsilon.$$

Proof. Let us write (23) in the form

$$(28) \qquad \Theta[f_i(t) - a] = \Phi_i(t),$$

and seek the probability that

$$(29) \qquad s < f_i(m_i) < s + ds.$$

Take

(30) $$t = g_i(s), \qquad s = f_i(t),$$

and as equivalent to (29) write

(31) $$t < m_i < t + dt.$$

From (22), (28), and (30), the probability for (31) and thus (29) is

(32) $$\Phi'(t)\,(dt) = (1/\pi^{1/2})\, e^{-(s-a)^2} ds.$$

Hence, by a well-known theorem,* the probability that

(33) $$s < (1/n) \sum_1^n f_i(m_i) < s + ds$$

is

(34) $$(n^{1/2}/\pi^{1/2})\, e^{-n(s-a)^2} ds.$$

We may now choose this s as identical with the v in (24), and write

(35) $$(n^{1/2}/\pi^{1/2})\, e^{-n(v-a)^2} dv$$

as the probability that

(36) $$u < M < u + du.$$

In (35) we may set $v = f(u)$ from (23), and integrate between u_1 and u_2 as required by (25), using a simple change of variable; and this leads to (26) on account of (22).

Furthermore, if we take $0 < \epsilon \leqq \delta$, we may conclude from (16), (21), (22), (23), that

(37) $$\Phi_i(a - \epsilon) < \tfrac{1}{2} - k\epsilon, \qquad \Phi_i(a + \epsilon) > \tfrac{1}{2} + k\epsilon,$$

(38) $$f(a - \epsilon) < a + \Theta^{-1}(\tfrac{1}{2} - k\epsilon), \qquad f(a + \epsilon) > a + \Theta^{-1}(\tfrac{1}{2} + k\epsilon).$$

and thus, from (26), $P(a - \epsilon, a + \epsilon)$ approaches unity with increasing n.

Corollary. Under the conditions of the theorem, it is possible to take n large enough so that if ξ is a preassigned positive number, the probability that

(39) $$M < z$$

is given by

(40) $$p(z) = \Theta\{n^{1/2}[f(z) - a]\} + \xi', \qquad 0 \leqq \xi' \leqq \xi.$$

Proof. In (37) and (38) take $\epsilon = \delta$, and determine n so that in (26), $P(a - \delta, a + \delta) > 1 - \xi$. Then in (26) take $u_2 = z$, $u_1 = a - \delta$.

2. *Illustration.* Suppose that for independent measurements m_i necessarily positive, the probability that $m_i < u$ is given by

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* Czuber, *Wahrscheinlichkeitsrechnung*, Vol. I, third edition (1914), p. 304.

(41) $$\Omega(u, h) = \Phi(u) = (h/\pi^{1/2}) \int_0^u e^{-(h \log t/a)^2}\, dt/t,$$

and that M is the geometric mean

(42) $$M = (m_1 m_2 \cdots m_n)^{1/n}.$$

Then the probability that $M < z$ is given by

(43) $$\Omega(z, n^{1/2} h) = \Theta(n^{1/2} h \log z/a),$$

where Θ is the common probability function in (22). To show this, use in (23),

(44) $$v = f(u) = a + h(\log u - \log a).$$

3. *Summary and Discussion.* Under rather general conditions, Therem II establishes the existence of means of n independent measurements converging with asymptotic certainty to the true value as n increases.

With an added requirement that the probability of a negative error shall be 1/2, it is shown in Theorem III that a mean may be constructed subject to a frequency law depending in a simple manner upon the usual probability integral. The computations here, indeed, involve for the most part simple arithmetic operations, to be supplemented by direct and inverse use of a probability table. This table may be based as well upon $e^{-t^2/2}$ as upon e^{-t^2}; since a change of t to $t\sqrt{2}$ in (22) would not involve any change in the reasoning. The work is especially simple if all the measurements are subject to the same frequency law.

The above condition that the probability of a negative error shall be 1/2 may be illustrated by the lateral distribution of errors of a golf ball aimed at the top of a mound with slopes which may be different on the two sides. It does not require any symmetry for the frequency curve.

Theorem III would seem to be of special interest in the case where the frequency functions $f_i(u)$ in (23) undergo a cyclic change yielding $f(u)$ as an average.

That this paper does not clear up all difficulties may be illustrated by referring to Rietz's example in (1). Under this law the arithmetic mean tends toward 76, thus creating an error of 12 if 64 is regarded as the normal value. But, if in the determination of the parameter in (1) by fitting the data, an error is made so that the parameter is written 4.236 instead of 4, then the cube of the mean cube-root, as given by (2) tends towards the same 76. An error of 12 for 64 is bad, and almost forces us to abandon the arithmetic mean. But caution must be exercised in choosing a substitute when the number of measurements is too scanty to establish within narrow bounds the parameters of the associated frequency curve.

On the Interpolatory Properties of a Linear Combination of Continuous Functions.

By D. V. Widder.

1. *Introduction.* In his treatment of the mean-value theorems belonging to a linear differential equation, G. Pólya* obtained a condition on a set of functions

$$(1) \qquad u_i(x) \qquad (i=1,2,3,\cdots,n)$$

which insured that any linear combination of them should enjoy the more important interpolatory properties of a polynomial of degree less than n. He designated the condition as the *property W.* We give the definition with a slight modification.†

Definition. If it is possible to determine constants c_{ij} *so that the* functions

$$h_i(x)=\sum_{j=1}^{n} c_{ij}\,u_j(x), \qquad (i=1,2,3,\cdots,n)$$

satisfy the inequalities

$$\begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_k(x) \\ h_1'(x) & h_2'(x) & \cdots & h_k'(x) \\ \cdot & \cdot & \cdot & \cdot \\ h_1^{(k-1)}(x) & h_2^{(k-1)}(x) & \cdots & h_k^{(k-1)}(x) \end{vmatrix} > 0, \qquad (k=1,2,3,\cdots,n)$$

in an interval (a,b), *then the set* (1) *possesses the property W in that interval. The functions* $u_i(x)$ *are assumed to have continuous derivatives of order n throughout the interval.*

It was shown, for example, that this condition on the functions (1) is necessary and sufficient that no linear combination of them, except the identically vanishing one, shall vanish n times in (a,b).

This condition has the disadvantage that it demands the existence of n

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* "On the Mean-Value Theorem Corresponding to a Given Linear Homogeneous Differential Equation," *Transactions of the American Mathematical Society,* Vol. 24 (1922), pp. 312-324.

† This modification is made in order to make the condition independent of the differential equation treated by Pólya.

continuous derivatives for the functions in question. In certain problems of interpolation these functions are known only to be continuous. A case in point is that of the generalized problem of Tschebycheff.* It would be desirable then to impose a condition on the functions $u_i(x)$ that will not involve the existence of the derivatives, but which will guarantee that any linear combination of these functions shall resemble a polynomial in its interpolatory properties.

It is the purpose of the present paper to obtain such a condition and to apply it to the solution of several problems.

2. *The Property V.* Consider a set of functions (1), all continuous in an interval $a \leqq x \leqq b$ of the x-axis.

Definition. If for every value d, $a < d < b$, it is possible to determine constants c_{ij} so that the functions

$$h_i(x) = \sum_{j=1}^{n} c_{ij}\, u_j(x), \qquad (i = 1, 2, 3, \cdots, n)$$

satisfy the inequalities

$$V_k(x) = V[h_1(x), h_2(x), \cdots, h_k(x)] =$$

$$\begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_k(x) \\ h_1(x+\delta) & h_2(x+\delta) & \cdots & h_k(x+\delta) \\ \cdot & \cdot & \cdot & \cdot \\ h_1(x+\overline{k-1}\,\delta) & h_2(x+\overline{k-1}\,\delta) & \cdots & h_k(x+\overline{k-1}\,\delta) \end{vmatrix} > 0$$

$$(k = 1, 2, 3, \cdots, n)$$

for every value of x in the interval $a \leqq x \leqq d$ and for every value of δ in the interval $0 < \delta \leqq (b-d)/n$, then the set (1) *is said to possess the property V in* (a, b).

As an example of such a set of functions take

$$h_i(x) = u_i(x) = x^{i-1}, \qquad (i = 1, 2, 3, \cdots, n).$$

Following the usual notation we set

$$\Delta^{\delta}_n f(x) = f(x+n\delta) - nf(x+\overline{n-1}\,\delta) + \frac{n(n-1)}{2!} f(x+\overline{n-2}\,\delta) + \cdots$$

$$+ (-1)^n f(x), \quad \Delta^{\delta}_1 f = \Delta f.$$

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* See § 6.

The functions $V_k(x)$ may then be written

$$V_k(x) = \begin{vmatrix} h_1 & h_2 & \cdots & h_k \\ \Delta^{\delta}{}_1 h_1 & \Delta^{\delta}{}_1 h_2 & \cdots & \Delta^{\delta}{}_1 h_k \\ \cdot & \cdot & \cdot & \cdot \\ \Delta^{\delta}{}_{k-1} h_1 & \Delta^{\delta}{}_{k-1} h_2 & \cdots & \Delta^{\delta}{}_{k-1} h_k \end{vmatrix} .$$

Now

$$\Delta^{\delta}{}_j\, x^{k-1} = \begin{cases} \delta^{k-1}(k-1)\,!, & j = k-1 \\ 0, & j > k-1 \end{cases}$$

Hence it is readily verified that

$$V_k(x) = (k-1)\,!\,(k-2)\,!\cdots 2\,!\;\delta^{(k-1)k/2}.$$

This is positive for any positive value of δ and is independent of x, so that the property V holds in any interval of the x-axis.

It is to be noted that the functions $h_i(x)$ may depend on d. As an example take the set of functions

$$u_1(x) = \sin x, \qquad u_2(x) = \cos x.$$

This set possesses the property V in the interval $(a, a+\pi)$, a being arbitrary. For, take

$$V_1(x) = h_1(x) = \sin[x + (\pi - a - d)/2]$$
$$h_2(x) = -\cos[x + (\pi - a - d)/2], \qquad V_2(x) = \sin\delta.$$

$V_1(x)$ and $V_2(x)$ are positive in the interval $a \leqq x \leqq d < b$ for all values of d. Clearly it would be impossible to find V_1 independent of d, since every linear combination of the two given functions has at least one zero in any closed interval of length π. More generally it may be shown that the property V holds for the set

$$\sin x,\ \cos x;\ \sin 2x,\ \cos 2x,\ \cdots,\ \sin nx,\ \cos nx$$

in the same interval. If a constant not zero is added to the set, the resulting set will enjoy the property in any interval of length not greater than 2π.

It will appear later that the property W implies the property V. That the converse is not true becomes evident by an example. Let $\phi(x)$ be any continuous monotonic increasing function (not constant in any interval however small) in an interval (a, b). Consider the set of functions

$$(2) \qquad h_i(x) = [\phi(x)]^{i-1}, \qquad (i = 1, 2, 3, \cdots, n).$$

Then

$$V_k(x) = \begin{vmatrix} 1 & \phi(x) & [\phi(x)]^2 & \cdots & [\phi(x)]^{k-1} \\ 1 & \phi(x+\delta) & [\phi(x+\delta)]^2 & \cdots & [\phi(x+\delta)]^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \phi(x+\overline{k-1}\,\delta) & [\phi(x+\overline{k-1}\,\delta)]^2 & \cdot & [\phi(x+\overline{k-1}\,\delta)]^{k-1} \end{vmatrix}$$

This is a Vandermonde determinant, and may be expanded as follows:

$$\begin{aligned} V_k(x) = {} & [\phi(x+\overline{k-1}\,\delta) - \phi(x+\overline{k-2}\,\delta)] \\ & [\phi(x+\overline{k-1}\,\delta) - \phi(x+\overline{k-3}\,\delta)] \cdots \\ & [\phi(x+\overline{k-1}\,\delta) - \phi(x)] \cdots [\phi(x+\delta) - \phi(x)]. \end{aligned}$$

This function is defined and positive in the interval $a \leqq x \leqq d < b$ for any positive δ at most equal to $(b-d)/n$. Each factor of V_k is positive since $\phi(x)$ is monotonic increasing. Hence the property V holds in the interval (a, b). If $\phi(x)$, does not have a continuous derivative, the definition of the property W is not applicable. Another instructive example is obtained by setting

$$\phi(x) = x^3$$

in the set (2). The resulting set is one for which the property W does not hold in any interval including the origin, since the function x^{3n-3} vanishes more than $(n-1)$ times at the origin. Yet the property V, on the other hand, is seen to hold in any interval since x^3 is monotonic increasing.

The property V insures the linear independence of the functions (1). For, if the functions were linearly dependent, constants c_i, not all zero, would exist such that

$$\begin{aligned} & c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) = 0 \\ & c_1u_1(x+\delta) + c_2u_2(x+\delta) + \cdots + c_nu_n(x+\delta) = 0 \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & c_1u_1(x+\overline{n-1}\,\delta) + c_2u_2(x+\overline{n-1}\,\delta) + \cdots + c_nu_n(x+\overline{n-1}\,\delta) = 0 \end{aligned}$$

for all δ sufficiently small. Hence we should have

$$V[u_1, u_2, \cdots, u_n] \equiv 0.$$

But

$$(3) \qquad V[h_1, h_2, \cdots, h_n] = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} V[u_1, u_2, \cdots, u_n].$$

Since $V[h_1, h_2, \cdots, h_n] > 0$, it follows that neither factor on the right-hand side of this equation can vanish. The contradiction shows that the set (1) is a linearly independent set.

The following remark will be useful in subsequent work. If the property V holds for the set (1) in the interval (a, b), then the set of functions

$$h_1(x), h_2(x), \cdots, h_k(x), \qquad (k \leqq n)$$

corresponding to the constant d, also possesses the property in the interval (a, d). For in this case the constants c_{ij} of the definition may be taken as

$$c_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

The functions

$$V[h_1], \quad V[h_1, h_2], \quad \cdots, \quad V[h_1, h_2, \cdots, h_k]$$

are all positive in $a \leqq x \leqq d$, and hence also in any sub-interval.

3. *The Operator L^δ.* We now define a linear difference operator L^δ by means of the following expression

$$L^\delta f(x) = V[u_1, u_2, u_3, \cdots, u_n, f].$$

If $f(x)$ is continuous in the closed interval (a, b), then $L^\delta f(x)$ is defined and continuous in the closed interval $(a, b - n\delta)$, supposing $\delta \leqq (b-a)/n$. It is well known* that a linear homogeneous difference operator of order n can be expressed by n successive applications of the operator Δ. We give a new proof of this fact which seems to be simpler than those heretofore given.

Consider the determinant $V[h_1, h_2, \cdots, h_n, f]$ and its adjoint determinant. Form the following minor from the adjoint:

$$H = \left| \begin{matrix} V[h_1(x+\delta), & \cdots, & h_{n-1}(x+\delta), & f(x+\delta)] \\ V[h_1(x), & \cdots, & h_{n-1}(x), & f(x)] \end{matrix} \quad \begin{matrix} V[h_1(x+\delta), & h_2(x+\delta), & \cdots, & h_n(x+\delta)] \\ V[h_1(x), & h_2(x), & \cdots, & h_n(x)] \end{matrix} \right| .$$

The four elements of this determinant, H, are the minors of $h_n(x)$, $f(x)$, $h_n(x+n\delta)$, and $f(x+n\delta)$ in the determinant $V[h_1, h_2, \cdots, h_n, f]$. By use of the familiar theorem * about the minors of the adjoint determinant we obtain

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* See, for example, Guldberg and Wallenberg, *Theorie der Linearen Differenzengleichungen*, p. 76.

$$H = V[h_1(x+\delta),\ h_2(x+\delta),\ \cdots,\ h_{n-1}(x+\delta)]\quad V[h_1(x),\ h_2(x),\ \cdots,\ h_n(x),\ f(x)].$$

Simple computation shows that †

$$\Delta\ \frac{V[h_1,\ h_2,\ \cdots,\ h_{n-1},\ f]_x}{V[h_1,\ h_2,\ \cdots,\ h_n]_x} = \frac{H}{V[h_1,\ h_2,\ \cdots,\ h_n]_x\ V[h_1,\ h_2,\ \cdots,\ h_n]_{x+\delta}}\ .$$

This leads at once to the formula

$$(4)\qquad \Delta\ \frac{V[h_1,\ h_2,\ \cdots,\ h_{n-1},\ f]_x}{V[h_1,\ h_2,\ \cdots,\ h_n]_x} = \frac{V[h_1,\ h_2,\ \cdots,\ h_{n-1}]_{x+\delta}\ V[h_1,\ h_2,\ \cdots,\ h_n,\ f]_x}{V[h_1,\ h_2,\ \cdots,\ h_n]_x\ V[h_1,\ h_2,\ \cdots,\ h_n]_{x+\delta}}\ .$$

By successive applications of this formula we arrive at the desired result. First note that $L^\delta f$ may be written as follows:

$$L^\delta f = V[u_1,\ u_2,\ \cdots,\ u_n,\ f] = KV[h_1,\ h_2,\ \cdots,\ h_n,\ f],\quad K = 1/|\,c_{ij}\,|.$$

From formula (4) we obtain directly

$$L^\delta f = K\ \frac{V[h_1,\ h_2,\ \cdots,\ h_n]_x\,.\,V[h_1,\ h_2,\ \cdots,\ h_n]_{x+\delta}}{V[h_1,\ h_2,\ \cdots,\ h_{n-1}]_{x+\delta}}\ \Delta\ \frac{V[h_1,\ h_2,\ \cdots,\ h_{n-1},\ f]_x}{V[h_1,\ h_2,\ \cdots,\ h_n]_x}\ .$$

Applying (4) to $V[h_1,\ h_2,\ \cdots,\ h_{n-1},\ f]$, we obtain

$$L^\delta f = K\ \frac{V_n(x+\delta)\ V_n(x)}{V_{n-1}(x+\delta)}\ \Delta\ \frac{V_{n-1}(x+\delta)\ V_{n-1}(x)}{V_n(x)\ V_{n-2}(x+\delta)}\ \Delta\ \frac{V[h_1,\ h_2,\ \cdots,\ h_{n-2},\ f]_x}{V[h_1,\ h_2,\ \cdots,\ h_{n-1}]_x}\ .$$

Now apply (4) to $V[h_1,\ h_2,\ \cdots,\ h_{n-2},\ f]_x$, and continue the process. The result is

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* See, for example, M. Bôcher, *Introduction to Higher Algebra*, p. 31.

† For brevity we write

$$V[h_1(x),\ h_2(x),\ \cdots,\ h_n(x),\ f(x)] = V[h_1,\ h_2,\ \cdots,\ h_n,\ f]_x.$$

$$(5)\qquad L^\delta f = K\frac{V_n(x+\delta)\,V_n(x)}{V_{n-1}(x+\delta)}\ \Delta\ \frac{V_{n-1}(x+\delta)\,V_{n-1}(x)}{V_n(x)\,V_{n-2}(x+\delta)}$$
$$\Delta\cdots\Delta\ \frac{V_1(x+\delta)\,V_1(x)}{V_2(x)\,V_0(x+\delta)}\ \Delta\ \frac{f(x)}{V_1(x)}\ .$$

Here $V_0(x)=1$. The validity of this formal work depends, of course, on the non-vanishing of the functions $V_i(x)$, $i=1, 2, \cdots, n$.

3. *The Mean-Value Theorem.*

THEOREM I. *If the function $f(x)$, continuous in the interval $a\leqq x\leqq b$, vanishes at the $(n+1)$ distinct points x_i*

$$a\leqq x_1<x_2<\cdots<x_{n+1}<b,$$

and if the set (1) *enjoys the property V in (a,b), then*

$$L^\delta f(\xi)=0,\qquad x_1<\xi<x_{n+1}-n\delta$$

for any value of δ satisfying the relations

$$(6)\qquad 0<\delta<x_{i+1}-x_i,\qquad (i=1, 2, \cdots, n)$$
$$0<\delta\leqq\frac{b-x_{n+1}}{n}\ .$$

To establish this theorem we make use of the following Lemma, the proof of which is easily supplied.

LEMMA. *If $f(x)$, continuous in the interval $a\leqq x\leqq b$, vanishes at a and at b, and if $0<\delta<b-a$, then there exists a value ξ such that*

$$\Delta f(\xi)=f(\xi+\delta)-f(\xi)=0,\qquad a<\xi<b-\delta.$$

To prove the theorem we make use of formula (5). Take $d=x_{n+1}$, and form the corresponding functions $h_i(x)$. The functions $V_i(x)$ are positive in the interval $a\leqq x\leqq x_{n+1}$ if $0<\delta\leqq(b-x_{n+1})/n$. Choose a fixed value of δ satisfying the relations (6). Since $V_1(x)$ is positive in $a\leqq x\leqq x_{n+1}$, it follows that the function $f(x)/V_1(x)$ is continuous in that interval, vanishing at the points x_i. We may consequently apply the Lemma to this function. Then

$$\Delta[f(\xi_{1i})/V_1(\xi_{1i})]=0,\qquad x_i<\xi_{1i}<x_{i+1}-\delta,\qquad (i=1, 2, \cdots, n).$$

The function $V_2(x)V_0(x+\delta)$ is positive in the interval $a\leqq x\leqq d$ so that the function

$$\frac{V_1(x+\delta)\,V_1(x)}{V_2(x)\,V_0(x+\delta)}\ \Delta\ \frac{f(x)}{V_1(x)}$$

is continuous in the interval $a \leqq x \leqq x_{n+1} - \delta$. Moreover the distance between two successive zeros ξ_{1i} of this function is greater than δ, so that the Lemma is applicable to this function. It follows that

$$\Delta \frac{V_1(x)\ V_1(x+\delta)}{V_2(x)\ V_0(x+\delta)} \ \Delta \ \frac{f(x)}{V_1(x)} \Bigg|_{x=\xi_{2i}} = 0, \quad \xi_{1i} < \xi_{2i} < \xi_{1i+1} - \delta,$$
$$(i = 1, 2, \cdots, n-1).$$

If we continue in this way, building up the formula (5) step by step, we see that

$$L^\delta f(\xi_{n1}) = 0, \qquad x_1 < \xi_{n1} < \xi_{n-1\,2} - \delta < x_{n+1} - n\delta.$$

The theorem is thus established.

COROLLARY. *If a function $f(x)$, continuous in $a \leqq x \leqq b$, assumes the same value as a function*

$$c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x)$$

at $(n+1)$ distinct points of $a \leqq x < b$, and if the set (1) *enjoys the property V in (a, b), then $L^\delta f(x)$ vanishes at an intermediate point for any positive value of δ sufficiently small.*

The proof is made by applying the theorem to the function

$$\bar{f}(x) = f(x) - c_1 u_1(x) - c_2 u_2(x) - \cdots - c_n u_n(x) \ ,$$

noting that

$$L^\delta \bar{f}(x) \equiv L^\delta f(x).$$

4. *The Interpolatory Properties of the Functions $u_i(x)$.*

THEOREM II. *A necessary and sufficient condition that the vanishing of the function*

$$c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x)$$

at n distinct points of the interval $a \leqq x < b$ should imply the identical vanishing of this function is that the set (1) *possess the property V in (a, b).*

We begin with the sufficiency of the condition. Suppose then that the set (1) enjoys the property V in (a, b), and that a function

$$\phi(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x),$$

not identically zero, vanishes at n distinct points x_i:

$$a \leqq x_1 < x_2 < x_3 < \cdots < x_n < b.$$

Determine a constant d greater than x_n and less than b, and form the corresponding functions $h_i(x)$. Then

$$\phi(x) = a_1h_1(x) + a_2h_2(x) + \cdots + a_nh_n(x).$$

Suppose that

$$a_k \neq 0, \quad a_{k+1} = a_{k+2} = \cdots = a_n = 0, \quad k \leqq n.$$

By a remark in the introduction the property V holds for the functions $h_1(x), h_2(x), \cdots, h_{k-1}(x)$ in the interval (a, d). We may consequently apply the corollary of Theorem I to the difference operator $L^\delta{}_{k-1}$ defined by the relation

$$L^\delta{}_{k-1}f = V[h_1, h_2, \cdots, h_{k-1}, f].$$

Since $\phi(x)$ vanishes at n distinct points of (a, d), $a_kh_k(x)$ assumes the value of the function

$$-[a_1h_1 + a_2h_2 + \cdots + a_{k-1}h_{k-1}]$$

at these points. Since $n > k - 1$, it follows that

$$V[h_1, h_2, \cdots, h_{k-1}, a_kh_k], \quad \text{or} \quad a_kV_k(x)$$

vanishes at an interior point of (a, d) for any value of δ sufficiently small. This, however, is impossible since $V_k(x) > 0$ in (a, d). The sufficiency of the condition is thus established.

To prove the necessity of the condition we begin by assuming that no linear combination of the functions $u_i(x)$, which is not identically zero, vanishes at n distinct points of (a, b). It follows then that the determinant

$$| u_i(x_j) | = \begin{vmatrix} u_1(x_1) & u_1(x_2) & \cdots & u_1(x_n) \\ u_2(x_1) & u_2(x_2) & \cdots & u_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ u_n(x_1) & u_n(x_2) & \cdots & u_n(x_n) \end{vmatrix}$$

is different from zero, no matter what positions the distinct points x_i assume in (a, b). Now choose an arbitrary constant d, $a < d < b$, and choose n distinct points x_i of the interval $b - (b - d)/n < x < b$. Determine linear combinations of the functions (1), $h_i(x)$, satisfying the conditions

$$h_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

This is possible since the determinant $| u_i(x_j) | \neq 0$. Now form the functions

$$V[h_1, h_2, \cdots, h_k], \qquad (k = 1, 2, \cdots, n).$$

We can show that all of these functions are different from zero in the interval $a \leqq x \leqq d$, if $\delta \leqq (b-d)/n$. For, suppose that $V[h_1, h_2, \cdots, h_k]$ vanished at a point x_0 of that interval. It would then be possible to determine a function

$$c_1h_1(x) + c_2h_2(x) + \cdots + c_kh_k(x)$$

not identically zero and vanishing at the points $x_0, x_0+\delta, \cdots, x_0+\overline{k-1}\,\delta$ of the interval $a \leqq x \leqq b-(b-d)/n$. This function would also vanish at the points

$$x_i \qquad (i=k+1, k+2, \cdots, n)$$

of the interval $b-(b-d)/n < x < b$. That is, the function would vanish at n distinct points of (a, b), contrary to assumption. If the functions $V_k(x)$ are not all positive, we have only to change the signs of certain of the functions $h_i(x)$. The proof is thus complete.

COROLLARY 1. *If the set* (1) *possesses the property* V *in* (a, b), *there exists a unique linear combination of the functions of the set taking on prescribed values at* n *arbitrary distinct points of* $a \leqq x < b$.

COROLLARY 2. *The property* W *implies the property* V.

For, Pólya showed that if the property W holds for the set (1), no linear combination of the functions of the set vanishes at n distinct (or coincident) points, unless it is identically zero.

We point out that it may be desirable to make use of the property V rather than the property W even though the functions (1) possess all the necessary derivatives. A case in point is that in which one wishes to distinguish between distinct and coincident zeros. For example, Theorem II shows that the function

$$c_1 + c_2x^3 \qquad (c_1^2 + c_2^2 \neq 0)$$

can not vanish at two distinct points of any interval. Yet the function x^3 has three coincident zeros at the origin.

THEOREM III. *If the set* (1) *possesses the property* V *in* (a, b), *it is possible to determine a linear combination of the functions of the set* changing sign *at* k, $[k < n]$, *distinct points of the interval* $a < x < b$. *Moreover, every linear combination which vanishes at* $n-1$ *points of that interval changes sign at each zero unless it is identically zero.*

Denote the k distinct points by x_i:

$$a < x_1 < x_2 < \cdots < x_k < b.$$

Choose $d > x_k$, and form the functions $h_i(x)$ corresponding. Consider the function

$$u(x) = F(x;\, x_1, x_2, \cdots, x_k) = \begin{vmatrix} h_1(x) & h_2(x) & \cdots & h_{k+1}(x) \\ h_1(x_1) & h_2(x_1) & \cdots & h_{k+1}(x_1) \\ \cdot & \cdot & \cdot & \cdot \\ h_1(x_k) & h_2(x_k) & \cdots & h_{k+1}(x_k) \end{vmatrix}.$$

This function is not identically zero, and vanishes at the points x_i. Since the property V holds for the set $h_1, h_2, \cdots, h_{k+1}$ in (a, d), u vanishes at no other points of (a, d). Furthermore, if we regard the x_i as variables, the function $F(x, x_1, x_2, \cdots, x_k)$ is different from zero if the variables remain distinct. F is a continuous function of its $(k+1)$ variables. Let the points $x, x_1, \cdots, x_k$ vary, always satisfying the inequalities

$$a < x < x_1 < x_2 < \cdots < x_k < d\ ,$$

until they coincide with the points

$$x_0,\ x_0 + \delta,\ x_0 + 2\delta,\ \cdots,\ x_0 + k\delta$$

respectively, all in the interval $a < x < d$. F then reduces to $V_k(x_0)$, which is positive. Hence $u(x)$ is positive if $x < x_1$. A similar proof shows that $u(x) < 0$ if $x_1 < x < x_2$. In this way we see that u changes sign at each of the points x_i.

If $k = n - 1$, the only linear combination of the given functions vanishing at the points x_i is the function $cF(x, x_1, x_2, \cdots, x_{n-1})$, the constant c being arbitrary. Every such function changes sign at the points x_i if $c \neq 0$, so that the theorem is completely established.'

5. *The Remainder Formula.*

THEOREM IV. *Let the set* (1) *possess the property* V *in* (a, b); *let* $f(x)$ *be continuous in* $a \leqq x \leqq b$; *and let* $x_1, x_2, \cdots, x_{n-1}$ *be arbitrary points of* $a < x < b$. *Then there exist functions* $\phi(x)$ *and* $\psi(x)$ *not identically zero satisfying the conditions*

$$\phi(x) = \sum_{i=1}^{n} c_i u_i(x), \qquad \phi(x_i) = f(x_i), \qquad (i = 1, 2, \cdots, n-1)$$

$$\psi(x) = \sum_{i=1}^{n} a_i u_i(x), \qquad \psi(x_i) = 0.$$

If x_0 is any point of $a < x < b$, then

$$f(x_0) = \phi(x_0) + \frac{L^\delta_{n-1} f(\xi)}{L^\delta_{n-1} \psi(\xi)} \psi(x_0) \qquad (a < \xi < b) \tag{7}$$

for any value of δ sufficiently small. Here

$$L^\delta_{n-1} f = V[h_1, h_2, \cdots, h_{n-1}, f], \qquad h_i = \sum_{j=1}^{n} c_{ij} u_j.$$

Choose a constant d greater than all the values x_i, $i = 0, 1, 2, \cdots, n-1$, and form the corresponding functions h_i. The property V holds for the set $h_1, h_2, \cdots, h_{n-1}$ in (a, d), and hence by Corollary 1 to Theorem II we may determine $\phi(x)$ as a linear combination of these functions:

$$\phi(x) = \sum_{i=1}^{n-1} k_i h_i(x), \qquad \psi(x) = \sum_{i=1}^{n} l_i h_i(x).$$

Now form the function

$$F(x) = f(x) - \phi(x) + C\psi(x),$$

where C is a constant to be determined. $F(x)$ vanishes at the $(n-1)$ points x_i, $i = 1, 2, \cdots, n-1$. C is to be determined so that $F(x_0) = 0$. This is possible since $\psi(x_0) \neq 0$. [Theorem II]. We now apply Theorem I to the function F, replacing the operator L^δ of that theorem by the operator L^δ_{n-1} defined by the equation

$$L^\delta_{n-1} f = V[h_1, h_2, \cdots, h_{n-1}, f].$$

We obtain in this way the following equation

$$L^\delta_{n-1} F(\xi) = 0 = L^\delta_{n-1} f(\xi) + C\, L^\delta_{n-1} \psi(\xi), \qquad a < \xi < b,$$

δ being any positive constant sufficiently small. This equation determines C since

$$L^\delta_{n-1} \psi(x) = l_n V_n(x) \neq 0.$$

$V_n > 0$ by hypothesis, and if l_n were zero, ψ would be a linear combination of $h_1, h_2, \cdots, h_{n-1}$, and would be identically zero since it vanishes at $n-1$ points of (a, d). We are thus led to formula (7) of the theorem.

6. *Applications.* As an example of formula (7) take

$$h_k(x) = u_k(x) = x^{k-1}, \qquad (k = 1, 2, 3, \cdots, n+1).$$

A function $\phi(x)$ satisfying the conditions

$$\phi(k\delta)=f(k\delta), \qquad (k=0,\,1,\,2,\,\cdots,\,n-1)$$

is seen to be

$$f(0)+\frac{x}{\delta}\,\Delta^{\delta}{}_1(0)+\cdots+\frac{x(x-\delta)\cdots(x-n-2\delta)}{(n-1)!\;\delta^{n-1}}\Delta^{\delta}{}_{n-1}(0).$$

A function $\psi(x)$ vanishing at the points $0,\,\delta,\,2\delta,\,\cdots,\,\overline{n-1}\,\delta$ is

$$x(x-\delta)(x-2\delta)\cdots(x-\overline{n-1}\,\delta).$$

In this case

$$L_n{}^{\delta_1}\psi=\Delta_n{}^{\delta_1}\psi=n!\;\delta_1{}^n.$$

Formula (7) gives us the relation

$$f(x)=f(0)+\cdots+\frac{x(x-\delta)\cdots(x-\overline{n-2}\,\delta)}{(n-1)!\;\delta^{n-1}}\,\Delta^{\delta}{}_{n-1}(0)$$
$$+\frac{\Delta_n{}^{\delta_1}f(\xi)}{n!\;\delta_1{}^n}\;x(x-\delta)\cdots(x-\overline{n-1}\,\delta),$$
$$0<\xi<(n-1)\delta, \qquad \xi<x.$$

This is Newton's interpolation formula with a remainder which was obtained by S. Bernstein.* The formula was used by him in order to affirm the analyticity of a real function not known to have derivatives.

In a similar way we might obtain for Lagrange's interpolation formula a remainder involving no derivatives.

As a further application we state without proof the following theorem:

THÉOREM V.† *If the set* (1) *possesses the properpyt V in (a,b), and if the functions $h_i(x)$ corresponding to a value d in $a<x<b$ form an orthogonal set on (a,d), then the set $h_i(x)$ forms a set of oscillating functions; that is, $h_i(x)$ vanishes just $(i-1)$ times in $a<x<d$, and the zeros of $h_i(x)$ and of $h_{i-1}(x)$ occur alternately.*

Finally we state the following theorem, the proof of which will be given in a later article.

THEOREM VI.‡ *If the functions $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$, $f(x)$ are*

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* "Sur la définition et les propriétés des fonctions analytiques d'une variable réele," *Mathematische Annalen*, Vol. 75 (1914), p. 452.

† Compare O. D. Kellogg, "The oscillations of functions of an orthogonal set," *American Journal of Mathematics*, Vol. 38 (1916), p. 1.

‡ Compare A. Haar, "Die Minkowskische Geometrie und die Annäherung an stetige Funktionen," *Mathematische Annalen*, Vol. 78 (1917-18), pp. 294-311.

all continuous in the interval $a \leqq x \leqq b$, *then a necessary and sufficient condition that there exist a unique function of approximation (in the sense of Tschebycheff*)*

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

to the function $f(x)$ *in an arbitrary interval* $a \leqq x \leqq d$ *of* (a, b) *is that the set of functions* $u_i(x)$ *possess the property V in* (a, b).

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* $\phi(x)$ is a function of approximation if and only if
$\max | f(x) - \phi(x) | \leqq \max | f(x) - c_1 u_1(x) - c_2 u_2(x) - \cdots - c_n u_n(x) |$
in (a, b), no matter what the constants c_i may be.

Note on Tschebycheff Approximation.

By D. V. Widder.

The problem of the representation of a continuous function $f(x)$ by Tschebycheff polynomials has been discussed in detail by various authors. The more general problem of the representation of $f(x)$ by a linear combination of prescribed continuous functions $u_1(x), u_2(x), \cdots, u_n(x)$, has received less consideration. J. W. Young* has discussed the general problem, and has imposed conditions on the functions $u_i(x)$ that guarantee the existence and uniqueness of the function of approximation (in the sense of Tschebycheff). These conditions are not necessary for the mere existence of the function of approximation. A necessary condition for the uniqueness was not discussed. The problem was also treated by A. Haar,† and it was found that a function of approximation always exists if the functions $u_i(x)$ are continuous. A necessary and sufficient condition for the uniqueness of the function was also obtained. The proofs given, however, are dependent on the theory of the geometry of Minkowski. Since the results are analytic, it seems desirable that a purely analytic proof be given. It is the purpose of the present note to give such a proof and to state the results in a new form.

In the author's paper on the interpolatory properties of a linear combination of continuous functions ‡ it was found that if the functions $u_i(x)$ possess a certain property, there designated as the property V, then every linear combination of them enjoys the more important interpolatory properties of a polynomial. We shall now show that this property gives a necessary and sufficient condition for the uniqueness of the function of approximation.

1. *The Existence of a Function of Approximation.*

Definition. If the functions

$$u_1(x), u_2(x), \cdots, u_n(x), \tag{1}$$

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* "General theory of approximation by functions involving a given number of arbitrary parameters," *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 331-344.

† "Die Minkowskische Geometrie und die Annäherung an stetige Funktionen," *Mathematische Annalen*, Vol. 78 (1917-18), p. 294.

‡ This number of the *American Journal of Mathematics*, p. 221.

and $f(x)$ are continuous in the interval $a \leqq x \leqq b$, then the function

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

is a function of approximation to $f(x)$ in (a, b) if and only if *

$$\max | f(x) - \phi(x) | \leqq \max | f(x) - c_1 u_1(x) - c_2 u_2(x) - \cdots - c_n u_n(x) |$$

for all real values of the constants c_i, the a_i being real constants.

THEOREM I. *There exists a function of approximation for every function $f(x)$ that is continuous in the interval $a \leqq x \leqq b$.*

Set

$$F(x) = \sum_{i=1}^{n} c_i u_i(x), \qquad y = f(x) - F(x).$$

Then the maximum of $| y |$ in $a \leqq x \leqq b$ depends in general on the values of c_i, and will be denoted by $m(c_1, c_2, \cdots, c_n)$. If $f(x)$ is a linear combination of the functions (1), $\phi(x)$ may be taken equal to $f(x)$, and the existence of the function of approximation is obvious. Otherwise $m(c_1, c_2, \cdots, c_n) > 0$ no matter what values the c_i assume. Now the function $m(c_1, c_2, \cdots, c_n)$ is continuous, since a small change in the c_i produces a small change in y and hence also in $\max | y |$.

Let M be the maximum of $| f(x) |$ in $a \leqq x \leqq b$, and for the present let us confine our attention to those functions $F(x)$ for which

$$| F(x) | \leqq 2M.$$

We may assume, without loss of generality, that the functions $u_i(x)$ are linearly independent. For, either they are all identically zero (in which case the theorem is true but trivial), or else there is some sub-set which is linearly independent. In the latter case the subsequent developments will apply for a smaller value of n. It follows then that the determinant

$$\Delta = | u_i(x_j) | \qquad (i, j = 1, 2, \cdots, n)$$

is not identically zero in the n variables x_j. Choose $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ so that $\Delta \neq 0$ for these values of the variables. It is then possible to determine functions

$$s_i(x) = \sum_{j=1}^{n} \alpha_{ij} u_j(x), \qquad (i = 1, 2, \cdots, n)$$

satisfying the conditions

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* The notation $\max | F(x) |$ means the maximum absolute value of $F(x)$ in $a \leqq x \leqq b$.

$$s_i(\overline{x}_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n)$$

the constants α_{ij} being uniquely determined.

We can now express the arbitrary constants c_i on which $F(x)$ depends in terms of the values $F(\overline{x}_i)$ as follows:

$$\begin{aligned} F(x) &= c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) \\ &= F(\overline{x}_1)s_1(x) + F(\overline{x}_2)s_2(x) + \cdots + F(\overline{x}_n)s_n(x) \\ &= F(\overline{x}_1) \sum_{j=1}^{n} \alpha_{1j}u_j(x) + F(\overline{x}_2) \sum_{j=1}^{n} \alpha_{2j}u_j(x) + \cdots + \\ &\qquad\qquad + F(\overline{x}_n) \sum_{j=1}^{n} \alpha_{nj}u_j(x) \\ &= u_1(x) \sum_{j=1}^{n} \alpha_{j1} F(\overline{x}_j) + u_2(x) \sum_{j=1}^{n} \alpha_{j2} F(\overline{x}_j) + \cdots + \\ &\qquad\qquad + u_n(x) \sum_{j=1}^{n} \alpha_{jn} F(\overline{x}_j). \end{aligned}$$

Hence

$$c_i = \sum_{j=1}^{n} \alpha_{ji} F(\overline{x}_j), \qquad (i = 1, 2, \cdots, n).$$

Since $| F(\overline{x}_j) | \leqq 2M$ it follows that

$$(2) \qquad | c_i | \leqq 2M \sum_{j=1}^{n} | \alpha_{ji} | = M_i, \qquad (i = 1, 2, \cdots, n).$$

That is, if we confine our attentions to functions $F(x)$ in absolute value not greater than $2M$, then the constants c_i are no longer arbitrary, but are restricted by the relations (2). The continuous function $m(c_1, c_2, \cdots, c_n)$ takes on a minimum value in the closed domain defined by these inequalities. If the minimum value is

$$m(a_1, a_2, \cdots, a_n) = \mu,$$

we shall show that the function

$$\phi(x) = \sum_{j=1}^{n} a_j u_j(x)$$

is a function of approximation. By the manner in which μ was obtained it is clear that

$$(3) \quad \mu = \max | f(x) - \phi(x) | \leqq \max | f(x) - F(x) | = m(c_1, c_2, \cdots, c_n)$$

provided that $| F(x) | \leqq 2M$. In particular

$$\mu \leqq \max | f(x) | = m(0, 0, \cdots, 0) = M.$$

It remains to show that the relation (3) holds even if the c_i are not restricted by (2). In this case we may have

$$\max | F(x) | > 2M$$
$$\max | f(x) - F(x) | > 2M - M \geqq \mu.$$

Hence (3) holds in all cases, and $\phi(x)$ is in fact a function of approximation.

2. *Uniqueness of the Function of Approximation.*

THEOREM II. *If the functions $u_1(x), u_2(x), \cdots, u_n(x), f(x)$ are continuous in the interval $a \leqq x \leqq b$, then a necessary and sufficient condition that there exist a unique function of approximation*

$$\phi(x) = \sum_{i=1}^{n} a_i u_i(x)$$

to the function $f(x)$ in the arbitrary interval $a \leqq x \leqq d$ of (a, b) is that the set of functions $u_i(x)$ possess the property V in (a, b).[*]

We begin with the sufficiency of the condition. Assuming the property V in (a, b) we wish to show the uniqueness of the function of approximation, $\phi(x)$, to $f(x)$ in the interval $a \leqq x \leqq d$. Here d is an arbitrary point in the open interval (a, b). Since the function

$$y = f(x) - \phi(x) \tag{4}$$

is continuous, $| y |$ takes on its maximum, m, at least once in $a \leqq x \leqq d$. Consider the set of points (x, y) on the curve defined by (4) for which $y = \pm m$. We shall say that the set presents a change of sign between (x', y') and (x'', y'') if $y' = -y'' = \pm m$, and if there exists no point (x, y) of the set for which $x' < x < x''$. It will now be shown that the set of points presents at least n changes of sign. For, suppose that there were only k changes of sign $[k < n]$. If the set has a change of sign between (x', y') and (x'', y''), choose a value x_i such that $x' < x_i < x''$. We thus obtain k values

$$x_1 < x_2 < \cdots < x_k.$$

It was shown in the author's paper already cited (Theorem III) that if the set (1) possesses the property V in (a, b), then it is possible to determine a linear combination of the functions of the set,

$$H(x) = \sum_{i=1}^{n} c_i u_i(x),$$

changing signs at each of the distinct points x_i. If $H(x)$ has further zeros they will all be greater than x_k.

The points x_i divide the interval (a, d) into $k + 1$ sub-intervals such that y can not take on the values m and $-m$ in the same interval and takes on both values in two adjoining intervals. We can now determine a constant η such that the function

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[*] For the precise definition of the property V, see the author's paper already cited.

$$y - \eta H(x)$$

is less than m in absolute value in (a, d). Suppose for definiteness that $y = +m$ at some point of (a, x_1). Then the following relations hold:

$$(5)\qquad \begin{array}{lll} -m+\epsilon \leqq y \leqq m & \text{in} & a \leqq x \leqq x_1 \\ -m \leqq y \leqq m-\epsilon & \text{in} & x_1 \leqq x \leqq x_2 \\ -m+\epsilon \leqq y \leqq m & \text{in} & x_2 \leqq x \leqq x_3 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array}$$

Here ϵ is some positive constant sufficiently small. Determine η so that $\eta H(x) > 0$ in (a, x_1), and such that

$$|\eta H(x)| < \epsilon \qquad \text{in} \qquad a \leqq x \leqq b.$$

Then

$$(6)\qquad \begin{array}{lll} -\epsilon < -\eta H(x) < 0 & \text{in} & a < x < x_1 \\ 0 < -\eta H(x) < \epsilon & \text{in} & x_1 < x < x_2 \\ -\epsilon < -\eta H(x) < 0 & \text{in} & x_2 < x < x_3 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array}$$

Combining the relations (5) with the inequalities (6) we see that

$$-m < y - \eta H(x) < m \qquad \text{in} \qquad a \leqq x \leqq b.$$

Hence $\phi(x) + \eta H(x)$ is a linear combination of the functions (1) which has closer approximation to $f(x)$ than $\phi(x)$. This contradicts the hypothesis that $\phi(x)$ is a function of approximation.

It is now a simple matter to show that $\phi(x)$ is unique. For, suppose that there were another function of approximation $\bar{\phi}(x)$. Form the function

$$\psi(x) = [f(x) - \phi(x)] - [f(x) - \bar{\phi}(x)].$$

When $y = m$, $\psi(x) \geqq 0$ since $f(x) - \bar{\phi}(x)$ is not greater than m; and when $y = -m$, $\psi(x) \leqq 0$. Since the set of points (x, y) presents at least n changes of sign, it follows that $\psi(x)$ vanishes at least n times in $a \leqq x \leqq d$. This is impossible unless $\psi(x)$ vanishes identically, since it is a linear combination of the functions (1)*. The contradiction shows that $\phi(x)$ is unique.

We turn now to the necessity of the condition. We shall show that if the function of approximation to $f(x)$ in the arbitrary interval $a \leqq x \leqq d < b$ is unique, then no linear combination of the functions (1) vanishes n times in

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* See Theorem II of the author's paper already cited.

$a \leqq x \leqq d$, unless it is identically zero, and hence that the property V holds in (a, b).* For, suppose that the function

$$h(x) = \sum_{i=1}^{n} b_i u_i(x),$$

not identically zero, vanishes at n distinct points x_i, $(i = 1, 2, \cdots, n)$ of $a \leqq x \leqq d$. Then the determinant

$$| u_i(x_j) | \qquad (i, j = 1, 2, \cdots, n)$$

vanishes. We may consequently determine constants $a_1, a_2, \cdots, a_n$, not all zero, such that

$$\sum_{i=1}^{n} a_i u_j(x_i) = 0, \qquad (j = 1, 2, \cdots, n)$$

Then if

$$F(x) = \sum_{i=1}^{n} c_i u_i(x),$$

it follows that

$$\sum_{i=1}^{n} a_i F(x_i) = 0 \tag{7}$$

no matter what value the constants c_i assume. We can now show that for a suitable choice of the functon $f(x)$ two functions of approximation exist. Choose $f(x)$ such that

$$\begin{aligned} f(x_i) &= a/| a_i | && \text{if} && a_i \neq 0, \\ f(x_i) &= 0 && \text{if} && a_i = 0, \\ | f(x) | &\leqq 1. \end{aligned}$$

Then

$$\max | f(x) - F(x) | \geqq 1.$$

For if this were not the case, $F(x)$ would have the same sign as $f(x)$ at each of the points x_i [for which $f(x_i) \neq 0$], and hence we should have

$$\sum_{i=1}^{n} a_i F(x_i) > 0.$$

But this contradicts the equation (7). Hence any function $F(x)$ such that $| f(x) - F(x) | \leqq 1$ is a function of approximation. In particular $F(x) \equiv 0$ is a function of approximation since $| f(x) | \leqq 1$. We now restrict $f(x)$ further so that

$$| f(x) - h(x) | \leqq 1.$$

This restriction is compatible with the foregoing restrictions since $h(x)$ vanishes at the points x_i. Then $h(x)$ is also a function of approximation, not identically zero, and we have obtained a contradiction. The theorem is thus completely established.

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* See Theorem II of the author's paper already cited.

Generalizations of Waring's Theorem on Fourth, Sixth, and Eighth Powers.

BY L. E. DICKSON.

In a recent number of this Journal, I investigated positive quadratic forms which represent every positive integer p. The fact that there are so many such forms led me to conjecture the existence of numerous forms of every degree n which represent every p. The form $a_1x_1^n + \cdots + a_mx_m^n$ is said to be of order m and *weight* $a_1 + \cdots + a_m$. I soon found[*] all such forms M of minimum weight when $n = 3, 4, 5$, while the cases $n = 6, 7$ have been treated in Chicago theses. Most of these forms M have the important property that their order is considerably smaller than their weight, while order and weight are equal for $x_1^n + \cdots + x_m^n$. The new generalizations of Waring's theorem on the latter form therefore involve a material reduction in order.

The theorems for $n = 4, 6, 8$ proved here have the like improvement of a reduction in order over the known results for Waring's theorem for the same values of n.

For $s = 0, 1, \cdots, 10$, I prove that every p is a sum of s doubles of biquadrates and $37 - 2s$ biquadrates, and hence p is represented by a form of order $37 - s$. The case $s = 0$ gives the best known complete result for Waring's theorem on biquadrates.[*] Hence there is a reduction of the order by s, whose maximum is 10.

Fleck showed that 2451 sixth powers suffice. Kempner gave a reduction to 970, and Baer to 478. My Theorems 2, 3, 4 give reductions to forms of orders 223, 277, 190, respectively.

Hurwitz proved that 36119 eighth powers suffice. By a simple remark, Kempner reduced this to 31353. My final theorem gives a reduction to a form of order 2690.

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[*] Recent numbers of the *Bulletin of the American Mathematical Society*, *American Mathematical Monthly*, *Annals of Mathematics*.

[†] Liouville proved that 53 biquadratets suffice. Reductions were made in turn by Réalis, Lucas, Fleck, Landau, and Wieferich (to 37). See the writer's *History of the Theory of Numbers*, Vol. 2, pp. 717-20. In his thesis, *Beiträge zum Waringschen Problem*, Göttingen, 1913, Baer modified the Landau-Wieferich type of proof and showed that 34 biquadrates suffice for $48k + 1$ and $48k + 33$. Some computations made in that proof are avoided in the present proof (Lemma 8).

PART I. LEMMAS ON *n*TH POWERS.

LEMMA 1. *If s and t are positive, n is a positive integer, and if $s \geqq g + t$, there exists a positive integer i such that*

$$g \leqq s - ti^n < g + tR, \quad R = r^n - (r-1)^n, \quad r = [s-g)/t]^{1/n}. \tag{1}$$

With the understanding that r is the positive real root, we have $r \geqq 1$. Write $r = i + f$, where $0 \leqq f < 1$, and i is an integer $\geqq 1$. Since $i \leqq r$,

$$g = s - tr^n \leqq s - ti^n,$$

as desired in (1). Next,

$$s - ti^n - g = tw, \quad w = r^n - (r-f)^n,$$
$$R - w = (r-f)^n - (r-1)^n > 0,$$

since $r - f$ exceeds $r - 1$, which is $\geqq 0$. This proves Lemma 1. To prove that

$$R < n\, r^{n-1}, \tag{2}$$

write $\rho = r - 1$. Then

$$R = (r - \rho) Q = Q = \sum_{i=0}^{n-1} r^{n-1-i} \rho^i.$$

Since $\rho < r$, each of the n terms of Q is $\leqq r^{n-1}$. When $g \geqq 0$, then $r^n \leqq s/t$ and we have

LEMMA 2. *If s and t are positive, n is a positive integer, $g \geqq 0$, and $s \geqq g + t$, there exists a positive integer i such that*

$$g \leqq s - t\, i^n < g + n(ts^{n-1})^{1/n}. \tag{3}$$

If $s < t$, then $s^n < t\, s^{n-1}$, and s is less than the radical and hence less than its product by n. Then (3) holds with $g = 0$, $i = 0$. Thus Lemma 2 implies

LEMMA 3. *If s and t are positive and n is a positive integer, there exists an integer $i \geqq 0$ such that*

$$0 \leqq s - t\, i^n < n(ts^{n-1})^{1/n}. \tag{4}$$

Hence there exists an integer $i_1 \geqq 0$ such that

$$0 \leqq s_1 = s - t\, i_1{}^n < m\, s^\nu, \qquad m = n\, t^{1/n}, \quad \nu = (n-1)/n.$$

Applying this to s_1 in place of s, we see that there exists an integer $i_2 \geqq 0$ such that

$$0 \leqq s_2 - s_1 - t i_2^{\,n} < m s_1^{\,\nu} < m^{1+\nu} s^{\nu^2}.$$

By induction on r, we obtain

LEMMA 4. *If s and t are positive and n is a positive integer, there exist integers $i_1, \cdots, i_r$, each $\geqq 0$, such that*

$$s = t i_1^{\,n} + \cdots + t i_r^{\,n} + s_r, \qquad \nu = (n-1)/n, \tag{5}$$

$$0 \leqq s_r < (n t^{1/n})^e s^{\nu^r}, \qquad e = 1 + \nu + \nu^2 + \cdots + \nu^{r-1}. \tag{6}$$

PART II. FOURTH POWERS.

We write $[b, d]$ for a sum of b biquadrates and d doubles of biquadrates. Our goal is

THEOREM 1. *Every positive integer is represented by* $[17, 10]$.

The main part of the proof applies to all integers exceeding a limit just under 95^4. We first prove

LEMMA 5. *Every positive integer $s \leqq 95^4$ is represented by* $[17, 10]$.

By Lemma 4 with $t = 2$, $n = 4$, $r = 5$, we have

$$\begin{gathered} s = 2 i_1^{\,4} + \cdots + 2 i_5^{\,4} + s_5, \qquad 0 \leqq s_5 < cd, \\ c = 2^{9e/4}, \qquad d = 95^f, \qquad f = 3^5/4^4, \qquad e = 3.050781. \end{gathered} \tag{7}$$

But $d < 78$ since $95^{243} < 78^{256}$, and $c < 2^7$. Hence $s_5 < 9984$. We next prove[*] that every positive integer $p \leqq 9986$ is represented by $[17, 5]$. If $p \geqq 2 \cdot 7^4 = 4802$, $p = 2 \cdot 7^4 + q$, $0 \leqq q \leqq 5184 = 4 \cdot 6^4$. Subtracting $2 \cdot 6^4$ from q when possible, we see that it remains to prove that every positive integer $\leqq 2 \cdot 6^4 = 2592$ is represented by $[17, 3]$. It is either $\leqq 2 \cdot 5^4 = 1250$ or exceeds the latter by an integer $u \leqq 1342$. If $u \geqq 1250$, $u \leqq 2 \cdot 5^4 + 92$. If $2 \cdot 4^4 = 512 \leqq u < 1250$, $u \leqq 2 \cdot 4^4 + 738$. Hence it remains only to show that every integer $\leqq 738$ is represented by $[17, 1]$. While a direct proof is not tedious, it has been verified in several ways.[†]

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[*] Hence Lemma 5 holds when 95 is replaced by 108.756, and therefore for $s < 1.399 \times 10^8$. See Lemmas 10, 11.

[†] Since $[17, 1]$ is a partition of all the forms in *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 319-27, in particular of that in § 8.

LEMMA 6. *If p is odd, $6p^2$ is represented by* $[3,4]$. *For any positive integer k, $6k^2$ is represented by* $[6,3]$.

It is known that every positive integer p, which is not of the form $q=4^r\,(16s+14)$ is represented by $a^2+b^2+2c^2$. The identity

$$(8)\qquad 6(a^2+b^2+2c^2)^2\equiv(a+b)^4+(a-b)^4+(2c^4)+2(a+c)^4 \\ +2(a-c)^4+2(b+c)^4+2(b-c)^4$$

shows that $6p^2$ is represented by $[3,4]$. Next, $16s+14$ is congruent to 6 modulo 8 and is known to be a sum of three squares. If we multiply all their roots by 2^r, we see that q is a sum of three squares. The identity

$$(9)\qquad 6(a^2+b^2+c^2)^2\equiv(a+b)^4+(a-b)^4+(a+c)^4+(a-c)^4 \\ +(b+c)^4+(b-c)^4+2a^4+2b^4+2c^4$$

shows that $6q^2$ is represented by $[6,3]$. Since any positive integer k is either a p or a q, and since representation by $[3,4]$ implies that by $[5,3]$ and hence by $[6,3]$, with one biquadrate zero, Lemma 6 is proved.

LEMMA 7. *Every positive integer $24k+6$ is represented by* $[15,10]$. *Every $24k+12$ is represented by* $[12,11]$.

For, every positive integer $4k+1$ or $4k+2$ is a sum of three squares, exactly 1 or 2 of which are odd, respectively. We apply Lemma 6. Since $6(4k+2)=6r^2+6s^2+6p^2$, where r and s are odd, it is represented by the sum $[12,11]$ of $[3,4]$, $[3,4]$, and $[6,3]$. Similarly, $6(4k+1)$ is represented by the sum $[15,10]$ of $[3,4]$, $[6,3]$, $[6,3]$.

LEMMA 8. *Every positive integer $48k+1$ or $48k+33$ is represented by* $[10,12]$ *if it exceeds r^4-480, where $r=95$ or 93, respectively.*

For $a=12n\pm1$, a^4-1 is the product of

$$a\mp1=12n,\quad a\pm1=2(6n\pm1),\quad a^2+1=2(72n^2\pm12n+1).$$

Hence

$$1-a^4=-48q,\quad q=n(\pm144n^2+18n\pm1)\equiv n(2n\pm1) \pmod{16}.$$

We readily prove that q ranges with n over a complete set of residues modulo 16. For, if $x(2x\pm1)\equiv n(2n\pm1) \pmod{16}$, evidently $x\equiv n \pmod 2$, whence $2x^2\equiv2n^2 \pmod 8$. Then the initial congruence gives $x\equiv n \pmod 8$ and therefore also $x\equiv n \pmod{16}$.

Since $n=16-m$ implies $n(2n+1)\equiv m(2m-1) \pmod{16}$, the values of $n(2n+1)$ for $n=0, 1, \cdots, 7$, and the values of $n(2n-1)$ for $n=1, \cdots, 8$, together form a complete set of residues modulo 16. The corresponding a's have the maximum $12 \cdot 8 - 1 = 95$.

Hence there exists a positive integer $a \leqq 95$ such that $1 - a^4 \equiv 48\alpha$ (mod $48 \cdot 16$), where α is any prescribed residue modulo 16. Take $\alpha = 6 - k$, where $s = 48k + 1$ is the first integer in Lemma 8. Then there is an integer h such that

$$1 - a^4 = 48(6-k) + 48 \cdot 16\, h, \quad s - a^4 = 48(6 + 16\, h) = 2^4 \cdot 6\, (8\, h + 3).$$

Since $s > 95^4 - 480$, we have $h > -1$. Thus $h \geqq 0$, and $8h+3$ is a sum of three odd squares. Lemma 6 shows that $6(8h+3)$ is represented by $[9, 12]$. Since the same is true of its product $s - a^4$ by 2^4, s is represented by $[10, 12]$.

Second, let $s = 48k + 33$. For $a = 3t$, $t = 4m \pm 1$,

$$33 - a^4 = 48Q, \quad Q = -1 - 27\,(16m^4 \pm 16m^3 + 6m^2 \pm m),$$
$$Q \equiv (7m \mp 1)(2m \pm 1) \equiv 7n(2n \mp 1) \pmod{16},$$

where $n = m \mp 7$.

As in the first case, the values of $n(2n+1)$ for $n = 8, \cdots, 15$ and of $n(2n-1)$ for $n = 0, -1, \cdots, -7$ together form a complete set of residues modulo 16. For the former, $m = n - 7 = 1, \cdots, 8$; $t = 4m - 1 = 3, 7, \cdots, 31$. For the latter, $m = n + 7 = 0, 1, \cdots, 7$; $t = 4m + 1 = 1, 5, \cdots, 29$. Hence * when t ranges over the 16 positive odd integers $\leqq 31$, Q ranges over a complete set of residues modulo 16. The proof now proceeds as in the first case.

Lemma 9. *Every positive integer* $s = 48k + 9$ *or* $48k + 25$ *is represented by* $[13, 11]$ *if it exceeds* $r^4 - 456$, *where* $r = 93$ *or* 95, *respectively.*

Since $s - 24 \equiv 33$ or 1 (mod 48), the proof of Lemma 8 shows the existence of a positive integer a such that

$$s - 24 - a^4 = 2^4 \cdot 6(8h+3), \quad s - a^4 = 24(16l + 13),$$

where h is an integer $\geqq 0$, and $l = 2h$. Since $16l + 13$ is $\equiv 1$ (mod 4), it is a sum of an odd and two even squares. The odd one is $\equiv 1$ (mod 8) and hence is $\equiv 1$ or 9 (mod 16). An even square is $\equiv 0$ or 4 (mod 16). Hence any sum of two even squares is $\equiv 0, 4$, or 8. One of the latter increased by 1 or 9 shall give 13 (mod 16), whence the summands are 4 and 9. Hence

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* A second proof follows from the fact that $t^4 \equiv \tau^4$, $\tau \equiv tx$ (mod 16^2) imply $x^4 \equiv 1$ (mod 16^2), whence $x \equiv \pm 1$ (mod 64).

$$16l+3=u^2+v^2+w^2,\quad u^2\equiv 9,\quad v^2\equiv 4,\quad w^2\equiv 0\pmod{16}.$$

We may take u, v, w all $\geqq 0$. Since $u=4n\pm 1$, $9\equiv \pm 8n+1 \pmod{16}$ and n is odd. Thus $u\equiv 3$ or $5 \pmod 8$. Next, v is the double of an odd integer. Hence both u and v are sums of three squares. The identity

$$(10)\quad 24(a^2+b^2+c^2)^2\equiv 2(a+b+c)^4+2(a+b-c)^4+2(a-b+c)^4 \\ +2(a-b-c)^4+(2a)^4+(2b)^4+(2c)^4$$

shows that $24u^2$ and $24v^2$ are both represented by $[3,4]$. Since $w=2m$, $24w^2=2^4\cdot 6m^2$ is represented by $[6,3]$ by Lemma 6. Hence $s-a^4$ is represented by

$$[3,4]+[3,4]+[6,3]=[12,11].$$

Proof of Theorem 1. Any integer $24n+1$ is of one of the forms $48k+1$, $48k+25$. Hence Lemmas 8 and 9 together state that every positive integer $24n+1$ or $24n+9$, exceeding 95^4-456, is represented by $[13,11]$ and hence by $[15,10]$. By Lemma 7, every positive $24n+6$ is represented by $[15,10]$, and every $24n+12$ by $[14,10]$.

By adding 1^4 or 1^4+1^4 to these types $24n+1, 6, 9, 12$, we get $24n+2$ or 3, $24n+7$ or 8, $24n+10$ or 11, $24n+13$ or 14. Next,

$$\begin{aligned}
24n &= \{24(n-7)+6\}+3^4+3^4,\\
24n+4 &= \{24(n-1)+12\}+2^4,\\
24n+15 &= \{24(n-3)+6\}+3^4,\\
24n+17 &= \{24n+1\}+2^4,\\
24n+18 &= \{24(n-3)+9\}+3^4,\\
24n+20 &= \{24(n-1)+12\}+2^4+2^4,\\
24n+21 &= \{24(n-3)+12\}+3^4.
\end{aligned}$$

The numbers involved are all positive if $n\geqq 7$. Those with $n<7$ are $\leqq 24\cdot 6+23=137$. By adding 1^4 to $24n+4, 15, 18, 21$, we get $24n+5$, 16, 19, 22. Hence the latter may be obtained by adding just two biquadrates to numbers of the types $24f+6, 9, 12$. We have now reached all the types $24n+0, 1, \cdots, 22$. Every such number N exceeding 95^4-294 is represented by $[17,10]$. For it was obtained by adding at most two biquadrates with the maximum sum 3^4+3^4 to a number $p=24k+1, 6, 9, 12$. Since $N\leqq p+162$, p exceeds 95^4-456 and hence is represented by $[15,10]$.

Hence N is represented by $[17, 10]$. Finally, $j = 24n + 23$ is obtained by adding 2 to $24n + 21$ and hence by adding $3^4 + 1^4 + 1^4$ to $24(n-3) + 12$. Thus j is represented by $[3, 0] + [14, 10] = [17, 10]$.

We have now proved that every integer exceeding $95^4 - 294$ is represented by $[17, 10]$. The same is true of those $\leqq 95^4$ by Lemma 5. Hence all are represented, as stated in Theorem 1.

The form $[17, k]$. Since this probably represents all positive integers when $k \geqq 1$, it is desirable to find the largest integer L_k for which we can prove by Lemmas 1-3 that every integer $< L_k$ is represented by $[17, k]$.

LEMMA 10. *Every positive integer* $p < 7848$ *is represented by* $[17, 1]$.

This is known for $p \leqq 4100$ (*Bulletin, l. c.*, § 8). We apply Lemma 1 with $n = 4$, $t = 2$, $g = 1249$, and write $\rho = 2r - 1$. Hence if $s \geqq 1251$, there exists a positive integer i such that

$$(11) \qquad 1249 \leqq \sigma < 1249 + \rho^3 + \rho, \quad \sigma = s - 2i^4, \quad 2r^4 = s - 1249.$$

The table of Bretschneider * shows that 1248 is the last integer $\leqq 4100$ which is not a sum of 17 biquadrates. Hence σ will be such a sum if the upper limit in (11) is 4101. Then

$$\rho^3 + \rho = 2852, \quad \rho = 14.1578, \quad r^4 = 3299.32, \quad s = 7847.6.$$

LEMMA 11. *Every positive integer* $< L_k$ *is represented by* $[17, k]$, *where* $L_2 = 20226$, $L_3 = 74425$, $L_4 = 421450$, $L_5 = 4148791$, $L_6 = 85388000$.

Taking the upper limit in (11) to be 7848, L_2, L_3, $\cdots$, we get

$\rho^3 + \rho$	ρ	r	r^4	s
6599	18.7391	9.8695	9488.13	20225.26
18977	26.66075	13.83038	36587.8	74424.6
73176	41.8190	21.4095	210100.1	421449.2
420201	74.896227	37.94811	2073771	4148791
4147542	160.66623	80.83312	42693500	85388000

PART III. SIXTH POWERS.

We write $(1_r, 8_s)$ for a sum of r sixth powers and the products of s sixth powers by 8.

THEOREM 2. *Every positive integer is represented by* $f = (1_{115}, 8_{108})$ *in sixth powers.*

Journal für Mathematik, Vol. 46 (1853), pp. 1-23.

We employ the identity due to A. J. Kempner:*

$$(12) \qquad 120(a^2+b^2+c^2+d^2)^3 \\ = \sum_8 (a \pm b \pm c \pm d)^6 + \sum_4 (2a)^6 + 8 \sum_{12} (a \mp b)^6.$$

Hence the product of 120 by the cube of any integer $\geqq 0$ is represented by $(1_{12}, 8_{12})$. Any positive integer p is a sum of nine cubes. Hence $120p$ is represented by $(1_k, 8_k)$, $k = 9 \cdot 12$. It is readily shown that any integer is congruent to a sum of 7 sixth powers with respect to 3, 5, or 8 as modulus, and hence with respect to their product 120 as modulus. Hence if A is an integer $\geqq 7 \cdot 119^6$, $A = \sigma + 120p$, where $p \geqq 0$ and σ is a sum of 7 sixth powers. Hence A is represented by f.

It remains to prove this also when $A < 7 \cdot 119^6$. Note that $7 < 2^3$, $119 < 128 = 2^7$. We shall prove that every positive integer $s \leqq 2^{45}$ is represented by $(1_{43}, 8_{16})$ and hence by f.

We apply Lemma 4 with $t = 1$, $n = 6$, $r = 15$, whence $\nu = 5/6$. The inequalities (6) still hold if we increase s to 2^{45} and increase e to 6 by extending the geometrical progression to infinity. Thus

$$0 \leqq s_{15} < 6^6\, 2^m, \quad m = 45(5/6)^{15} < 3,$$

since

$$15 \log 1.2 = 1.187718 > 1.176091 = \log 15, \quad (6/5)^{15} > 15.$$

But $6^6\, 2^3 < 9^6$ since $2^3 < 3^2$. Hence by (5), s is a sum of 15 sixth powers $s \geqq 0$ and an integer m, where $0 \leqq m < 9^6$. Then $m = 3^6\alpha + \beta$, $0 \leqq \alpha \leqq 3^6$. $0 \leqq \beta \leqq 3^6$. We shall show that α and β are each represented by $(1_{14}, 8_8)$, whence m is represented by $(1_{28}, 8_{16})$. For, $3^6 < 12 \cdot 64$. When u, v, w are chosen from $0, 1, \cdots, 7$, $u + 8v + 2^6 w$ is represented by $(1_7, 8_7, 1_7)$. When $u, v = 0, 1, \cdots, 7$ and $z = 1, 2, 3$, $u + 8v + 2^6(8 + z)$ is represented by $(1_7, 8_7, 8_1, 1_3)$.

THEOREM 3. *Every positive integer is represented by* $(1_{178}, 8_{99})$ *in sixth powers.*

In the proof of Lemma 6 it was shown that every positive integer k is represented by at least one of $a^2 + b^2 + 2c^2$ and $a^2 + b^2 + c^2$. In the respective cases we apply (12) with $d = c$ or $d = 0$ and see that one term vanishes, whence $120k^3$ is represented by $(1_{12}, 8_{11})$ or $(1_{11}, 8_{12})$, and hence always by $(1_{19}, 8_{11})$. The rest of the proof of Theorem 2 applies here.

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* *Über das Waringsche Problem*, Dissertation, Göttingen, 1912, p. 47. Extract in *Mathematische Annalen*, Vol. 72 (1912), p. 396.

THEOREM 4. *Every positive integer S is represented by $g = (1_{10}, 2_{90}, 3_{72}, 4_{18})$ in sixth powers.*

We employ facts proved by Baer, *l. c.*, pp. 41-51. For at least one of the values 273 and 281 of σ, $S - \sigma$ is congruent to a sum $\sum_7$ of 7 sixth powers modulo 360, each $< 360^6 < 2^{9 \cdot 6}$. Assume first that $S \geqq 2^{968}$. Write $S = \sum_7 + l$. Then $l > 2^{968} - 7 \cdot 2^{54} > 2^{968} - 2^{57} > 2^{967}$, and $l \equiv \sigma \pmod{360}$. By an intricate proof, Baer showed that every such l is a sum of three sixth powers together with the product by 1440 of the sum of the cubes of six doubles of positive odd integers. Each such double is a sum of three squares. The identity

$$1440\,(a^2 + b^2 + c^2)^3$$

$$\equiv 2 \sum_{12} (2a \pm b \pm c)^6 + 2 \sum_{3} (2a)^6 + 3 \sum_{12} (2a \pm b)^6 + 4 \sum_{3} a^6$$

employed six times shows that l is represented by $(1_3, 2_{96}, 3_{72}, 4_{18})$. Hence $S = \sum_7 + l$ is represented by g.

We next prove that S is represted by g if $0 \leqq S \leqq 2^{968}$. Write $S = k^6 + 2s$, where $k = 0$ or 1 according as S is even or odd. Apply Lemma 4 with $t = 1$, $n = 6$, $r = 40$. The inequalities (6) still hold if we increase s to 2^{967} and increase e to 6 by extending the geometrical progression to infinity. Thus

$$s = \sum_{j=1}^{40} i_j^6 + s_{40}, \quad 0 \leqq s_{40} < 6^6\, 2^m, \quad m = 967(5/6)^{40}.$$

But

$$625 = 5^4 < 3^4\, 2^3 = 648, \quad (5/6)^4 < \tfrac{1}{2}, \quad 2^{10} = 1024 > 967, \quad m < 1;$$

$$\sigma = 2s_{40} < 4 \cdot 6^6, \quad \sigma = 3^7 \alpha + \beta, \quad 0 \leqq \beta < 3^7, \quad \alpha < \frac{4 \cdot 6^6}{3^7} = \frac{4 \cdot 2^6}{3} < 86.$$

Now α is a sum of 63 sixth powers, which are all 0 or 1 if $\alpha < 64$, while one is 2^6 if $\alpha \geqq 64$. Hence $3^6 \cdot 3\alpha$ is a sum of 63 triples of sixth powers. Next, $\beta = p \cdot 3^6 + \delta$, $p = 0$, 1 or 2, $0 \leqq \delta < 3^6$. Also, $\delta = 3 \cdot 2^6 q + \epsilon$, $q = 0$, 1, 2, or 3, $0 \leqq \epsilon < 3 \cdot 2^6$. Evidently $\epsilon = \lambda$, $2^6 + \lambda$ or $2 \cdot 2^5 + \lambda$, where $0 \leqq \lambda \leqq 63$. An even λ is $2u$, where $u \leqq 31$, and is a sum of u doubles of sixth powers, all unity. An odd λ is $1 + 2u$. Hence every λ, ϵ, δ, β, σ is represented by respectively $(1, 2_{31})$, $(1_3, 2_{31})$, $(1_3, 2_{31}, 3_3)$, $(1_5, 2_{31}, 3_3)$, $(1_5, 2_{31}, 3_{66})$. Hence every $S \leqq 2^{968}$ is represented by $(1_6, 2_{71}, 3_{66})$ and therefore by f.

PART IV. EIGHTH POWERS.

A. Hurwitz [*] gave the identity

$$5040\,(a^2+b^2+c^2+d^2)^4 \equiv \sum_{48} (2a \pm b \pm c)^8 + 6 \sum_{4} (2a)^8$$
$$+ 6 \sum_{8} (a \pm b \pm c \pm d)^8 + 60 \sum_{12} (a \pm b)^8.$$

Any positive integer is of the form $5040\,Q + R$, where $0 \leqq R \leqq 5039$. Since Q is a sum of the fourth powers of 37 integers (Part II), each a sum of four squares, $5040\,Q$ is represented by 37 sums of $(1_{48}, 6_{12}, 60_{12})$. Each R is of the form $2^8 u + v$, $0 \leqq u \leqq 19$, $0 \leqq v \leqq 225 = 2^8 - 1$. Hence R is represented by 1_{274}.

We obtain a better result as follows: [†]

$$\begin{array}{lll} R = 6\cdot 2^8 e + f, & e = 0, 1, 2, 3; & 0 \leqq f < 6\cdot 2^8; \\ f = 2^8 g + h, & 0 \leqq g \leqq 5, & 0 \leqq h < 2^8; \\ h = 60 j + k, & 0 \leqq j \leqq 4, & 0 \leqq k < 60; \\ k = 6 l + m, & 0 \leqq l \leqq 9, & 0 \leqq m \leqq 5. \end{array}$$

Hence k, h, f, R are represented by respectively

$$(1_5, 6_9), \quad (1_5, 6_9, 60_4), \quad (1_{10}, 6_9, 60_4), \quad (1_{10}, 6_{12}, 60_4),$$

the last being of order 26. This proves

THEOREM 5. *Every positive integer is represented by* $(1_{1786}, 6_{456}, 60_{448})$ *of order* 2690 *in eighth powers.*

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[*] *Mathematische Annalen*, Vol. 65 (1908), pp. 424-7.

[†] Avoiding 60, we may use $h = 6n + p$, $n \leqq 42$, $p \leqq 5$. Hence R is represented by $(1_{10}, 6_{54})$, of order 55.

On Complete Systems of Irrational Invariants of Associated Point Sets.

BY CLYDE M. HUBER.

Introduction. The theory of invariants of algebraic forms has been developed to a great extent by the symbolic methods of Aronhold and Clebsch along the line of rational integral invariants, rational in the domain of the coefficients of the given form. In the early days expressions in terms of the differences of the roots were used and the English writers (as in Elliott's treatise) developed the theory to a well organized form but more particularly from the standpoint of obtaining rational integral invariants of a binary form in terms of irrational invariants (rational in the domain of the roots of the form). The products of the differences of the roots of a binary sextic were used by Joubert * and Richmond † to obtain rational integral invariants of the sextic but not to prove the completeness of the system. The completeness of the system was established from this point of view by A. B. Coble ‡ in a paper to which reference will be made frequently. Miss Whelan, in *American Journal of Mathematics*, Vol. 48 (1926), p. 73, obtained rational integral invariants (rational in the domain of the coefficients) of the binary octavic in terms of the irrational invariants.

In several papers, references to which are given in an address § at a meeting of the Chicago Section of the American Mathematical Society at Cincinnati, December 28, 1923, Professor Coble points out the connection between linear systems of irrational invariants of two types of point sets in S_1 and S_p respectively and the solution of the equation of degree $2p+2$. Numerous geometric and algebraic applications of the irrational invariants are made in these papers as well as the outline of their connection with hyperelliptic modular functions of genus p.

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* Joubert, "Sur l'équation du sixième degré," *Comptes Rendus*, tome 64 (1867).

† H. W. Richmond, "Note on the Invariants of a Binary Sextic," *Quarterly Journal of Mathematics*, Vol. 31 (1899), p. 57.

‡ A. B. Coble, "Point Sets and Allied Cremona Groups," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 155.

§ A. B. Coble, "The Equation of the Eighth Degree," *Bulletin of the American Mathematical Society*, Vol. 30 (1924), p. 301.

It is the purpose of this paper to investigate the linear systems of irrational invariants with the object of determining complete systems of such invariants. In § 1 a résumé of some of the elementary notions and definitions used throughout the paper are given. Section 2 gives a method of expressing irrational invariants of a class called *polycyclic* in terms of a class called *monocyclic* and in case of the even degree equation the polycyclic invariants are expressed in terms of products of the differences of the roots linear in each root, that is, in terms of *linear* invariants. In § 3 it is found that if all invariants of degree 4 are expressible in terms of invariants of degree 2 for an odd degree equation, the invariant of any degree for the odd degree equation may be expressed in terms of those of degree 2, i. e. a complete system is made up of a number of linearly independent *cyclic* invariants. The main object of the paper is then somewhat interrupted to give applications of this theorem to the quintic and septimic in §§ 4, 5 respectively as well as a discussion for the case of the septimic of a general mapping problem. In § 6 the principal aim of the paper is accomplished, namely to prove that for the set of points P^1_{2p+2} defined by a binary equation of even degree, a complete system of irrational invariants is made up of linearly independent linear invariants. For n odd it would appear that a complete system is made up from the cyclic invariants; however, the writer has succeeded in establishing this for specific cases only. A comparatively simple method of exhaustion is given whereby the conjectured theorem can be verified in any specific case. In the last section invariants of a linear system on a set of points in S_p are expressed in terms of the linear invariants of a set of points in S_1, each set defined, to within projective modification, by the binary equation of degree $2p+2$.

1. *Preliminary Definitions and Remarks.* The general form of a rational integral invariant of a set of points $P^1{}_n$ on a line, which may be regarded as the roots of a binary form of order n, has been generalized by Professor Coble ‡ as follows: A rational integral invariant of the point set $P_n{}^\kappa$ of n points in S_κ is a rational integral function of the $n/(\kappa+1)$ determinants $P(i_1, i_2, \cdot\cdot\cdot, i_{\kappa+1})$ which is (A) homogeneous and of the same degree in the coordinates of the n points and (B) unaltered in value however the points be permuted. An irrational invariant of $P_n{}^\kappa$ is a function which satisfies all of the requirements of the foregoing definition except (B). An incomplete integral invariant (rational or irrational) of $P_n{}^\kappa$ is one which satisfies the proper requirements for a set $P_{n'}{}^\kappa$ $(n' < n)$ contained in $P_n{}^\kappa$.

For a set of n points in S_1 such a form is in general

$$(1) \qquad I = \sum \prod (12)^{a_{12}}(13)^{a_{13}}(14)^{a_{14}} \cdots (\overline{n-1\,n})^{a_{n-1,n}}$$

where (ij) is an abbreviation for the determinant $p_{i1}p_{j2} - p_{i2}p_{j1}$ and p_{i1}, p_{i2} $(i = 1, 2, \cdots, n)$ are the coordinates of the points of the set $P^1{}_n$, and the a_{ij}'s are positive integers, or zero such that

$$(2) \qquad \sum_{j=1}^{n} a_{ij} = N, \quad (i = 1, 2, \cdots, n), \quad a_{ij} = a_{ji}, \quad a_{ii} = 0.$$

N is the degree of the invariant in the coordinates of each point.

In the same paper it is shown that for every invariant of any type of $P_n{}^\kappa$ there is obtained an invariant J of the same type of the associated set $Q_n{}^{n-\kappa-2}$ and the two are equal to within a factor which is a power, whose index is the weight, w, of I equal to $nN/(\kappa+1)$ of an undetermined constant. To a complete system of invariants I, there corresponds a complete system of invariants J.

To find a complete system of irrational invariants of the set of points $P^1{}_n$ and hence of its associated set $Q_n{}^{n-3}$, we have then to solve the system of diophantine equations (2). The different possible types of independent solutions of this system then yield the different types of products π entering into the invariant I.

Now it is known that in the case of $P^1{}_4$ and $P^1{}_6$ complete systems of irrational invariants are made up of certain elementary products. For $P^1{}_4$ a complete system is $(12)(34)$, $(13)(24)$, $(14)(23)$ only two of which are linearly independent. For the set $P^1{}_5$ we shall find later (§ 4) that the complete system is made up from the invariants of degree two, called cyclic invariants, which are of the following types:

$$(3) \qquad \begin{gathered}(ij)(jk)(kl)(lm)(mi);\\ (ij)^2(kl)(lm)(mk).\end{gathered}$$

All products π entering into the invariant I of $P^1{}_5$ are expressible directly as products of factors of the type (3).

The first product of (3) we designate as *monocyclic,* that is consisting of one cycle. The general *polycyclic* invariant with k cycles for the set $P^1{}_n$ is an invariant of degree two in the coordinates of each point which has the following form:

$$(4) \qquad \{(a_1a_2)(a_2a_3)\cdots(a_{a-1}a_a)(a_aa_1)\}\{(b_1b_2)(b_2b_3)\cdots(b_\beta b_1)\}\cdots \\ \{(k_1k_2)\cdots(k_\lambda k_1)\},$$

where $\alpha + \beta + \cdots + \lambda = n$. The number of types of cyclic invariants for $P^1{}_n$ is then the number of different partitions of n, including of course the number n, and exclusive of the number one, since from the definition we must have the coordinates of at least two points in any single cycle.

A linear invariant we define as a product linear in the coordinates of each point and hence of the form $(ij)(kl) \cdots (rs)$. Such linear invariants clearly exist only in the cases in which n is even. Moreover it is evident that for n odd we can have no invariant of odd degree, since the sum of all the a_{ij}'s in the system (2) is even and on the right the sum would give an odd multiple of N, hence N itself would have to be even. The treatment of the two cases will be different and the results are of a different character.

The complete systems for $P^1{}_6$ and $P^2{}_6$ have been determined.* For $P^1{}_6$ the product π can be expressed rationally and integrally by the use of the binary determinant identity $(12)(34) = (13)(24) + (14)(32)$ in terms of the fifteen products of the type $(ij)(kl)(mn)$ of which only 5 are linearly independent.

2. *Expression of Polycyclic Invariants in Terms of Monocyclic Invariants in General.* We now show how to express the general polycyclic invariant on any number of roots in terms of monocyclic invariants. Such a polycyclic invariant may be written in the form (4). It is sufficient to show that two adjacent cycles may be combined into a sum of monocycles, since this process could be repeated as long as two or more cycles are left in any single term of the expression for the original polycyclic invariant. Consider any two adjacent cycles without any restriction as to the number of roots occurring in either cycle (save that the number in both shall not exceed n) as the following:

$$\{(a_1a_2)(a_2a_3) \cdots (a_{\alpha-1}a_\alpha)(a_\alpha a_1)\} \; \{(b_1b_2)(b_2b_3) \cdots (b_{\beta-1}b_\beta)(b_\beta b_1)\}.$$

Now apply the binary determinant identity to the product $(a_\alpha a_1)(b_1b_2)$. We then form a sum of two cycles as follows:

$$(a_1a_2)(a_2a_3)(a_3a_4) \cdots (a_{\alpha-1}a_\alpha)(a_\alpha b_1)(b_1b_\beta) \cdots (b_3b_2)(b_2a_1) + (a_1a_2)(a_2a_3) \cdots (a_{\alpha-1}a_\alpha)(a_\alpha b_2)(b_2b_3) \cdots (b_{\beta-1}b_\beta)(b_\beta b_1)(b_1a_1).$$

Proceeding in this way as long as two adjacent cycles remain we eventually obtain only monocyclic products.

We now show that in case n is even, that the general cyclic invariant can be expressed in terms of linear invariants. We first express the cyclic invariant in the above manner in terms of monocyclic invariants containing an even number of roots. It is evident that every monocyclic invariant containing an even number of roots can be expressed as a product of two linear factors as follows:

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* See "Point Sets" paper of Coble, referred to above.

$$(a_1a_2)(a_2a_3)(a_3a_4)(a_4a_5)\cdots(a_{2p-1}a_{2p})(a_{2p}a_{2p+1})(a_{2p+1}a_{2p+2})(a_{2p+2}a_1)=$$
$$\{(a_1a_2)(a_3a_4)\cdots(a_{2p-1}a_{2p})(a_{2p+1}a_{2p+2})\}$$
$$\{(a_2a_3)(a_4a_5)\cdots(a_{2p}a_{2p+1})(a_{2p+2}a_1)\}.$$

Hence the general cyclic invariant on an even number of points is a rational, integral function of the linear invariants.

THEOREM I. *The general polycyclic invariant* (4) *for the point set* P^1_n *can be expressed as a sum of monocyclic invariants by use of the binary identity. If* $n=2p+2$ *the general cyclic invariant can be expressed rationally and integrally in terms of the linear invariants.*

3. *Conversion of any Invariant Product* π *into a Reducible Product* π'. Consider now any product π of degree N, for any value of n. If any a_{ij}, say without loss of generality a_{12}, equals N the indices 1 and 2 will not occur again in π and the remaining product will be an invariant for a set of points contained in P^1_n. Then if the invariants of the set P^1_{n-2} are reducible, the invariant product π for P^1_n is reducible. We can easily verify that a complete system for the set P^1_3 is the single variant (12)(23)(31) by a direct solution of the diophantine system (2) for the value $n=3$. Moreover we know that all invariants of the set P^1_4 are reducible as referred to previously. If then we can take any product π and carry out on its exponents a series of operations which modify them so that eventually one exponent becomes equal to N, then, by induction, the modified product π' is reducible.

Suppose then that every a_{ij} is less than N, and let a_{ij} and a'_{ij} be any two solutions of the diophantine system

$$(6)\qquad \begin{array}{l} a_{12}+a_{13}+a_{14}+\cdots+a_{1n}=N, \\ a_{21}+a_{23}+a_{24}+\cdots+a_{2n}=N, \\ a_{31}+a_{32}+a_{34}+\cdots+a_{3n}=N, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}+a_{n2}+a_{n3}+a_{n4}+\cdots+a_{n,n-1}=N, \end{array}$$

for the same value of N.

Set $\delta = a_{ij} - a'_{ij}$, so that the δ's satisfy the system

$$
(7)\qquad
\begin{array}{lllllll}
 & \delta_{12} + \delta_{13} + \delta_{14} & + & \cdots & + & \delta_{1n} = 0, \\
\delta_{21} & \quad\;\; + \delta_{23} + \delta_{24} & + & \cdots & + & \delta_{2n} = 0, \\
\delta_{31} + \delta_{32} & \qquad\qquad\; + \delta_{34} & + & \cdots & + & \delta_{3n} = 0, \\
\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot & \cdot & \cdot & \cdot & \cdot\quad\cdot\ , \\
\delta_{n1} + \delta_{n2} & + \delta_{n3} + \delta_{n4} & + & \cdots & + & \delta_{n,n-1} = 0.
\end{array}
$$

Adding the last $n-1$ equations and subtracting from this result the first equation, we have

$$
(8)\qquad \delta_{23} + \delta_{24} + \delta_{25} + \cdots + \delta_{2n} + \delta_{34} + \cdots \\
+ \delta_{3n} + \delta_{45} + \cdots + \delta_{n,n-1} = 0,
$$

which we may write as

$$
(8') \qquad \delta_1 + \delta_2 + \cdots + \delta_{(n-1)(n-2)/2} = 0.
$$

It is evident that all solutions of this equation are compounded from these elementary solutions

$$
\delta_i = 1,\ \delta_j = -1,\quad (i, j = 1, 2, \cdots, (n-1)(n-2)/2), \quad (i \neq j), \\
\text{all other } \delta\text{'s} = 0.
$$

But these are not all independent since from $n(n-3)/2$ like $\delta_1 = 1$, $\delta_j = -1$, $[j = 2, 3, \cdots, (n-1)(n-2)/2]$ the remaining elementary solutions can be obtained. These solutions lead in (8) to two distinct types of solutions, namely the types $\delta_{23} = 1$, $\delta_{24} = -1$ in which the δ's have a common subscript and the type $\delta_{23} = 1$, $\delta_{45} = -1$ in which the δ's have no common subscript. Hence for the solutions of the system (7) we obtain the two types

$$
(9)\qquad
\begin{array}{ll}
\text{Type I:} & \left.\begin{array}{l} \delta_{23} = \delta_{41} = 1 \\ \delta_{24} = \delta_{31} = -1 \end{array}\right\} \text{ all other } \delta\text{'s} = 0, \\
\text{Type II:} & \left.\begin{array}{l} \delta_{23} = \delta_{14} = \delta_{15} = 1 \\ \delta_{46} = \delta_{12} = \delta_{13} = -1 \end{array}\right\} \text{ all other } \delta\text{'s} = 0.
\end{array}
$$

The difference between the exponents of like factors of any two products π for the same value of N can then be expressed in terms of elementary differences of two types. Each of these types of elementary differences then leads to an operation on the exponents which we denote by O_1 and O_2. O_1 is an operation which reduces a_{ij} and a_{kl} by one and increases a_{ik} and a_{jl} by one. O_2 is an operation which reduces a_{ij}, k_{kl}, a_{km} by one and increases a_{kl}; a_{kj}; a_{lm} by one.

We observe that the operation O_2 can be expressed as a product of operations of type O_1, which may be shown as follows. The operation of type O_1 replaces the product $(ij)(kl)$ by the product $(ik)(jl)$; the operation of type O_2 replaces the product $(ij)(kl)(km)$ by the product $(ik)(kj)(lm)$. Let a specific operation of the type O_1 be $O_{ik,jl}$ which replaces $(ik)(jl)$ by $(ij)(kl)$. If we operate upon $(ik)(jl)(km)$ with $O_{ij,kl}$ we obtain the product $(ik)(jl)(km)$. Now we operate with $O_{jl,km}$ upon the resulting product and obtain $(ik)(jk)(lm)$, which gives the same result as if we had operated on the original product with the operation $O_{ij,kl,km}$ of type O_2, or

$$O_{ij,kl,km} = O_{ij,kl} \cdot O_{jl,km} .$$

THEOREM II. *Operations of type O_2 can be replaced by products of operations of type O_1.*

Since however only positive or zero values of the a_{ij}'s are in question, these operations are to be applied only to such products as will yield products with positive or zero exponents.

For a given value of N let π be the product

$$(12)^{\alpha}(13)^{\beta}(14)^{\gamma}\cdots(\overline{n-1,n})^{\sigma},$$

where we now assume $\alpha, \beta, \cdots, \sigma < N$. Now the indices 1 and 2 must occur again in the product and suppose that each occurs with a different index, say with 3 and 4 respectively. We can by an operation of type O_1 increase the exponent of (12) and at the same time decrease the exponents of the factors (13) and (24). We may then by repeated applications of operations of type O_1 increase the degree of the factor (12) and decrease the degrees of the other factors containing the indices 1 and 2 so long as 1 and 2 each occur again in factors with different roots. We can by a finite number of applications of operations of type O_1 come to the stage where either (a) the degree of the factor (12) is N, or (b) we have 1 and 2 each occurring with the same index, say 3.

In case (a) the product π' would then be reducible.

If case (b) occurs we will have a product of the form

$$(12)^{r}(13)^{s}(23)^{t}\cdots(\overline{n-1\,n})^{u},$$

where $r+s=N$, and $r+t=N$, from which it follows that $t=s$, but we may not have $s+t=N$. If the latter were true we would have the cyclic factor $(12)^{v}(23)^{v}(31)^{v}$ and the remaining factor would be an invariant on $n-3$ of the points and we would then be led to a reducible product π'. This clearly can occur only in case N is even. Suppose then that $s+t<N$. By means of the operation $O_{13,23,lm}$ we can decrease the exponents of the factors (13) and (23) and at the same time the exponent of some other factor, say (45), containing none of the indices 1, 2, or 3. Now such a factor must clearly be of degree greater than $s+t$, or there must be at least $s+t$ such factors that do not contain the indices 1, 2, or 3. Hence at no time will there be negative exponents introduced by s applications of the operation $O_{13,23,45}$. At the same time we will be increasing the degree of the factor (12), until eventually its degree, a_{12}, will be equal to N, the degree of the invariant π. Hence by previous argument, the resulting product π' is expressible in terms of the cyclic or linear invariants, or π' is reducible.

THEOREM III. *Every invariant product* π *for the point set* $P^1{}_n$ *can be converted by means of the operations of type* O_1 *into a sequence of products* $\pi, \pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(r-1)}, \pi^r = \pi'$ *in which* π' *is a reducible product.*

If further we can show that the *reverse* of the above series of operations applied to a reducible product always leads to a reducible product, then the original product upon which we operated must have been reducible. We have shown that any operation of type O_2 can be expressed as a product of operations of type O_1. Moreover the reverse of an operation of type O_1 can be applied to a single cyclic invariant, or to a product of two cyclic invariants; also to a single linear invariant, to a product of two linear invariants, or to a product of a linear and a cyclic invariant. That is, the operations of type O_1 can affect at most in a single application an invariant product of degree four. If then we show that all solutions of the system (6) for N equal to 3, and 4 are reducible, then the product π is reducible for any value of N.

This proceedure can be further shortened by a separation of the cases in which n is odd or even. We first consider the case in which n is odd giving two methods of proving the reducibility of any product π for the set $P^1{}_n$ for this case. We defer the treatment of the even case until § 6. It is evident that the reverse of an operation O_1 applied to a single cyclic product yields such a product; moreover, there are no invariants of degree 3 for n odd. (1) Hence it is sufficient to show that the reverse of operations of type O_1 applied to a product of two cyclic invariants always yields a product of cyclic invariants. For in that case in recovering the original product π we would pass back through a sequence of reducible products at each operation and hence the original product π would be reducible. (2) It is sufficient to show that all invariants of degree 4 are reducible, as in that case the application of the reverse of operations of type O_1 would always yield invariants of degree

4 which by hypothesis are reducible and therefore by the first alternative the original product π must be reducible. Both of these conditions are obviously necessary.

THEOREM IV. *A necessary and sufficient condition for the reducibility of any invariant product π for the point set $P^1{}_n$ when n is odd is that all invariants of degree 4 shall be reducible.*

It will be observed that the above two methods are each equivalent to Theorem IV, however the proceedure for verification in the two cases is quite different. In the following two sections we shall apply the first of the above methods to determine the complete system for the set $P^1{}_5$ and the second method to determine the complete system for the set $P^1{}_7$. We shall show in each case that the cyclic invariants form a complete system of irrational invariants.

4. *Complete System for $P^1{}_5$.* For the set $P^1{}_5$ we have two types of cyclic invariants, namely, $(ik)^2(jl)(lm)(mj)$ and $(ik)(kj)(jl)(lm)(mi)$. The reverse of an operation of type O_1 has the effect of interchanging two roots one of which we assume occurs in the first invariant of a product of two cyclic invariants, and the other in the second invariant factor. Such an interchange sends the product $(ik)(jl)$ into the product $(ij)(kl)$. With respect to such an operation the product of two cyclic invariants may be affected as follows:

$$
(10) \qquad
\begin{aligned}
&(1). \quad (ik)^2(i_3i_4i_5)\cdot(jl)^2(j_1j_2j_5),\\
&(2). \quad (ik)^2(i_3i_4i_5)\cdot(j_1j_2)^2(jlj_5),\\
&(3). \quad (ik)^2(i_3i_4i_5)\cdot(jlj_1j_2j_5),\\
&(4). \quad (i_3i_4)^2(iki_5)\cdot(j_1j_2)^2(jlj_5),\\
&(5). \quad (i_3i_4)^2(iki_5)\cdot(jlj_1j_2j_5),\\
&(6). \quad (iki_3i_4i_5)\cdot(jlj_1j_2j_5),
\end{aligned}
$$

$$\text{where } (i_li_mi_n) = (i_li_m)(i_mi_n)(i_ni_l).$$

The effect of the interchange of the indices j and k may be observed in each of these products, noting that the resulting product is reducible in each case. We illustrate the type of verification for the first case.

We may without loss of generality take $(ik)(jl)$ as the product $(12)(34)$.

(1). $(12)^2(i_3i_4i_5)(34)^2(j_1j_2j_5)$ goes by the above interchange into the product $(13)(12)(i_3i_4)(i_4i_5)(i_5i_3)(24)(34)(j_1j_2)(j_2j_5)(j_5j_1)$. Now the index 3 must occur in the second cycle of the first factor and similarly the index 1 must occur in the second cycle of the second factor. We may take each of these indices in the leading position in the cycle, so that the modified product has the form $(13)(12)(3i_4)(i_4i_5)(i_53)\cdot(24)(34)(1j_2)(j_2j_5)(j_51)$. The factor (i_4i_5) must be either (45) or (54) and (j_2j_5) must be either (25) or (52). In either case we have the cyclic factor $(12)(25)(54)(43)(31)$, so that the remaining factor must necessarily be cyclic, and hence the product is reducible after the application of the reverse of an operation of type O_1.

The argument may be carried out in a similar manner for the remaining cases as the reader may verify and need not be given here. The reverse of the operations required to modify π so that it becomes a product π' which we know to be reducible then yields reducible products at each stage, so that the original product must have been reducible. A complete system of irrational invariants for P^1_5 is made up from products of the types

$$(11)\qquad \begin{aligned} &(a).\quad (ij)(jk)(kl)(lm)(mi);\\ &(b).\quad (ij)^2(kl)(lm)(mk). \end{aligned}$$

The invariants of type (b) are expressible by Theorem I as sums of invariants of type (a), and conversely. It is shown in the "Point-Sets" paper of Coble (p. 190) referred to above that there are just six products of either type that are linearly independent.

THEOREM V. *A complete system of irrational invariants for the set P^1_5 consists of six linearly independent products of the type* (11, a).

These six linearly independent products map binary quintics and thus points of the plane upon a quintic two way in S_5. This is a fairly well-known surface being also the map of the plane by a linear system of cubic curves with simple points at four base points in the plane.

5. *Complete System for P^1_7.* In this case we show that every invariant of degree 4 is reducible and hence by Theorem IV the general invariant must be reducible.

Consider the product π for the set P^1_7 of degree 4. If any $a_{ij}=4$, then the product π is reducible, since the remaining factor would be an invariant on the set P^1_5 contained in the set P^1_7 which we have shown to be reducible in § 4. Let us assume then that no exponent in the product π is equal to 4. We now list the possible types of products π in which we have three and no more exponents equal to 3, two exponents equal to 3, etc., showing in every case that π contains a cyclic factor, and hence every invariant of degree 4 for P^1_7 is reducible as required. It may be easily verified that these are the only types, and that all other products obtained are either excluded (because of coming under a previous classification), or impossible (i. e. a case in which the factor (ii) would have to be introduced in order to complete the degree of the invariant in the coordinates of the point whose index is i). This method of verification we illustrate, the process being similar for other cases.

Let three exponents of π be equal to 3, we must then have a factor of the *type* $(12)^3(34)^3(56)^3$. We then consider the different possible ways of completing the product π in the coordinates of the point whose index is 1 to the required degree, then the point whose index is 2, etc., as follows:

$34)^3(56)^3$	(13)	(24)	(57)	(67)	requiring the factor (77) to complete;
		(27)	(45)	(67)	requiring the factor (77) to complete;
			(47)	(57) (67)	has the cyclic factor $(12)^2(34)^2(56)(57)(67)$;
	(17)	(23)	(45)	(67)	an impossible type;
			(47)	(57) (67)	has the cyclic factor $(12)^2(34)^2(56)(67)(75)$;
		(27)	(35)	(46)	impossible type;
				(47) (67)	has the cyclic factor $(12)^2(34)(47)(76)(65)(53)$;
			(37)	(45) (67)	has the cyclic factor $(12)^2(34)(45)(56)(67)(73)$.

In a similar way we could find the remaining types of products all of which contain a cyclic factor and hence are reducible. Now every product of degree four which is an invariant of the set P^1_7 must come under some one of the above types. But each product of any one of these types is reducible, therefore all products π of degree four are reducible. Hence, by Theorem IV it follows that any invariant product π for the set P^1_7 is reducible.

THEOREM VI. *Every invariant product π for the point set P^1_7 is reducible, that is, can be expressed as a product of the following types of cyclic invariants:*

(12′) 360 of type $(ij)(jk)(kl)(lm)(mn)(no)(oi)$;
(12″) 252 of type $(ij)^2(kl)(lm)(mn)(no)(ok)$;
(12) (12‴) 105 of type $(ij)^2(kl)^2(mn)(no)(om)$;
(12^{IV}) 105 of type $(ij)(jk)(ki)(lm)(mn)(no)(ol)$.

By Theorem I each of the cyclic invariants of the last three types may be expressed rationally and integrally in terms of invariants of the first type, that is, the monocyclic type. A reduced complete system of irrational invariants for P^1_7 is then made up from certain linearly independent products out of the 360 products (12′). We now determine the number of linearly independent invariants in the complete system.

If in the 105 invariants of the third type in (12) we set $6 = x$, $7 = y$ we obtain the following types of irrational covariants of a binary quintic:

$$
(13) \quad
\begin{array}{ll}
\text{(a)} & 10 \text{ of type } (xy)^2(12)^2(34)(35)(45);\\
\text{(b)} & 15 \text{ of type } (xy)(1x)(1y)(25)^2(34)^2;\\
\text{(c)} & 20 \text{ of type } (1x)^2(2y)^2(34)(35)(45);\\
\text{(d)} & 60 \text{ of type } (1y)^2(4x)(5x)(23)^2(45).
\end{array}
$$

These forms can be expressed in terms of powers of the determinant (xy) whose coefficients are elementary covariants and invariants arising from polars of the given forms in a unique manner.* In the expansion of the above forms there arise aside from numerical coefficients

(a′) 10 invariants of the type $(12)^2(34)(35)(45)$;

(b′) quadratic covariants of the types $(1x)^2(25)^2(34)^2$,
$(1x)(2x)(12)(34)(35)(45)$,
$(1x)(2x)(13)(24)(35)(45)$; $(1x)(2x)(13)(23)(45)^2$,

(c′) quartic covariants of the types $(1x)^2(2x)(3x)(23)(45)^2$,
$(1x)^2(2x)^2(34)(35)(45)$, $(1x)(2x)(3x)(4x)(15)(25)(34)$.

There are six linearly independent invariants of the type (a′) as noted above.

By use of the six functions of Joubert ‡ subject to the relation $A + B + C + D + E + F = 0$, the quadratic covariants can all be expressed in terms of the 15 products of the functions A, etc. These products are all linearly independent so that there are in all only 15 linearly independent quadratic covariants.

The quartic covariants can all be expressed in terms of the 30 quartic covariants of type $(1x)(2x)(3x)(4x)(15)(25)(34)$, which we designate as $q_{34,5}$. These are grouped in five sets of 6 each according to the root that is isolated, as in the set $q_{12,5}$; $q_{13,5}$; $q_{14,5}$; $q_{23,5}$; $q_{24,5}$; $q_{34,5}$. The q's in each set are subject to three relations of the type $q_{34,5} + q_{41,5} + q_{13,5} \equiv 0$, analogous to the relations on the six differences of the four indices 1, 2, 3, 4 from which we conclude that in each set of six covariants there are three linearly independent ones.

---

* A. Clebsch, *Theorie der binären Formen*, Leipzig, Teubner (1872), § 7.

There can be no other relations existing containing covariants from different sets of six for suppose there were some relation such as

$$(14) \qquad \lambda_1(2x)(3x)(4x)(5x)+\lambda_2(1x)(3x)(4x)(5x)+\cdots \\ +\lambda_5(1x))2x)(3x)(4x)\equiv 0,$$

where the λ's represent any coefficients whatever. Now let $x=1$ and all terms but the first in relation (14) vanish. Since in general the five roots are distinct, $\lambda_1=0$. Similarly we may show that the remaining λ's must be zero, that is, no relation of the type (14) could exist. The only linear relations satisfied by the 30 quartic covariants are those arising from the relations within the sets of six with one root isolated, hence there are just 15 linearly independent quartic covariants.

Furthermore there can be no relations existing among covariants of different order. Hence there are 36 linearly independent invariants and covariants. Since the 105 covariants are expressible in terms of these elementary ones we conclude that there are 36 linearly independent covariants of types (13).

Now replacing x and y by the roots 6 and 7 we have 36 linearly independent invariants of the type (12'''). These are all expressible in terms of those of type (12') and conversely, hence the number of linearly independent invariants in the two systems is the same.

THEOREM VII. *The simplest reduced complete system of irrational invariants of the point set $P^1{}_7$ consists of* 36 *linearly independent invariants from the set of* 360 *invariants* (12').

It would seem that in general the invariants of the odd degree equation are expressible in terms of the monocyclic invariants. The methods sketched above will suffice to verify this in any particular case. We now give for the particular case of $P^1{}_7$ a map determined by the irrational invariants.

The 36 linearly independent cyclic invariants for $P^1{}_7$ map binary septimics upon a four way in S_{35}. We now determine the order of this spread which is invariant under the linear transformation group of order 7! in S_{35} corresponding to the symmetric group of the permutations of the roots of the binary septimic.

Let t_7, a root of the binary septimic, be transformed to infinity and $t_1+t_2+\cdots+t_6=0$. Every binary septimic determines a point in S_4, after the above transformation, which is then mapped upon the spread in S_{35} by the 36 linearly independent invariants. In S_4 we have certain spreads defined by the cyclic invariants themselves. These spreads evidently have multiple points at the points of a set P_6^4 as follows:

$$\begin{array}{rrrrrr}
-5 & 1 & 1 & 1 & 1 & 1, \\
1 & -5 & 1 & 1 & 1 & 1, \\
1 & 1 & -5 & 1 & 1 & 1, \\
1 & 1 & 1 & -5 & 1 & 1, \\
1 & 1 & 1 & 1 & -5 & 1, \\
1 & 1 & 1 & 1 & 1 & -5.
\end{array}$$

The multiplicity is of order 3 at each of these points, the order of the spread being 5. We then have quintic three ways in S_4, four of which intersect in a finite number of variable points, the number of which gives the order of the invariant spread in S_{35}. A line on two of the points of P_6^4 would cut the M_3^5 in six points hence the M_3^5 would contain the entire line, i. e., M_3^5 contains the 15 lines joining the points of P_6^4 as a fixed part.

Three of these M_3^5's would intersect in a curve of order 5^3 with multiple points of order 3 at the points of P_6^4, but the 15 lines are fixed, hence the order of the variable part will be $5^3 - 15$ with multiple points of order 22 at each point of P_6^4. A fourth M_3^5 would cut this variable curve in $5(5^3 - 15)$ points of which $6 \cdot 22 \cdot 3$ are fixed at P_6^4, hence there remain 154 variable points of intersection which gives the order of the invariant spread in S_{35}.

THEOREM VIII. *The 36 linearly independent invariants of the complete system for* P^1_7 *map projectively distinct binary septimics, and hence points of the linear space* S_4 *upon a spread of order* 154 *and dimension* 4 *in* S_{35} *which is invariant under the linear transformation group* $G_{7!}$ *which corresponds to the permutation group of the roots of the binary septimic.*

6. *Complete System for* P^1_{2p+2}. By Theorem III the product π, an invariant product for the point set P^1_{2p+2}, can be converted by a sequence of r operations described in § 3 into a product π' which is reducible. We have then a sequence of $r+1$ products $\pi, \pi^{(1)}, \cdots, \pi^{(r)} = \pi'$ such that π' factors into a product of cyclic and linear invariants. Moreover Theorem I enables us to express each of the cyclic invariant factors of π', by use of the binary determinant identity, as a sum of products of linear invariants. The product π' is then equal to a rational integral function Σ', of linear invariants like $(ij)(kl) \cdots (rs)$. We are thus led from the original invariant product π to a rational integral function Σ' of the linear invariants.

Now to recover from the product $\pi' = \Sigma'$ the original product π, we should apply the reverse of the operations necessary to convert π into the reducible product π'. These operations by Theorem II are all expressible as products of operations of type O_1. If we show that the reverse of an operation of type O_1 applied to Σ' always yields a rational integral function of linear invariants, we would then have a sequence of the functions Σ such that $\pi' = \Sigma'$, $\pi^{(r-1)} = \Sigma^{(r-1)}, \cdots, \pi = \Sigma$. Hence the original product π would then be expressible as a rational integral function of linear invariants.

The operations of type O_1 have the effect of merely interchanging two indices j and k in a pair of factors of given type. In the application of the binary determinant identity to the cyclic invariant products in π' we can follow any pair of indices j and k and determine their position in the function Σ'. The interchange of the indices j and k in a given pair of factors of π' would then be equivalent in Σ' to the interchange of two definite indices j and k of each and every term of Σ'. In any term of Σ' such an interchange may occur within a single linear invariant, in which case a linear invariant would clearly result so that the reverse of O_1 would turn a rational function of linear invariants into such a function in this case. Or the indices may occur in different linear factors of the same term, in which case the resulting product will appear either as two linear factors or as a cyclic invariant which may be further expressed as a function of linear invariants by the method of § 2. Hence in any case the result of an operation interchanging a definite pair j, k in π'; when carried out on the function Σ' yields a new $\Sigma^{(r-1)}$ which is again a rational integral function of linear invariants. We then have the equalities

$$\pi' = \Sigma',$$
$$\pi^{(r-1)} = \Sigma^{(r-1)}.$$

Now from $\pi^{(r-1)}$ we go to $\pi^{(r-2)}$ by an interchange of indices as described above, hence from $\Sigma^{(r-1)}$ we get a new $\Sigma^{(r-2)}$, etc., finally we get back to the original invariant product $\pi = \Sigma$. Hence every invariant product π for the even degree equation is equal to a rational integral function of the linear irrational invariants.

THEOREM IX. *A complete system of irrational invariants of the point set P^1_{2p+2} consists of a set of linearly independent products like $(ij)(kl)\cdots(rs)$ out of the $\dfrac{(2p+2)!}{(p+1)!\,2^{p+1}}$ such products.*

The number of such linearly independent invariants, which we call invariants of type (A), is $\dfrac{1}{p+2}\dbinom{2p+2}{p+1}$.[*]

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[*] A. B. Coble, "The Equation of the Eighth Degree," *Bulletin of the American Mathematical Society*, Vol. 30 (1924), p. 302.

7. *Expression of Invariants* (*B*) *in terms of Invariants of Type* (*A*). The binary $(2p+2)$-ic defines three sets of points $P^1{}_{2p+2}$ on S_1, $Q^p{}_{2p+2}$ on a rational norm curve N^p in S_p, and R^{2p-1}_{2p+2} on N^{2p-1} in S_{2p-1}. The two extreme sets are associated sets,* and hence the invariants of R^{2p-1}_{2p+2} are proportional to the invariants formed from the complimentary determinants of the set $P^1{}_{2p+2}$. The set $Q^p{}_{2p+2}$ gives rise to invariant products of the type

$$(1, 2, \cdot\cdot\cdot, p+1)(p+2, \cdot\cdot\cdot, 2p+2), \quad \text{where}$$

$$(15) \qquad (i_1 i_2 i_3 \cdot\cdot\cdot i_{p+1}) \equiv \begin{vmatrix} t_{i_1}{}^p & t_{i_1}{}^{p-1} & \cdot\cdot\cdot & 1 \\ t_{i_2}{}^p & t_{i_2}{}^{p-1} & \cdot\cdot\cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ t^p_{i_{p+1}} & t^{p-1}_{i_{p+1}} & \cdot\cdot\cdot & 1 \end{vmatrix}$$

which we designate as an invariant of type (B). This determinant may also be expressed in the form

$$(16) \qquad (i_1 i_2 i_3 \cdot\cdot\cdot i_{p+1}) = \prod (i_r i_s); \quad r, s = 1, 2, \cdot\cdot\cdot, p+1; \quad r < s.$$

In this form we see that an invariant (B) is equal to an invariant of degree p of the set $P^1{}_{2p+2}$, which by Theorem IX of the preceding section is equal to a rational integral function of the linear invariants (A). We now give a method for expressing the invariants (B) in terms of the linear invariants (A).

The determinant product $(i_1 i_2 i_3 \cdot\cdot\cdot i_{p+1})$ can be arranged according to the following scheme:

$$(17) \qquad \begin{array}{cccccccccc} (i_1i_2) & (i_1i_3) & (i_1i_4) & (i_1i_5) & (i_1i_6) & \cdots & (i_1i_{p-2}) & (i_1i_{p-1}) & (i_1i_p) & (i_1i_{p+1}) \\ & (i_2i_3) & (i_2i_4) & (i_2i_5) & (i_2i_6) & \cdots & (i_2i_{p-2}) & (i_2i_{p-1}) & (i_2i_p) & (i_2i_{p+1}) \\ & & (i_3i_4) & (i_3i_5) & (i_3i_6) & \cdots & (i_3i_{p-2}) & (i_3i_{p-1}) & (i_3i_p) & (i_3i_{p+1}) \\ & & & (i_4i_5) & (i_4i_6) & \cdots & (i_4i_{p-2}) & (i_4i_{p-1}) & (i_4i_p) & (i_4i_{p+1}) \\ & & & & (i_5i_6) & \cdots & (i_5i_{p-2}) & (i_5i_{p-1}) & (i_5i_p) & (i_5i_{p+1}) \\ & & & & & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ & & & & & & (i_{p-3}i_{p-2}) & (i_{p-3}i_{p-1}) & (i_{p-3}i_p) & (i_{p-3}i_{p+1}) \\ & & & & & & & (i_{p-2}i_{p-1}) & (i_{p-2}i_p) & (i_{p-2}i_{p+1}) \\ & & & & & & & & (i_{p-1}i_p) & (i_{p-1}i_{p+1}) \\ & & & & & & & & & (i_pi_{p+1}). \end{array}$$

Consider the product of all the factors on a diagonal beginning with the upper left and proceeding downward to the right. If we combine the first diagonal product with the last, we have a cyclic invariant, say C_1, since each index occurs twice and only twice. Similarly, the second diagonal product with the next to the last one together form a cyclic invariant, etc. We now distinguish two cases, p odd and p even.

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* See "Point Sets" paper of Coble, referred to above.

If $p = 2k + 1$, we have an odd number of diagonal products, the middle one of which in the above scheme is clearly a linear invariant on the $p + 1 = 2k + 2$ points. The remaining diagonal products combine in pairs to form cyclic invariants, so that we have

$$(i_1 i_2 i_3 \cdots i_{p+1}) = C_1 C_2 \cdots C_k \cdot L. \tag{18}$$

If $p = 2k$ we have an even number of diagonal products which combine as above in pairs into cyclic invariants giving

$$(i_1 i_2 i_3 \cdots i_{2k+2}) = C_1 C_2 \cdots C_k . \tag{18'}$$

Now the remaining determinant factor of the invariant (B) may be expressed as above in each case respectively as

$$(j_1 j_2 j_3 \cdots j_{2k+2}) = C_1' C_2' \cdots C_k' \cdot L',$$

or

$$(j_1 j_2 j_3 \cdots j_{2k+1}) = C_1' C_2' \cdots C_k'.$$

Any one of the C's multiplied by any one of the C''s gives a cyclic invariant on the set P^1_{2p+2} which by the method of section 2 can be expressed rationally and integrally in terms of invariants (A). Similarly the remaining factors may be paired and expressed in terms of invariants (A). In case $p = 2k + 1$, the product $L \cdot L'$ is an invariant (A) so that the method holds for any invariant (B), giving in either case

$$(i_1 i_2 i_3 \cdots i_{p+1}) (j_1 j_2 j_3 \cdots j_{p+1}) = R^p(A).$$

URBANA, ILLINOIS,
June 1, 1926.

Regular Maps and Their Groups.

By H. R. Brahana.

The geometrical representation of a finite group by means of fundamental regions, due to Dyck,[*] leads to a map that is transformed into itself by every operation of the group. The group may be represented transitively on symbols for regions, but in general it may not be represented transitively on symbols for edges or vertices. On the other hand it is well known [†] that the polyhedral groups may be represented transitively on symbols for regions, for edges, or for vertices of the regular polyhedra or the corresponding maps on a sphere. In the one case the number of regions of the map is equal to twice the order of the group and in the other the number of regions is equal to the order of the group divided by the number of sides of a region. Maps of the latter type we have called regular maps.

The polyhedral groups are the only ones that may be represented on regular maps on a sphere. There has been no attempt to determine the kinds of groups that may be represented on regular maps on surfaces of higher genus. Heffter [‡] showed that the metacyclic groups may be so represented. In a recent paper [§] all the maps of twelve five-sided regions with a group of order 120 containing an icosahedral subgroup were exhibited. In this last cited investigation the surfaces were allowed to be one-sided or two-sided and when the surface was two-sided a transformation of the map into itself in such a way as to reverse the sense of the boundary of each region was permitted. Lastly, the maps on a surface of genus one and the groups connected with them were considered.[*] It was shown that any such regular map must be made up of triangles, of quadrangles, or of hexagons and their groups must be generated by two operators of orders two and three with product of order six, or by two operators of order two and four with product of order four, and conversely that to every such group there corresponds a regular map on an anchor ring.

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[*] "Gruppentheoretischen Studien," *Mathematische Annalen*, Vol. 20 (1882), pp. 1-44.

[†] Klein, *Vorlesungen ueber das Ikosaeder*, I, § 13.

[‡] "Ueber metacyklische Gruppen und Nachbarconfigurationen," *Mathematische Annalen*, Vol. 50 (1897), pp. 261-268.

[§] Brahana and Coble, "Maps of twelve countries, etc.", *American Journal of Mathematics*, Vol. 48 (1926), pp. 1-20.

[*] "Regular Maps on an Anchor Ring," *American Journal of Mathematics*, Vol. 48 (1926), pp. 225-240.

In § 1 we recall a condition that has been stated previously as necessary that a given group be the group of a regular map and prove that it is sufficient by giving a method of constructing the map corresponding to the given group. In § 2 we note four classes of groups that satisfy the given condition and determine the genera of the surfaces on which the corresponding maps lie. In § 3 we apply the earlier considerations to determine the regular maps on a surface of genus two.

1. *The Group of a Regular Map.* A *map* is a finite set of distinct 0-cells, 1-cells, and 2-cells which constitutes a closed, connected, two-sided, two-dimensional manifold. We shall hereafter use the terms *vertices, edges* or *lines,* and *regions* for 0-cells, 1-cells, and 2-cells.

If the interior of a region of the map is imaged continuously on the interior of a circle there will be a finite * set of points on the boundary of the circle which correspond to the vertices of the map that lie on the boundary of the given region. The number of points in this set is the number of *vertices of the region.* The number of vertices of a region is the same as the number of *sides or edges of the region.*

A map of k n-sided regions will be said to be *regular* if the number of ways in which the surface can be put into $(1-1)$ correspondence with itself so that regions correspond to regions, edges to edges, and vertices to vertices, without reversal † of the sense of the boundaries of regions is kn.

The operations of transforming a map into itself in such a way as to preserve the sense of a region obviously constitute a group; this group will be called *the group of the map.* A map of k n-sided regions will be regular if its group is of order kn.

It is immediately evident and has been noted otherwhere * that the group of a regular map may be generated by two operators, viz.: S which leaves a given region fixed permuting its edges cyclically, and T which leaves an edge of this region fixed interchanging the two regions which have the edge in

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[*] We exclude the case of a map defined by designating a single point on a sphere as the boundary of the remainder of the surface.

[†] It is well-known that the regions of a map on a two-sided surface may be sensed alike in two distinct ways, cf. Veblen and Young, *Projective Geometry,* Vol. II, p. 495. We require that there be kn ways of transforming k positively sensed regions into the same k positively sensed regions.

[*] Brahana and Coble, *loc. cit.,* p. 5.

common. T is of order two. By means of these two operations and combinations of them any region of the map may be transformed into any other region with a particular vertex of the first going into any vertex of the second.

In demonstrating the sufficiency of the above condition, we shall distinguish two cases: (1) T is not permutable with S or any group generated by a power of S; and (2) there exists a group generated by a power of S which is permutable with T. The tetrahedral, octahedral, and icosahedrol groups come within the first category and the dihedral groups are in the second.

We consider a group G generated by two operators S and T, the latter of order two and not permutable with any group generated by a power of the former. We distribute the operators of G in k right co-sets with respect to the subgroup H consisting of S and its powers and denote each co-set by a letter. Multiplication of all the operators of G on the right by any operator of G interchanges the co-sets and so determines a substitution on the k letters. The resulting substitution group will be transitive and will be simply isomorphic with G. We note for future use that the only substitutions which omit the letter corresponding to H are those which correspond to operators of H.

Let S_a, T_{ab}, and S_b be the substitutions corresponding to S, T, and TST respectively, b being the letter into which a is transformed by T_{ab}. The letter b appears in a cycle of S_a, otherwise S_b would omit a contrary to the hypothesis that T is not permutable with any subgroup generated by a power of S. The cycle C of S_a which contains b is of degree n, otherwise some power of S_a lower than the nth would leave b fixed; this would imply that $ST \cdot S^m = S^lT$ or $S^m = TS^{l-1}T$ which is impossible for the same reason.

We may now construct a regular map corresponding to G. A polygon of n sides may be denoted by a and bounded cyclically by n n-sided polygons named from C. The transform of a and C by T_{ab} gives b and its boundary. Continuing this process we get k bounded n-sided polygons. Since there are n substitutions leaving a fixed there will be n substitutions leaving b fixed and b will appear n times in the conjugates of C. Therefore, each region appears on the boundaries of n other regions. We may join the k polygons into a simply connected polygon in a plane with its edges paired in the ordinary manner. The two dimensional manifold so defined is two-sided, for any operation leaving a fixed is a power of S_a and so leaves C unchanged making it impossible to transform a into itself with its boundary reversed. Hence, *to every group in the first class there corresponds a regular map.*

The groups of the second category will be examined in two distinct classes according as (a) T is permutable with S, or (b) T is not permutable with S but is permutable with a group generated by a power of S.

The groups of class (a) are cyclic and of even order if T is a power of S. A map corresponding to such a group is obtained by taking a polygon of n sides and joining opposite sides so as to give a two-sided surface.* An example of such a map is a four-sided region on an anchor ring; it contains one region, one vertex, and two edges.

If T is not a power of S the group is Abelian and of order $2n$. If there exists a regular map corresponding to it and containing an n-sided region it must contain two such regions. We may take two n-sided polygons and letter their sides $(abc \cdots f)$ and $(\alpha\beta\gamma \cdots \zeta)$ respectively. We join them together along a and α, making b correspond to β, c to γ, and so on, the members of a pair being oppositely sensed with respect to the double polygon. Such a map admits the operations $(abc \cdots f)(\alpha\beta\gamma \cdots \zeta)$ and $(a\alpha)(b\beta)(c\gamma) \cdots (f\zeta)$. It may be readily verified that the number of vertices is 1 or 2 according as n is odd or even and that the genus of the resulting surface is $(n-1)/2$ or $(n-2)/2$.

In the groups of class (b) the subgroup $1, T$ is not invariant and we may represent the group as a substitution group on symbols for the co-sets with respect to this subgroup by the method used for groups of the first category. If a is the letter corresponding to the set $1, T$ the substitution corresponding to S contains a in a cycle C of n letters, for the n co-sets $\begin{matrix} 1 & S & S^2 & \cdots S^{n-1} \\ T & TS & TS^2 & \cdots TS^{n-1} \end{matrix}$ are all distinct. If we denote the order of G by kn as before, we see that the substitution group is of degree $kn/2$. The number of letters in C and its conjugates is kn so that each of the letters appears twice. We may construct a map corresponding to G by taking an n-sided polygon for each of the conjugates of C and bounding it according to the letters of the conjugate. This time we name edges on the boundary instead of regions across the boundary. The polygons may be joined into a single two-sided surface by coalescing like-named edges with the usual precautions as to the senses of corresponding edges.

We note that the dihedral groups are contained in this class. If T transforms S into its inverse the group is dihedral. The resulting map contains two n-sided regions and lies on a sphere. It may be obtained by drawing an

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* The only requirement is that the pair of corresponding sides a and a' be oppositely sensed on the boundary of the polygon and that they be then joined so that the two senses coincide. See Brahana, "Systems of Circuits on Two-Dimensional Manifolds," *Annals of Math.*, Vol. 23 (1922), p. 146.

n-sided polygon on a sphere. T may transform S into some power of itself other than its inverse in which case the group is again of order $2n$ and the map consists of two n-sided polygons not on a sphere. If neither of the above conditions obtain the map still resembles the dihedral maps in that each region touches some other region along more than one edge. To see this let S_α and S_β be generators of the groups leaving the neighboring regions α and β fixed, and let $T_{\alpha\beta} S_\alpha T_{\alpha\beta} = S_\beta$, where S_α and $T_{\alpha\beta}$ are the substitutions corresponding to S and T respectively. Since $T S^m T = (S^m)^r$ for some m less than n, then $T_{\alpha\beta} S_\alpha^m T_{\alpha\beta} = (S_\alpha^m)^r = (S_\beta)^m$ leaves both α and β fixed and since $mr \neq n$ S_α must transform an edge common to α and β into another edge common to α and β.

We return from the digression of the last paragraph to state the principal result so far obtained in the following theorem:

A necessary and sufficient condition that G be the group of a regular map is that G be generated by two operators of which one is of order two.

2. *Some Types of Group that Give Regular Maps. The Symmetric and Alternating Groups.* It is well known that the tetrahedral, octahedral, and icosahedral groups are simply isomorphic with the alternating and symmetric groups of degree four and the alternating group of degree five. We extend the above result by means of two theorems, of which the first is:

To the symmetric group of degree n there corresponds a regular map of $(n-1)!$ n-sided regions on a surface of genus

$$p = 1 + (n-2)!\,(n^2 - 5n + 2)/4.$$

The existence of the map follows from the theorem of §1 and a theorem due to Moore[*] that the symmetric group of degree n is generated by two operators of orders n and two whose product is of order $(n-1)$. In order that we may determine the genus of the surface on which the map lies we shall recall a theorem that was used in *Regular Maps on an Anchor Ring.* In that paper it was proved (p. 227), though not explicitly stated, that

If S generates the group leaving a region fixed and T is the operator leaving an edge of the same region fixed, then ST generates the group leaving a vertex of the region fixed.

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[*] *Proceedings of the London Mathematical Soc.*, Vol. 28 (1896), pp. 357-366. See also Carmichael, *Quarterly Journal*, Vol. 49 (1922), p. 235.

From this theorem is follows that the number of regions at a vertex in a regular map is equal to the order of ST. In the maps under consideration there are thus $(n-1)$ regions at a vertex. The number of vertices is $n(n-2)!$, the number of edges is $n!/2$, and the number of regions is $(n-1)!$. From the Euler formula we obtain the genus $p=1+(n-2)!(n^2-5n+2)/4$. We give in the following table a list of the maps described by the theorem for $n=4$, 5, 6, and 7.

n	k	p	n	k	p
4	6	0	6	120	49
5	24	4	7	720	481

The first of these maps is the cube; the second is one of the maps described in *Maps of Twelve Five-Sided Regions,* etc. (*l. c.,* p. 19) and is the doubly covered figure II of that paper.

The alternating groups of degree greater than three are among the groups that give regular maps, for each group is generated by two operators one of which is of order two.* The generators may be chosen to be of orders 2 and $(n-1)$ with product of order $(n-1)$ if n is even, and of orders 2 and $(n-2)$ with product of order n if n is odd. When n is even we have $n(n-2)!/2$ vertices, $n!/4$ edges, and $n(n-2)!/2$ regions. The Euler formula takes the form

$$n(n-2)!/2-n!/4+n(n-2)!/2=2(1-p),$$

whence

$$p=1+n(n-2)!\,(n-5)/8.$$

When n is odd we have $(n-1)!/2$ vertices, $n!/4$ edges, and

$$n(n-1)(n-3)!/2$$

regions. Hence,

$$p=1+(n-1)(n-3)!\,(n^2-6n+4)/8.$$

We have the following theorem:

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* It may be shown readily that $S=(a_1, a_2, \cdots, a_{n-1})$ and $T=(a_1a_2)(a_3a_n)$ for n even, and $S=(a_1, a_2, \cdots, a_{n-2})$ and $T=(a_1a_{n-1})(a_2a_n)$ for n-odd, generates the alternating group of degree n.

To the alternating group of degree $n(>3)$ *there corresponds a regular map of* $n(n-2)!/2$ $(n-1)$*-sided regions or a map of* $n(n-1)(n-3)!/2$ $(n-2)$*-sided regions on a surface of genus* $1+n(n-2)!(n-5)/8$ *or* $1+(n-1)(n-3!)(n^2-6n+4)/8$ *according as n is even or odd.*

It is of some interest to note the genera of the maps for small values of n. In the following list n' is the number of sides of each region of the map given by the theorem above.

n	n'	k	p	n	n'	k	p
4	3	4	0	6	5	72	19
5	3	20	0	7	5	504	199.

The first two are the tetrahedron and the icosahedron respectively.

Every regular map determines a second regular map which we shall call its *dual.* The dual of a map is obtained by taking a point within each region and joining the points of every pair of neighboring regions by an arc across their common edge, or by an arc across each common edge if more than one exists, the arcs being chosen so that no two intersect. The resulting map has a region for each vertex and a vertex for each region of the original map; the number of edges is the same in both. The cube and the octahedron are dual to each other, as are also the dodecahedron and the icosahedron. The dual of the tetrahedron is a tetrahedron; such a map will be called *self-dual.*

If S and T are the generators of the group from which a given map is obtained by the methods of § 1, the generators of the same group which would give the dual map are (ST) and T. The number of sides of a region of a map is equal to the number of regions at a vertex of its dual. Hence, *a necessary and sufficient condition that the map corresponding to the generators* S *and* T *be self-dual is that the orders of* S *and* ST *be the same.*

We have immediately the following theorem:

The maps given by the theorem concerning alternating groups are self-dual whenever n is even.

Subgroups of the Metacyclic Groups. The metacyclic group $G_{p(p-1)}$ of degree p (p must be prime) is generated by an element Σ of order p and a cyclic element S of order $(p-1)$.* The element Σ generates an invariant subgroup and the remaining operators are the transforms of powers of S by Σ and its powers. S and Σ satisfy a relation $S^{-1}\Sigma S = \Sigma^{m}$, where $m^{p-1} \equiv 1$, mod p, and m satisfies no relation of the form $m^{x} \equiv 1$, mod p, where $x < p - 1$. Since p is a prime (> 2) S is of even order and $S^{(p-1)/2}$ is of order 2.

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* Netto, *Substitutiontheorie,* § 125.

If $p - 1$ contains any even factor ρ and $p - 1 = \rho\lambda$ then S^{λ} generates a cyclic group of order ρ which contains an element of order two. S^{λ} and Σ generate a group of order ρp. We shall show that this group $G_{\rho p}$ may be generated by $S' = S^{\lambda}$ and any of its elements of order two except $(S')^{\rho/2}$, e. g. by $T' = \Sigma^{-k} S'^{\rho/2} \Sigma^{k}$. From the fact that $S^{-1}\Sigma S = \Sigma^{m}$, we have $\Sigma S = S\Sigma^{m}$, and $\Sigma S' = S' \Sigma^{m^{\lambda}}$. Hence

$$T' = \Sigma^{-k}(S')^{\rho/2}\Sigma^{k} = (S')^{\rho/2}\Sigma^{-km^{(p-1)/2}}\Sigma^{k} = (S')^{\rho/2}\Sigma^{2k}.$$

This last relation is due to the fact that

$$m^{p-1} - 1 = (m^{(p-1)/2} - 1)\,(m^{(p-1)/2} + 1) \equiv 0, \text{ mod } p.$$

The invariant subgroup is generated by any power of Σ except identity and hence $[S', T']$ contains Σ. Therefore to every group $G_{\rho p}$ there corresponds a regular map.

In order to determine the genus of the surface on which the map corresponding to $G_{\rho p}$ lies we must find the order of $S'T'$. From considerations similar to those used above we see that

$$S'T' = (S')^{[(\rho/2)+1]} \quad \Sigma^{2k}$$

$$(S'T')^{2} = (S')^{2[(\rho/2)+1]} \quad \Sigma^{2k(m^{[(\rho/2)+1]\lambda}+1)}$$

$$= (S')^{2[(\rho/2)+1]} \quad \Sigma^{2k(-m^{\lambda}+1)}$$

$$(S'T')^{3} = (S')^{3[(\rho/2)+1]} \quad \Sigma^{2k(m^{2\lambda}-m^{\lambda}+1)}$$

and in general,

$$(S'T')^{n} = (S')^{n[(\rho/2)+1]} \quad \Sigma^{2k(1+m\lambda-m^{2\lambda}+\dots+(-1)^{n-1}m^{(n-1)\lambda})}$$

If we set $(S'T')^{n} = 1$ and seek the smallest value of n that will satisfy the relation, we shall have to find the smallest value of n that will satisfy the two congruences

(a) $\lambda n[(\rho/2)+1] \equiv 0$, mod $p - 1$, and

(b) $2k(1 - m^{\lambda} + m^{2\lambda} - m^{3\lambda} + \cdots + (-1)^{n-1} m^{(n-1)\lambda}) \equiv 0$, mod p.

If n is odd (b) takes the form $\frac{1+m^{n\lambda}}{1+m^{\lambda}}\equiv 0$, mod p, the $2k$ being dropped because p is prime. This implies $m^{n\lambda}=-1$, mod p, hence $n=\rho/2$ satisfies the congruence provided $\rho/2$ is odd. This value also satisfies (a) since $(\rho/2)+1$ is even and $n[(\rho/2)+1]$ is a multiple of ρ. If n is even (b) takes the form $\frac{1-m^{n\lambda}}{1+m^{\lambda}}\equiv 0$, mod. p, in which case $m^{n\lambda}\equiv 1$ mod p, and $n=\rho$. This value obviously satisfies (a). Therefore the order of $S'T'$ is ρ or $\rho/2$ according as $\rho/2$ is even or odd.

The corresponding map will have p regions, $p\rho/2$ edges, and p or $2p$ vertices. From the Euler formula we find the genus of the surface. We state the result in the following theorem:

To every group of even order $p\rho$ which is contained in the metacyclic group of degree p there corresponds a regular map of p ρ-sided regions on a surface of genus $1+(p/4)(\rho-4)$ or $1+(p/4)(\rho-6)$ according as $\rho/2$ is or is not a multiple of 4.

Since when ρ is a multiple of 4 S' and $S'T'$ are of the same order, we have

The maps corresponding to $G_{\rho p}$ are self-dual whenever ρ is a multiple of 4.

When $\rho=2$ the groups are dihedral and the maps lie on a sphere. When $\rho=p-1$ the groups are the metacyclic groups themselves and the maps are those given by Heffter (cf. the reference above). When $\rho=4$ or 6 the maps lie on an anchor ring (cf. above).

It is worthy of note that of all the maps whose existence we have proved very few lie on surfaces of low genus. The Euler formula $V-E+F=2(1-p)$ shows that if p is to be small V and F must be as large as possible, which for a group of given order requires that n and v, the orders of S and ST, be small. If we seek a map on a surface of low genus whose group is a subgroup of a metacyclic group the degree of the group or ρ or both must be small. If the genus is to be greater than 1 p must be at least 11 and ρ must be at least 8. The metacyclic group of degree 11 gives a map on a surface of genus 12; the map on the surface of lowest genus corresponding to a group G_{8p} is made up of 17 octagons on a surface of genus 18. The smallest map of decagons corresponds to a G_{31p} and lies on a surface of genus 32. We note in passing that whenever p is of the form $40h+1$ there exists a map of p octagons and a map of p decagons each on a surface of genus $p+1$. The more interesting maps, i. e. those on surfaces of low genus, are all missing except some of the maps on an anchor ring.

Subgroups of the Modular Group. The two-rowed unit matrices with integer elements constitute a group simply isomorphic with the modular group.* If the elements of each matrix are reduced modulo n there is obtained a finite set of matrices of determinant 1, mod n, which constitute a group isomorphic $(1 - \infty)$ with the modular group. This group, $G_{2\mu(n)}$, contains a single element of order two, viz.: $\begin{pmatrix} n-1 & 0 \\ 0 & n-1 \end{pmatrix}$, and so is not available for the group of a regular map. If, however, we make the further reduction of considering $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be equivalent to $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ we obtain $G_{\mu(n)}$ which is in $(1-2)$ isomorphism with $G_{2\mu(n)}$. The group $G_{\mu(n)}$ is generated by the two operators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ n-1 & 0 \end{pmatrix}$ which we denote by S and T. The order of S is n, the order of T is 2, and the order of ST is 3. The order of the group is given by the equation

$$\mu(n) = (n^3/2)(1-1/q_1^2)(1-1/q_2^2)\cdots \quad \text{where } n = q_1^{\gamma_1} q_2^{\gamma_2} \cdots$$

and the q's are distinct primes.

The group $G_{\mu(n)}$ determines a map of $\mu(n)/n$ n-sided regions, $\mu(n)/2$ edges, and $\mu(n)/3$ vertices. Using these values in the Euler formula we determine the genus of the surface. The result is the following theorem:

To every group $G_{\mu(n)}$ there corresponds a regular map of $\mu(n)/n$ n-sided regions on a surface of genus $1 + (1/6 - 1/n)\mu(n)$.

It is evident that the maps associated by Klein-Fricke with the groups $G_{\mu(n)}$ have a close resemblance to the regular maps we have obtained. Their maps consist of $2\mu(n)$ triangles in which the subgroups of order n are represented by the $2n$ triangles that come together at a vertex of one type, the subgroups of order three by the six triangles at a vertex of another type, and the elements of order two by the four triangles at a vertex of a third type. We represent a subgroup of order n by a single n-sided region, which amounts to combining the $2n$ triangles at a vertex of the first type into a single region. If this combination is made at each of the vertices of the first type their maps will obviously become regular maps in the sense in which we are using the term.

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* The facts of this paragraph are to be found in Klein-Fricke, *Theorie der Elliptischen Modulfunktionen*, Leipzig (1890), Chapter 7.

For certain composite values of n $G_{\mu(n)}$ will contain distinct elements of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$; for example, in addition to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have, when $n=8$, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and, when $n=15$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. These elements will constitute an invariant subgroup of $G_{\mu(n)}$ and if this subgroup is of order m we obtain a group $G_{\mu(n)}/m$ in $(1-m)$ isomorphism with $G_{\mu(n)}$ by considering the operators of this invariant subgroup as equivalent. The groups $G_{\mu(n)}/m$ satisfy the condition of § 1 and so give regular maps.

The following is the list of maps corresponding to groups $G_{\mu(n)}$ for small values of n; it is essentially the combination of two lists given by Klein-Fricke.

k	n	p	k	n	p
4	3	0	24	7	3
6	4	0	24	8	5
12	5	0	36	9	10
12	6	1	36	10	13

The first four are regular polyhedra and a map on an anchor ring.

The subgroups of $G_{\mu(n)}$ for n prime are described by Klein-Fricke. Every such subgroup, and the metacyclic groups are among them, which is generated by two operators one of which is of order two gives a regular map. We shall not pursue this question further but shall note two groups of low order that give maps on surfaces of low genus. The group already mentioned obtained by taking $G_{\mu(8)}$ and considering $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ to be equivalent gives a G_{96} which corresponds to a map of 12 octagons on a surface of genus 3. If we take $G_{\mu(8)}$ and consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. $\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 4 & 3 \end{pmatrix}$ equivalent we obtain a G_{48} which corresponds to a map of 6 octagons on a surface of genus 2. (Klein-Fricke, p. 652.) To see that the map is made up of octagons we note that $S=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is still of order 8, since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies $a \equiv 0$, mod 8. We note for a later reference that $S^4T = TS^4$, where $T=\begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}$, for $TS^4T = \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

3. *The Regular Maps on a Surface of Genus Two.* Our method of investigating this question is first to determine from the Euler formula the possible maps and then to examine the possibility of the existence of groups having the characteristics required by the maps. From the maps we are able to determine the order of the group, the orders of two generating operators, and the order of their product. Since the existence of a map implies the existence of a dual we may choose the generating operators of the group in two distinct ways whenever the proposed map is not self-dual. This is of appreciable advantage whenever the map is such that the generating operators of the group may be chosen so that S is of prime order, for it assures us that the order of the group must be twice the order of S if T is permutable with a group generated by a power of S, or else, in the opposite case, the group may be represented on symbols for regions of the proper one of the two dual maps. Having the order and degree of the group we are able to make use of the work that has been done in listing groups of low degree.

We shall consider first the cases of maps of one and two regions respectively. For a map of one region on a surface of genus two the Euler formula takes the following form $(n/v) - (n/2) + 1 = -2$ where v is the number of regions at a vertex and n is the number of sides of a region. This equation may be written in the form $(n-6)(v-2) = 12$. $(v-2)$ is a positive integer, $(n-6)$ is therefore positive and is an integer because n is an integer. The possible solutions are

$$\begin{array}{llllll} n = 7, & 8, & 9, & 10, & 12, & 18 \\ v = 14, & 8, & 6, & 5. & 4, & 3. \end{array}$$

The sides of the region are to be joined in pairs and so n cannot be odd. If $n = 8$ S is of order 8 and $T = S^4$. Then $ST = S^5$ and is of order 8. Since this is the value of v it follows that *there exists a map of a single octagon on a surface of genus two.*

If $n = 10$ we have $S^{10} = 1$, $T = S^5$, and $ST = S^6$ is of order 5 which is the value of the corresponding v. Hence, *there exists a map of a single decagon on a surface of genus two.*

If $n = 12$ we have $S^{12} = 1$, $T = S^6$, and $ST = S^7$ is of order 12. This is not the proper value for v, and hence there is no map. If $n = 18$ we have $S^{18} = 1$, $T = S^9$, and $ST = S^{10}$ is of order 9. There is no map in this case.*

If the map contains two regions we get in the same manner as above $(n-4)(v-2) = 8$. The possible solutions are

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* The last two groups give maps on surfaces of genera 3 and 4 respectively.

$$\begin{array}{lllll} n = 5, & 6, & 8, & 12 \\ v = 10, & 6, & 4, & 3. \end{array}$$

The dual of the map of a single decagon above corresponds to the case $n = 5$. Thus, *there exists a map of two pentagons on a surface of genus two.* We note that the second map given above is self-dual and therefore we do not expect to find its dual again. Since the group of order 10 is either cyclic or dihedral and since the map corresponding to the latter group is on a sphere there is no other map of two pentagons on a surface of genus two.

If $n = 6$ we note that the group must be Abelian and S and T must be independent generators, for $TST = S^m$ where $m^2 \equiv 1$, mod 6. The only solutions are $m = 1$, or 5. The latter gives a dihedral group and the map lies on a sphere. Hence, S and T are permutable and ST is of order 6 which is the corresponding value of v in the table above. Therefore, *there exists a map of two hexagons on a surface of genus two. The map is self-dual.*

If $n = 8$ we have $TST = S^m$ where $m^2 \equiv 1$, mod 8. The solutions are $m = 1$, 3, 5, and 7. The last gives the dihedral group of order 16 and need not be considered. The first gives ST of order 8 which is not the proper value for v. For $m = 3$ we may take $S = (abcdefgh)$ and $T = (bd)(cg)(fh)$, whence $ST = (adeh)(bgfc)$ which is of the proper order. Hence, *there exists a map of two octagons on a surface of genus two.* If $m = 5$ we may take S to be as above in which case T will be $(bf)(dh)$; their product is $(afgdebch)$ which is not of the proper order. Hence, *there is just one map of two octagons.*

Finally, if $n = 12$ $TST = S^m$ where $m^2 \equiv 1$, mod 12. The possible solutions are $m = 1$, 5, 7, 11. The first and last are impossible as in the preceding case. For the other two cases we may take S to be $(abcdefghijkl)$ and T and T' to be $(bf)(ck)(ei)(hl)$ and $(bh)(dj)(fl)$ respectively. The orders of ST and ST' are 4 and 6 and therefore the groups give maps on surfaces of genera 3 and 4 respectively. Hence, *there exists no map of two dodecagons on a surface of genus two.*

We have now considered all the possibilities when k, the number of regions, is less than 3. We tabulate the remaining possibilities in the following list.

In making out this list we may proceed according to values of k as we have done for the cases $k = 1$ and 2. To see that k cannot be greater than 28 we may put the equation given by the Euler formula in the form $2/v + 2/n + 4/kn = 1$. Since $k > 28$ and $n \geqq 3$, $1/v + 1/n \geqq 10/21$. Neither v nor n can be less than 3 and hence both must be less than 7, and if one of them is 4 the other must be less than 5. Possible pairs of values for v, n are 3, 3; 3, 4; 3, 5; 3, 6; 4, 3; 4, 4; 5, 3; 6, 3. The corresponding values of $n/2 - (n + v)/v$ are $-1/2$, $-1/4$, $-1/10$, 0, $-1/3$, 0, $-1/3$, 0. None of these values are positive as they must be to satisfy the Euler formula, since k is necessarily positive. Hence *there is no map of more than* 28 *regions on a surface of genus two.*

The columns headed r and g give respectively the number of vertices and the orders of the groups. The list contains just one of a pair of dual maps, the one given being the one with the smaller value of k, for example, the first map would contain 3 10-sided regions coming together 3 at a vertex and its dual which is not listed would be made up of 10 triangles coming together 10 at a vertex. The only exceptions we have made are in the cases where the dual map would contain 1 or 2 regions.

	k	n	v	r	g		k	n	v	r	g
1.	3	10	3	10	30	7.	6	8	3	16	48
2.	3	4	12	1	12	8.	6	3	18	1	18
3.	4	9	3	12	36	9.	8	5	4	10	40
4.	4	6	4	6	24	10.	8	3	12	2	24
5.	4	5	5	4	20	11.	12	7	3	28	84.
6.	4	4	8	2	16						

We may dispose immediately of nos. 2, 6, 8, and 10. Since there does not exist a map of one 12-sided region or one 18-sided region on a surface of genus two *there exists no map corresponding to no.* 2 *or no.* 8. There exists a map of two octagons and no map of two dodecagons, hence *there exists a map of* 4 *4-sided regions* and *there does not exist a map of* 8 *triangles on a surface of genus two.*

A brief consideration shows that there can exist no regular map of 4 pentagons corresponding to no. 5. No region can touch itself along an edge unless it touches itself along every edge, in which case the map would consist of a single region of an even number of sides. If a given region of a regular map touches another region more than once along an edge it must touch each of the regions on its boundary the same number of times and hence if the number of its edges is a prime it can touch but one other region. Such a map can contain but two regions. Hence *there exists no map of* 4 *pentagons on a surface of genus two.*

For the remaining possibilities we examine the substitution groups of the proper order and degree, all of which, with one exception, have been listed.*

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* These groups are to be found as follows:
Degrees 4–8, Miller, *American Journal of Mathematics*, Vol. 28 (1899), pp. 287-338;
Degree 10, Cole, *Quarterly Journal*, Vol. 27 (1894), pp. 39-50.
Degree 12, Miller, *Quarterly Journal*, Vol. 28 (1896), pp. 193-231.

The degree of the group may be taken to be either k or r, since r is the n ber of regions of the dual map. The exception is that of a possible grou order 48 and degree 6 or 16 corresponding to no. 7 of the list. It is evid that if such a map exists its group cannot be represented on symbols regions, but may be represented on symbols for regions of the dual n Thus we would seek a substitution group of order 48 and degree 16. groups of degree 16 have not been listed.

We take up the remaining cases in order.

1. We seek a group of order 30 and degree 10. No such group ex and hence *there exists no map.*

3. There are five groups of order 36 and degree 12.

The first two groups contain $(abcdef\cdot ghijkl)_{18}$ which is simply isomor with $(\alpha\beta\gamma)$ all $(\delta\epsilon\zeta)$ cyc. G_{18} thus contains operators of orders 2, 3, an The two groups are obtained by combining G_{18} with $T_1 = (ag\cdot bh\cdot ci\cdot dj\cdot ek\cdot$ and $T_2 = (aj\cdot bl\cdot ck\cdot dg\cdot ei\cdot fh)$ respectively. T_1 is permutable with every o ator of G_{18}, but T_2 is permutable with G_{18} without being permutable w every operator. Neither of these groups contains an operator of order 9, since $T_1ST_1 = S$ and $T_2ST_2 = S'$ for every S where S and S' are in $(T_1S)^9 = 1$ would imply $T_1S^9 = 1$ and $(T_2S)^9 = 1$ would imply $T_2S(S'$ $= 1$ which are impossible since neither T_1 nor T_2 is in G_{18}.

The third and fourth groups may be managed in the same manner. third is $\{(abcdef)$ cyc $(ghijkl)$ cyc$\}$ pos and $R_1 = (ajdg\cdot bkeh\cdot clfi)$ wh square is in G_{18}. The fourth is $\{(abcdef)_6\ (ghijkl)_6\}$ pos and $R_2 = (aj$ $blfh\cdot ckei)$ whose square is also in the corresponding G_{18}. The first G_{18} c tains operators of orders 2, 3, and 6; the second contains operators of or 2 and 3 since $(abcdef)_6$ is simply isomorphic with the symmetric grou degree 3 and the products of the type $(ad)(bf)(ce)(ghi)(jkl)$ are not in An operator of order 9 would have to be outside G_{18} in either case and he of the form RS where R is either R_1 or R_2 and S is in the corresponding Since $RSR = R^4\cdot RSR = R^2R^3SR = R^2S'' = S'$ where S'' and S' are in $(RS)^9 = 1$ would imply $RS(S'S)^4 = 1$ which is impossible since R is in G_{18}.

The fifth group is $(abcd\cdot efgh\cdot ijkl)$ pos which is simply isomorphic w the alternating group of degree 4, and $(aei\cdot bfj\cdot cgk\cdot dhl)$ which is permut with every operator of the G_{12}. G_{12} contains operators of orders 2 and 3 G_{36} contains operators of orders 2, 3, and 6. Hence, *there exists no ma twelve triangles on a surface of genus two.*

4. We seek a group of order 24 and degree 6. There are three transitive roups of the proper order and degree. The first two $(\pm abcdef)_{24}$ and $+ abcdef)_{24}$, are simply isomorphic with the symmetric group of degree 4 nd so contain no operators of order 6. Hence, neither of these can be the roup of a regular map of six 4-sided regions coming together six at a vertex.

The third group is $(abcdef)_{24_6}$ which is simply isomorphic with the direct roduct of the alternating group on four letters and an operator of order two. ts operators are of orders 2, 3, and 6 and so it could not be the group of a nap of 4-sided regions.

7. We seek a map of 16 triangles coming together 8 at a vertex. The ual map is composed of six octagons coming together three at a vertex. ince in the dual map $n > k$ each octagon would have to touch just four or wo others and it would be impossible to represent the group on symbols for egions. Therefore we have no hope of finding the group among those of rder 48 and degree 6. The groups of order 48 and degree 16 have not been sted. However, the group generated by S and T which satisfy the relations

$$S^8 = 1, \quad T^2 = 1, \quad (ST)^3 = 1, \quad \text{and} \quad (S^4T)^2 = 1$$

of order 48.[*] To see this we note that S,

$$S_1 = TST, \quad S_2 = S^{-1}S_1S = S^6T, \quad S_3 = S^{-1}S_2S = S^5TS,$$
$$S_4 = S^{-1}S_3S = TS^6, \quad \text{and} \quad S_5 = TS_3T = S^3$$

nstitute a complete conjugate set, S transforming S_1, S_2, S_3, S_4 cyclically nd leaving S_5 fixed and T transforming S, S_2, and S_3 into S_1, S_4, and S_5 spectively. S represented on symbols for its conjugates is of order four, is of order two, and ST is of order three transforming S, S_1, and S_4 cyclic-lly and S_2, S_5, and S_3 likewise cyclically. The group represented on symbols or conjugates of S is the group of the cube, or the octahedral group, being enerated by two operators of orders two and three whose product is of order ur. Hence, the group generated by S and T is of order 48. Therefore, *here exists a map of six octagons coming together three at a vertex on a urface of genus two.*

9. We seek a group of order 40 and degree 8 or 10. There is one group f order 40 and degree 10, viz. $(abcde \cdot fghij)_{20}$ and $T = (af \cdot bg \cdot ch \cdot di \cdot ej)$ hich is permutable with every element of the G_{20}. The G_{20} is simply iso-orphic with the metacyclic group of degree 5. The elements of G_{20} are of

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[*] This group is among the groups of genus two given by Burnside. It is also the oup described at the end of § 2.

orders 2, 4, and 5 and the elements outside of G_{20} are of orders 2, 4, and 10. The elements of order 5 with identity constitute a cyclic subgroup which is invariant under G_{20} and so under G_{40}. Let S and Σ be elements of orders 4 and 5 respectively of G_{20}. G_{40} is generated by S and an element of order two outside of G_{20} which must be of the form $T\cdot\Sigma^{-k}S^2\Sigma^k = T'$, for the group $[S, T']$ contains $S^2T' = TS^2\Sigma^{-k}S^2\Sigma^k = T\cdot\Sigma^{-k}$, since $S^{-1}\Sigma S = \Sigma^2$. $T\cdot\Sigma^{-k}$ is of order 10 and its square is a power of Σ, hence Σ is in $[S, T']$. The order of ST' is not 5 since all the operators of order 5 are in G_{20}. Hence, the map corresponding to such a choice of generators would not lie on a surface of genus two.* The generators might be chosen to be $S' = ST$ and $T'' = \Sigma^{-k}S^2\Sigma^k$ the first being any operator of order four outside of G_{20} and the second being in G_{20}. The above argument shows also that their product is not of order 5. Noting that there exists no group of order 40 and degree 8, we have the result that *there exists no regular map of ten 4-sided regions on a surface of genus two.*

11. This case is readily disposed of by observing that there exists no group of order 84 and degree 12.

Collecting our results we see that there exist just 8 regular maps on a surface of genus two. They are made up of

1 octagon, a self-dual map;
1 decagon and the dual map of
2 pentagons;
2 hexagons, a self-dual map;
2 octagons and the dual map of
4 quadrangles;
6 octagons and the dual map of
16 triangles.

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* For neither choice of generators could ST be of order two since G_{40} is not dihedral; ST must therefore be of order four and the map lies on an anchor ring. See *Regular Maps on an Anchor Ring*, p. 234.

The Groups Belonging to a Linear Associative Algebra.

By Arthur Ranum

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1. The present paper may be considered a generalization and extension of the ideas contained in a paper on "The Group-Membership of Singular Matrices" in the *American Journal of Mathematics*, vol. 31 (1909), pp. 18-41; but no knowledge of the earlier paper is here presupposed. Whereas there the algebra treated was that of all n-ary matrices whose elements belong to the field of all complex numbers, here we shall take a more general viewpoint and consider any linear associative algebra whatever in any field whatever.

Frequent references will be made to L. E. Dickson's *Algebras and Their Arithmetics,* University of Chicago Press (1923), and to G. Scorza's *Corpi Numerici e Algebre,* Messina (1921).

2. Groups and semi-groups * will be defined as follows. Given a set S of elements and a law of combination of the elements, under which they satisfy the associative law, then if the result of combining any two of the elements of S is an element of S, S is called a *semi-group.* If in addition S contains a unique identical element (identity) and for every element a unique inverse, then S is called a *group.*

Any algebra (and indeed any linear set) is evidently a group under addition, and any associative algebra is a semi-group, but not a group, under multiplication. It may be possible, however, to select certain sets of elements of an associative algebra, each of which is a group under multiplication. The present investigation is concerned with precisely these groups and with the relations existing between them and the remaining elements (if any), namely the non-group-members. We assume, therefore, that every algebra considered is associative.

The only case heretofore studied is that in which the algebra A has a modulus m, and the group concerned has m for its identical element and so consists either of the totality G of those elements x of A that have inverses x^{-1} such that $xx^{-1} = m$, or of the elements of a subgroup of G.

3. But it is evident that whether the algebra A has a modulus or not, any element u of A for which $u^2 = u$, that is any idempotent element or zero,

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* See H. Hilton's *Theory of Groups of a Finite Order* (1908), p. 51.

can be chosen as the identity of a group belonging to A; and conversely, any group belonging to A must have for its identity either an idempotent element or zero. Moreover, if u is any idempotent element, Scorza (Art. 160, p. 266) shows that an element x of A will satisfy the condition $ux = xu = x$, if and only if, it belongs to the sub-algebra uAu, having the modulus u. Hence any group having u for its identity must belong to uAu.

We shall use the term *entire group* for the largest such group G_u, of which all the others are sub-groups; G_u is then the totality of the elements x of uAu having inverses x^{-1} in uAu such that $xx^{-1} = u$; moreover $x^{-1}x = u$, and x uniquely determines x^{-1}, in uAu and therefore also in G_u. Now this is equivalent, as proved by Scorza (II, Art. 109), to saying that an element x of the sub-algebra uAu will belong to the group G_u, if and only if, it is neither zero nor a divisor of zero in uAu (it may be a divisor of zero in A). Among these divisors of zero there may, of course, be other idempotents and therefore other groups.

It is clear that the element zero is the identity of an entire group G_0, to which no other element of A belongs; that is, $G_0 = 0$. Since a nilpotent algebra contains no idempotents, it contains no groups except G_0; all its other elements are non-group-members.

We shall use small Greek letters to denote scalars, or numbers of the field F in which the algebra is defined, and small Roman letters to denote elements of the algebra. It is clear that if x is any *group-member*, belonging to the group G_u, then αx $(\alpha \neq 0)$ also belongs to G_u; its inverse is $(1/\alpha)x^{-1}$. Moreover, every element αu, where u is idempotent and $\alpha \neq 0$, is obviously invariant in the group G_u.

If the identity u of an entire group G_u is a principal or a primitive idempotent of A, we shall call G_u a *principal* or a *primitive* group of A, respectively. Hence, if u is any idempotent whatever, G_u is the principal group of the sub-algebra uAu.

4. We define the *left* (*right*) *characteristic* (Scorza II, Art. 104) of an element x of an algebra A to be the order of the linear set $xA\,(Ax)$. It may be mentioned in passing that when we introduce the left (right) matrix or linear transformation corresponding to x, or in Dickson's nomenclature (§ 58), the first (second) matrix of x, then the left (right) characteristic of x is equal to the *rank* of its left (right) matrix (Scorza II, Art. 193).

THEOREM: *All the elements of a group G_u belonging to an algebra A have the same left characteristic and the same right characteristic; and if x and y are any two such elements, then*

$$(1) \qquad xA = yA = uA, \qquad Ax = Ay = Au,$$

and therefore

$$(2) \qquad xAy = uAu.$$

Proof: Let z be the element of G_u for which $xz = y$, $x = yz^{-1}$. Then since $A \geqq zA$, $xA \geqq xzA = yA$; and since $A \geqq z^{-1}A$, $yA \geqq yz^{-1}A = xA$; hence $xA = yA$, and the rest of the theorem follows immediately.

Corollary: If x is any group-member belonging to A, then

$$(3) \qquad xA = x^2A \quad (Ax = Ax^2).$$

5. THEOREM: *Any two entire groups are mutually exclusive; they cannot have any element in common.*

Proof: It will evidently be sufficient to show that if two entire groups G_{u_1} and G_{u_2} have an element x in common, their idempotents u_1 and u_2 coincide. Since $xu_1 = x = xu_2$, therefore

$$(4) \qquad x(u_1 - u_2) = 0.$$

Let B denote the sub-algebra xA; in view of (3), we have

$$(5) \qquad xB = B.$$

Since (1) shows that $xA = u_1A = u_2A$, we see that $xA (= B)$ contains the elements u_1, u_2, and therefore $u_1 - u_2$. But it is known (Dickson, p. 30, lemma 3, and p. 45, 2nd paragraph) that in view of (5), any equation of the form $xz = y$, where y is an element of B, has a unique solution, in B, for z. Hence (4) implies that $u_1 = u_2$, and the theorem is proved.

6. THEOREM: *If the algebra A has an invariant sub-algebra B, any group G_u belonging to A either has no element in common with B or belongs entirely to B.* That is, an invariant sub-algebra either pays no attention to a group or swallows it whole.

Proof: Let x be an element common to G_u and B. Since B is invariant, the identical element $u = xx^{-1}$ of G_u will belong to B. For the same reason, *any* element $y = uy$ of G_u will belong to B. That is, G_u belongs entirely to B.

Now consider an algebra A of index r (Dickson, § 29), so that

$$A > A^2 > \cdots\cdots > A^r = A^{r+1} = \cdots$$

Let x be any group-member contained in A, and G_u the entire group to which it belongs. Then x^r will belong to G_u and also to the sub-algebra A^r. The latter, however, is obviously invariant in A, and so, by the last theorem, contains the entire group G_u and therefore the element x.

Hence *every group of an algebra A of index r belongs wholly to its subagebra A^r*; that is, every element belonging to A but not to A^r is a non group-member.

7. Since a division algebra has a modulus m and no divisors of zero all of its elements are group-members and the entire algebra is exhausted b two groups, namely G_0 and the principal group G_m. The question arises a to whether there exist other algebras that are exhausted by their group-members. The answer is given by the

THEOREM: *A necessary and sufficient condition that an algebra A consist entirely of group-members is that A be either a division-algebra or direct sum of division-algebras.* If t is the number of division-algebras in A then 2^t is the number of entire groups exhausting the alegbra.

Proof: Let A consist entirely of group-members. Then it contains n nilpotent elements and therefore no exceptional (maximal nilpotent invariant) sub-algebra. Hence (Dickson, §§ 37 and 40) it must be semi-simple and either A or each of its irreducible components A_i must be a simple algebr with a modulus. But by Wedderburn's theorem (Dickson, § 51) A_i is a direc product of a division-algebra B_i and a complete matric algebra* C_i, each o which is a sub-algebra of A_i and therefore of A. Since in our case C_i has n nilpotent elements, it must be of order 1, and $A_i = B_i$ which is a division algebra.

Conversely, let A be the direct sum of the division algebras A_i $(i = 1$ $\cdots, t)$, and let u_i be the modulus of A_i. Let x be any element of A; the $x = \sum_1^t x_i$, where x_i belongs to A_i. If every term $x_i = 0$, x is the group member 0; if at least one term $x_i \neq 0$, there is no loss of generality i writing $x = \sum_1^s x_i$, where $s \leq t$ and $x_i \neq 0$ $(i = 1, \cdots, s)$. If $u = \sum_1^s u$ and $y = \sum_1^s x_i^{-1}$, where x_i^{-1} is the inverse of x_i in the principal group of A we see at once, in view of the equations $A_i A_j = 0$ $(i \neq j)$, that $u^2 = u$ $xu = ux = x$, and $xy = u$. Hence x belongs to the entire group G_u. Th number of entire groups in A is evidently 2^t.

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[*] I prefer not to use Dickson's term "simple matric algebra" for C_i, becaus other matric algebras, sub-algebras of C_i, may also be simple.

COROLLARY 1: If the number of groups in an algebra is infinite, or is finite, but not a power of 2, then the groups cannot exhaust the algebra.

Since the proof we have given of the first part of the above theorem is based entirely on the absence of nilpotent elements from the algbra, we have

COROLLARY 2: If an algebra contains non-group-members, some of them, at least, must be nilpotent.

Since an algebra is nilpotent, if and only if, it has no idempotents (Dickson, § 31), we have the

THEOREM: *An algebra will contain no group-members except zero, if and only if, it is nilpotent.*

8. We shall now derive some criteria for determining whether a given element x of an algebra A is a group-member or not, and if it is, we shall show how to find the entire group to which it belongs. The corollary of § 4 gives a necessary condition for group-membership, which is not, however, sufficient in all cases.

In the first place it is known (Scorza, II, Arts. 105-109) that when A has a modulus, then x will belong to the principal group of A, if and only if, the left (right) characteristic of x is equal to the order of A, that is, if

$$xA = A \quad (Ax = A).$$

THEOREM: *If an algebra A has a modulus or merely a right (left) modulus, any element x of A will be a group-member or not, according as the left (right) characteristic of x^2 is equal to, or less than, the left (right) characteristic of x, that is, according as*

$$x^2A \overset{=}{<} xA \quad (Ax^2 \overset{=}{<} Ax). \tag{6}$$

By a *right modulus* (Scorza, II, Arts. 67, 70) we mean an element v such that $xv = x$, for every x in A. For a *left modulus,* similarly, we have $vx = x$. A modulus is therefore both a right modulus and a left modulus.

Proof: Let A have a right modulus v, and let x be any element of A such that $x^2A = xA$. The sub-algebra $xA = B$ will contain $xv = x$. Since $xB = B$, any equation

$$xy = z, \tag{7}$$

where z belongs to B, will have a unique solution, in B, for y exactly as in § 5. Putting $z = x$, we see that B contains a unique element u such that

$$xu = x. \tag{8}$$

Hence $xuz = xz$, $x(uz - z) = 0$, where z is *any* element of B. But this last equation is another special case of (7), and therefore implies

$$(9) \qquad uz = z,$$

so that u is a left modulus of B. Putting $z = x$ and also $z = u$, in (9), and combining the results with (8), we have $u^2 = u$, $ux = xu = x$. Hence u is idempotent and x belongs to the sub-algebra uAu, by § 3.

Finally, putting $z = u$ in (7), we see that B contains a unique element x^{-1}, such that $xx^{-1} = u$. Hence $xx^{-1}x = ux = xu$, $x(x^{-1}x - u) = 0$, which implies $x^{-1}x = u$. Moreover, x^{-1} evidently belongs not merely to B but also to its sub-algebra $Bu = xAu = uAu$. Hence x is a group-member and belongs to the group G_u. For the case where A has a left modulus and $Ax^2 = Ax$, the proof is exactly similar. In view of the corollary of § 4, the theorem is completely established.

9. In order to find a criterion applicable to the general case where A is any algebra, not necessarily possessing either right or left modulus, we make use of an *augmented algebra* A' (Dickson, p. 97), defined as follows: Letting $A = (u_1, \cdots, u_n)$, where $u_1, \cdots, u_n$ form a set of basic units of A, we put

$$(10) \qquad A' = (u_0, u_1, \cdots, u_n) = A + (u_0),$$

where u_0 is an annexed unit such that

$$u_0^2 = u_0, \quad u_0 u_i = u_i = u_i u_0 \qquad (i = 1, \cdots, n).$$

Evidently A' is an associative algebra as well as A, and it has the modulus u_0. Since A is an invariant sub-algebra of A', it follows (§ 6) that every entire group of A is an entire group of A'. By the last theorem, an element x of A is a group-member or not, according as $x^2A' \gtreqless xA'$.

In order to express this condition in terms of the original algebra A, we make use of (10) and the derived equations $xA' = xA + (x)$, $x^2A' = x^2A + (x^2)$, and so obtain the criterion

$$(11) \qquad x^2A + (x^2) \gtreqless xA + (x),$$

which can, however, be simplified and written in the form

$$(12) \qquad x^2A \gtreqless xA + (x).$$

For, let (11_1) be satisfied,* so that x belongs to a group, say to G_u. Then xA

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* By (11_1) and (11_2) we shall mean the formula (11) taken with the upper and lower signs, respectively.

contains $xu = x$, and $xA + (x) = xA$. Hence (12_1) reduces to (3) and is therefore satisfied. On the other hand, if (11_2) is satisfied, so that x is not a group-member, then since in every case $x^2A \gtreqless x^2A + (x^2)$, (12_2) is satisfied a fortiori. We have now proved the

THEOREM: *An element x of any algebra A is a group-member or not, according as it satisfies (12_1) or (12_2).*

$xA + (x)$ is obviously a sub-algebra of A. If A has a right modulus, (12) reduces to (6), as it should.

10. If x is an element of an algebra A, it is obvious (Scorza, II, Art. 113) that there always exists a positive integer r, such that

$$(13) \qquad A > xA > x^2A > \cdots > x^{r-1}A > x^rA = x^{r+1}A = \cdots,$$

except that for $r = 1$, $A \geqq xA = x^2A = \cdots$, so that the first inequality of (13) may possibly be replaced by an equality. If $r > 1$, this means that the left characteristics of $x, \cdots, x^r$ form a steadily decreasing set of integers, while the left characteristics of $x^r, x^{r+1}, \cdots$, are all equal.

We shall call r the *left index* of x in A. The *right index* is defined in a similar manner. If s is the least positive integer (if any) for which x^s is a group-member, we shall call s the *group-index* of x. Hence the group-index of a group-member is equal to 1.

If A has a right modulus and if x is an element of A whose left-index is r, then by the theorem of § 8, x^t is a group-member or not, according as $x^tA = x^{2t}A$ or $x^tA > x^{2t}A$, that is, in view of (13), according as $t \geqq r$ or $t < r$; hence r is the group-index of x. This gives us the

THEOREM: *If the algebra A has a right (left) modulus, the left (right) index of every element of A is equal to its group-index, and if A has a modulus, the left index, right index, and group-index of every element are all three equal.*

11. Now let A be any algebra whatever and A' its augmented algebra, as in § 9. Any element x of A will have the same group-index r in A' as in A, but not necessarily the same left-index. Since A' has a modulus, the theorem just proved shows that in A' the left-index of x is equal to its group-index r. Hence

$$(14) \qquad x^{r-1}A' > x^rA' = x^{r+1}A'$$

is a necessary and sufficient condition that x be of group-index $r > 1$. By means of the equations $x^rA' = x^rA + (x^r)$, etc., derived from (10), the condition (14) is expressible in terms of the original algebra A in the form

$$x^{r-1}\mathrm{A} + (x^{r-1}) > x^rA + (x^r) = x^{r+1}A + (x^{r+1}). \tag{15}$$

Finally we shall prove that (15) is equivalent to the simpler, though less symmetric, condition

$$x^{r-1}A + (x^{r-1}) > x^rA + (x^r) = x^{r+1}A. \tag{16}$$

For, if x satisfies (15), x^r is a group-member and is therefore contained in x^rA. Hence x^{r+1} is contained in $x^{r+1}A$, and (15) reduces to (16) and indeed to the still simpler form

$$x^{r-1}A + (x^{r-1}) > x^rA = x^{r+1}A, \tag{17}$$

which is a necessary, but not a sufficient, condition.

Conversely, if x satisfies (16),

$$x^rA + (x^r) \leqq x^{r+1}A + (x^{r+1}). \tag{18}$$

But since the left member of (18) always contains the right member, the two members are equal, and (15) is satisfied. We have now established the

THEOREM: *In any algebra A an element x is of group-index $r > 1$, if and only if, it satisfies the condition* (16).

If A has a right modulus, (16) reduces to (13), in agreement with the theorem of § 10.

12. THEOREM: *If in any algebra the group-index of an element x is r, its left* (*right*) *index must be either r or $r - 1$.*

Proof: Assuming (14) and noting that by (17) $x^rA = x^{r+1}A$, we see that it will be sufficient to show that

$$x^{r-2}A > x^{r-1}A \geqq x^rA,$$

or in other words that

$$x^{r-2}A > x^rA, \tag{19}$$

where $r > 2$; if $r = 2$, we write A in place of $x^{r-2}\mathrm{A}$. Now $x^{r-1} = x^{r-2}x$ is an element of $x^{r-2}A$, but is not an element of x^rA. For if it were, it would be an element of x^rA', and $(x^{r-1}) \leqq x^rA'$ would imply $x^{r-1}A' \leqq x^r(A')^2 = x^rA'$, which would contradict (14). This proves the theorem.

Incidentally, it follows that *every element x of an algebra has a finite group-index,* and *if x^r belongs to an entire group G, every higher power of x will also belong to G.*

13. *Definition:* The totality of the elements of an entire group G belonging to an algebra A, together with all those non-group-members, if any,

each of which has some power (with positive integral exponent) belonging to G, we shall call a *pseudo-group* $\bar{G}$, associated with G.

Obviously a pseudo-group G has the property that any positive integral power of an element of $\bar{G}$ is an element of $\bar{G}$; notice, however, that the product of two elements of $\bar{G}$ is not necessarily an element of $\bar{G}$, so that a pseudo-group is not necessarily a semi-group, much less a group.

But *every commutative pseudo-group is a semi-group.* For let r and s ($r \gtreqless s$) be the group-indices of any two elements x and y of the commutative pseudo-group $\bar{G}$ associated with the group G. Then $(xy)^r = x^r \cdot y^r$; and since x^r and y^r both belong to G, their product belongs to G, and xy belongs to $\bar{G}$.

Evidently a pseudo-group $\bar{G}$ is a group, if and only if, all its elements are of group-index 1, so that $\bar{G} = G$.

No two pseudo-groups can have an element x in common. For if they did, x^r, for a sufficiently high value of r, would belong to two distinct entire groups, which is impossible.

Thus the pseudo-groups of an algebra, like the entire groups, are mutually exclusive; but unlike the entire groups, they exhaust the algebra, as we see by reference to § 12. Hence *every element of an algebra belongs to one and only one pseudo-group.*

A nilpotent algebra clearly consists of just one pseudo-group, which is also a semi-group. If an algebra is not nilpotent, its nilpotent elements (if any) together with zero, constitute a pseudo-group, which is a semi-group, provided all its elements are properly nilpotent or zero (Dickson, § 32).

14. Examples.

(a) Let $A = (x, y)$, with the multiplication table

	x	y
x	x	y
y	0	0

Every element $u = x + \beta y$ is idempotent, and since $uAu = (u)$, G_u consists of the elements $\alpha(x + \beta y)$, where $\alpha \neq 0$. Besides the entire groups G_u, one for each value of β, and the group $G_0 = 0$, there is one pseudo-group $\bar{G}_0$ not a group, whose elements βy are nilpotent or zero.

(b) Let $A = (x, y, z)$, with the modulus $x + y$ and the multiplication table

	x	y	z
x	x	0	0
y	0	y	z
z	0	z	0

There are three idempotents: $x + y = u$, x, and y. Hence there are four entire groups: G_u, G_x, G_y, and G_0; and A is made up of four pseudo-groups $\bar{G}_u$, $\bar{G}_x$, $\bar{G}_y$, and $\bar{G}_0$. Let $v = \alpha x + \beta y + \gamma z$.

(1) If $\alpha\beta \neq 0$, v belongs to $G_u = \bar{G}_u$, the principal group.

(2) If $\beta = 0$, $\alpha \neq 0$, v belongs to $\bar{G}_x$, and if $\gamma = 0$, to G_x; if $\gamma \neq 0$, v is of group-index 2 and $v^2 = \alpha^2 x$ belongs to G_x.

(3) If $\alpha = 0$, $\beta \neq 0$, v belongs to $\bar{G}_y = G_y$.

(4) If $\alpha = \beta = 0$; v belongs to $\bar{G}_0$; if $\gamma \neq 0$, v is of group index 2 and $v^2 = 0$.

15. We shall now derive another criterion of group-membership, one that involves the minimum equation (Dickson, § 68) satisfied by an element x of an algebra A. In the first place it is known (Scorza, II, Arts. 117-125) that the minimum equation $\phi(\omega) = 0$ of x will have a constant term $\alpha \neq 0$, if and only if, A has a modulus m and x belongs to the principal group of A. In the corresponding equation $\phi(x) = 0$ the constant term α is to be replaced by αm.

Now consider any algebra A and any element x of A whose minimum equation $\phi(\omega) = 0$ lacks a constant term and is therefore of the form

$$\omega^s + \cdots + \gamma\omega^2 + \beta\omega = 0, \tag{20}$$

so that

$$x^s + \cdots + \gamma x^2 + \beta x = 0. \tag{21}$$

Suppose that x is a group-member, belonging to a group G_u, and that x^{-1} is its inverse, so that $xx^{-1} = u$. Then equation (21), multiplied through by x^{-1}, becomes

$$x^{s-1} + \cdots + \gamma x + \beta u = 0. \tag{22}$$

It follows that $\beta \neq 0$; for if it were zero, (22) shows that x would satisfy an equation of lower degree than its minimum equation (20).

Conversely, suppose that $\beta \neq 0$. In the augmented algebra A' (§ 9), with a modulus u_0, x will satisfy the same minimum equation (20) as in A. But in A' equation (21) can be written $x = x^2 y$, where

$$y = -(1/\beta)(x^{s-2} + \cdots + \gamma u_0).$$

Since $yA' \overset{=}{<} A'$, it follows that $xA' = x^2yA' \overset{=}{<} x^2A'$. But $xA' \overset{=}{>} x^2A'$, in every case; hence in this case $xA' = x^2A'$, from which we conclude (§ 8) that x is a group-member.

Combining these results, we have the

THEOREM: *In any algebra A an element x, whose minimum equation is*

$\phi(\omega)=0$, *will be a group-member, if and only if*, $\phi(\omega)$ *is not divisible by* ω^2; *the case in which* $\phi(\omega)$ *is not even divisible by* ω *will occur only when* A *has a modulus and* x *belongs to the principal group of* A.

Given a group-member x and its minimum equation (20), we can find the idempotent of the entire group to which it belongs, in view of (22), by the formula

$$u=-(1/\beta)\,(x^{s-1}+\cdot\cdot\cdot+\gamma x)\,; \tag{23}$$

and we can then obviously find the inverse of x by the formula

$$x^{-1}=-(1/\beta)\,(x^{s-2}+\cdot\cdot\cdot+\gamma u).$$

Moreover, the potential * algebra $B=(x, x^2, \cdot\cdot\cdot, x^{s-1})$, which is a commutative sub-algebra of A, and also of uAu, contains u, by (23), and u is its modulus. Hence if G_u is the entire group, in A, to which x belongs, and if H_u is the principal group of B, to which x also belongs, then H_u is an Abelian sub-group of G_u.

16. THEOREM: *In an algebra* A *a non-group-member* x, *whose minimum equation is* $\phi(\omega)=0$, *will be of group-index* $r(>1)$, *if and only if*, $\phi(\omega)$ *is divisible by* ω^r *but not by* ω^{r+1}.

Proof: Let the minimum equation of x be

$$\omega^s+\cdot\cdot\cdot+\beta\omega^{r+1}+\alpha\omega^r=0 \qquad (r>1,\ \alpha\neq 0). \tag{24}$$

Then in the augmented algebra A', with a modulus u_0, we have $x^r=x^{r+1}y$, where

$$y=-\,(1/\alpha)\,(x^{s-r-1}+\cdot\cdot\cdot+\beta u_0)\,;$$

and exactly as in § 15 we see that $x^rA'=x^{r+1}A'$, so that x^r is a group-member, by (14). But x^{r-1} is not a group-member. For if it were, *its* minimum equation $\psi(\omega)=0$, by the last theorem, would not be divisible by ω^2, so that x would satisfy an equation of the form

$$\omega^{t(r-1)}+\cdot\cdot\cdot+\beta_1\omega^{r-1}+\alpha_1=0 \qquad (\alpha_1,\ \beta_1 \text{ not both}\neq 0),$$

and the left member of this equation would have to be divisible (Scorza, II, Art. 125) by the left member of (24), which is impossible.

Hence r is the group-index of x. The converse is obvious and the theorem is proved.

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* See Scorza, II, Art. 117.

The potential sub-algebra

$$B = (x, \cdots, x^r, \cdots, x^{s-1})$$

is evidently of index r, and

$$B^r = (x^r, \cdots, x^{s-1}).$$

It is easy to prove that B^r has a modulus u, which is the idempotent of the entire group to which x^r belongs. By § 6, the group-members of B are all contained in B^r.

17. Let us now briefly consider an algebra A in a field F that is non-modular and therefore infinite. A member x of a group G_u belonging to A will usually be of infinite period (order), but may be of finite period m. In that case $x^m = u$ and $x^{m+1} - x = 0$, so that x satisfies the equation $\omega(\omega^m - 1) = 0$. Hence if $\phi(\omega) = 0$ is the minimum equation of x, $\phi(\omega)$ must divide $\omega(\omega^m - 1)$.

When we speak of the roots of an equation $\psi(\omega) = 0$, we shall understand that for the moment the field F has been extended sufficiently to make $\psi(\omega)$ completely reducible (resolvable into linear factors).

Since in a non-modular field the roots of the equation $\omega(\omega^m - 1) = 0$ are all distinct (simple), the roots of $\phi(\omega) = 0$ are also distinct and are roots of unity or zero.

Conversely, if x has a minimum equation $\phi(\omega) = 0$ whose roots are distinct and are roots of unity or zero, then by § 15 x belongs to a group, say G_u, and satisfies an equation of the form $\omega(\omega^m - 1) = 0$, so that $x^{m+1} = x$. This equation, multiplied by by x^{-1}, gives $x^m = u$. Hence we have the

THEOREM: *In order that an element of an algebra in non-modular field may be a group-member of finite period, it is necessary and sufficient that its minimum equation have distinct roots, which are roots of unity or zero.*

18. If m and r are the least positive integers for which the equation $x^{m+r} = x^r$ holds, then x is clearly of group-index r and its higher powers x^r, $x^{r+1}, \cdots, x^{m+r-1}$ form a cyclic group of order m; and conversely. We state without proof the

THEOREM: *An element of an algebra in a non-modular field will be a non-group-member, whose higher powers form a cyclic group, if and only if, its minimum equation is of the form* $\omega^r\psi(\omega) = 0$, *where* $r > 1$ *and the roots of* $\psi(\omega) = 0$ *are distinct roots of unity.*

19. Returning to the general case of any field, we shall now examine

more in detail the groups contained in an algebra, and show how their structure is related to the structure of the algebra.

If $x \neq 0$ and $y \neq 0$, while $xy = yx = 0$, we shall say that x and y are *mutually nilfacient.*

Suppose that the algebra A is the *direct sum* of two sub-algebras A_1 and A_2, $A = A_1 \oplus A_2$. The equations $A_1A_2 = A_2A_1 = 0$ express the fact that any two non-zero elements of A, one from each component, are mutually nilfacient. Let G_1 and G_2 be any two groups contained in A_1 and A_2, respectively, and let u_1 and u_2 be their respective identities.

By the equation $G = G_1 + G_2$ we shall mean that G consists of all the elements obtained by adding an element of G_1 to an element of G_2. G is evidently a group and its identity is $u_1 + u_2$. Conversely, any group whatever of A can be expressed uniquely in the form $G = G_1 + G_2$.

Now assume that neither G_1 nor G_2 is zero, so that G belongs to neither A_2 nor A_1. Notice that G_1 and G_2 are not sub-groups of G; they have no elements in common with G or with each other. Put

$$H_1 = G_1 + u_2,$$
$$H_2 = u_1 + G_2.$$

Then H_1 and H_2 are sub-groups of G having no common elements except the identity $u_1 + u_2$. Obviously they are simply isomorphic with G_1 and G_2, respectively. Moreover every element of H_1 is commutative with every element of H_2, and $H_1H_2 = G_1 + G_2 = G$. Hence G is the *direct product* of H_1 and H_2. By an easy generalization we arrive at the

THEOREM: *If an algebra A is the direct sum of two or more sub-algebras, any group belonging to A, but not to one of its components, is the direct product of two or more sub-groups.*

Hence the groups considered in the first theorem of § 7 are direct products.

20. If B is an invariant proper sub-algebra of A, consider the difference algebra $A' = A - B$, whose elements are the classes $[x]$ of elements of A, taken modulo B. If $u^2 = u$ and $xy = u$, then $[u]^2 = [u]$ and $[x][y] = [u]$. Hence if x runs through the elements of a group G belonging to A, $[x]$ will run through the elements of a group G' belonging to A'. In particular, if x lies in B, G lies wholly in B, by § 6, and $G' = [0]$.

On the other hand, if G' is a given group belonging to A', it seems difficult to determine whether there exists a group G belonging to A and having one or more elements in each class $[x]$ of G'. It is clear, however, that the set of *all* the elements of A belonging to the various classes of G' is a semi-group.

21. In the special case where B is nilpotent, we can get more precise results.

THEOREM: *If the algebra A has a modulus u and a nilpotent invariant sub-algebra E, and if G' is the principal group of the difference algebra $A' = A - E$, then the totality of the elements of A belonging to the various classes $[x]$ of G' is a group G, namely the principal group of A.*

Proof: It is clear that $[u]$ is the modulus of A'. Let $[x]$ be any element of G' and $[y]$ its inverse, so that $[x][y] = [u]$. Let x and y be *any* elements of A belonging to the classes $[x]$ and $[y]$ respectively. Then since $xy \equiv u \pmod{E}$, we have

$$xy = u - e, \tag{25}$$

where e is an element of E. E being nilpotent, $e^r = 0$, for a sufficiently high value of r. Since u and e are commutative,

$$(u - e)(u^{r-1} + eu^{r-2} + \cdot \cdot \cdot + e^{r-1}) = u^r - e^r = u;$$

that is, $(u - e)z = u$, where

$$z = u + e + e^2 + \cdot \cdot \cdot + e^{r-1},$$

and by (25), $xyz = u$. Hence x belongs to the principal group G of A, and its inverse is yz. The converse follows from § 20, and the theorem is proved.

COROLLARY: *The elements $u + e$ of the class $[u]$ constitute an invariant sub-group $u + E$ of G.*

For the equation $x^{-1}ux = u$ implies $[x^{-1}][u][x] = [u]$, whence

$$x^{-1}(u + e)x = u + e'.$$

Evidently G' is the quotient-group of G with respect to $u + E$. Hence if H' is any sub-group of G', the elements of A belonging to the classes of H' form a sub-group H of G containing $u + E$, and $H' = H/(u + E)$.

22. The following theorem will not be needed later and its proof will therefore be omitted.

THEOREM: *Let A be a non-nilpotent algebra having a nilpotent invariant sub-algebra E; let G' be any entire group $\neq [0]$ of the difference algebra*

$A - E$; and $[u]$ its idempotent. Then there exists in A at least one entire group G whose idempotent u lies in the class $[u]$, and if $B = uAu$, the elements of G are precisely those elements of B that belong to the various classes $[x]$ of G'.

If $B \wedge E = F$, so that F is a nilpotent invariant sub-algebra of B, and H' is the principal group of $B - F$, then by § 21 G can also be described as the set of elements of B that belong to the classes of H'.

23. We are now prepared to study the structure of any entire group G_u whatever in any algebra A whatever. The trivial case where $G_u = 0$, and therefore also the case where A is nilpotent, will be excluded.

If A is not semi-simple, we denote its exceptional (maximal nilpotent invariant) sub-algebra by E, and put $A' = A - E$; then A' is the direct sum of simple algebras $A'_1, \cdots, A'_t$, or if $t = 1$, is itself simple. The case in which A is semi-simple can be regarded as included by putting $E = 0$, $A' = A$ and $A'_i = A_i$ $(i = 1, \cdots, t)$.

The idempotent u and the group G_u are said to have the signature *

$$(p_1, \cdots, p_t),$$

provided $u = u_1 + \cdots + u_t$ $(u_i u_j = 0,\ i \neq j)$ and therefore

$$[u] = [u_1] + \cdots + [u_t],$$

where $[u_1]$ is an idempotent or zero of A_i' $(i = 1, \cdots, t)$, and is either expressible as the sum of p_i primitive idempotents, mutually nilfacient, or is itself primitive $(p_i = 1)$ or finally is zero $(p_i = 0)$. At least one of the integers p_i must be positive; if u is a principal idempotent, they are all positive (Dickson, p. 88), and the signature of u is said to be the *signature of the algebra A*.

Suppose that the number of positive integers p_i is r $(0 < r \leqq t)$; if $r < t$, we can rearrange the irreducible components A_i' of A' and write the signature of G_u in the form

$$(p_1, \cdots, p_r, 0, \cdots, 0), \tag{26}$$

so that $u = u_1 + \cdots + u_r$.

We know by § 3 that G_u is the principal group of the sub-algebra $A_u = uAu$, which has a modulus u. Moreover, A_u has the signature

$$(p_1, \cdots, p_r). \tag{27}$$

Hence our problem of finding the structure of an entire group of signature

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* Scorza, II, Arts. 249-252.

(26) in an algebra A reduces to a study of the principal group G_u of a sub-algebra A_u of signature (27). If $r > 1$, we shall show that G_u is expressible, in a certain sense, in terms of groups for which $r = 1$. The latter groups will be studied in §§ 31-34.

24. Continuing the notation of the last section, we now proceed to decompose the algebra A_u, employing Dickson's method (§ 57), except that his u is a principal idempotent of A, while ours is any idempotent for which $r > 1$. We put

$$A_{ij} = u_i A u_j, \qquad E_{ij} = u_i E u_j \qquad (i, j = 1, \cdots, r). \tag{28}$$

Let E_u be zero or the exceptional sub-algebra of A_u; then

$$\begin{gathered} A_u = \Sigma A_{ij}, \qquad E_u = \Sigma E_{ij}, \\ A_{ij} = E_{ij} \quad (i \neq j), \end{gathered} \tag{29}$$

and E_{ii} is zero or the exceptional sub-algebra of A_{ii}. Let

$$A'_u = A_u - E_u, \quad A'_{ii} = A_{ii} - E_{ii} \qquad (i = 1, \cdots, r); \tag{30}$$

then $A'_u = \Sigma A'_{ii}$, which is a direct sum. Let the principal groups of the various algebras

$$A_u, \quad A_{ii}, \quad A'_u, \quad A'_{ii}$$

be denoted by

$$G_u, \quad G_{ii}, \quad G'_u, \quad G'_{ii},$$

respectively. Then by § 19, $G'_u = \Sigma G'_{ii}$.

Now let x be any element of A_u; then $x = \Sigma x_{ij}$, where x_{ij} is in A_{ij}. Let $[x]$ be the corresponding element of A'_u, that is, a class of elements of A_u, taken modulo E_u. Then $[x] = \Sigma [x_{ii}]$, since $[x_{ij}] = [0]$, is $i \neq j$; and $[x_{ii}]$ is an element of A'_{ii}, that is, a class of elements of A_{ii}, taken modulo E_{ii}.

By the theorem of § 21, x will belong to the group G_u, if and only if, $[x]$ belongs to the group G'_u, which implies that $[x_{ii}]$ belongs to the group G'_{ii} $(i = 1, \cdots, r)$, and conversely. But by applying the same theorem (§ 21) to the algebra A_{ii}, we see that $[x_{ii}]$ will belong to G'_{ii}, if and only if, x_{ii} belongs to the principal group G_{ii} of A_{ii}. Notice that no condition is imposed on x_{ij} $(i \neq j)$, which is therefore free to roam anywhere in the algebra A_{ij} (if $A_{ij} \neq 0$); whereas x_{ii} must have an inverse x_{ii}^{-1} in the algebra A_{ii}, such that $x_{ii} x_{ii}^{-1} = u_i$.

We have now proved the

THEOREM: *Let G_u be an entire group of signature $(p_1, \cdots, p_r, 0, \cdots, 0)$ in an algebra A and let its idempotent be $u = u_1 + \cdots + u_r$ $(r > 1$; $u_i u_j = 0$, $i \neq j)$, where u_i is of signature $(0, \cdots, 0, p_i, 0, \cdots, 0)$; also let $A_{ij} = u_i A u_j$, so that $uAu = \Sigma A_{ij}$; then*

$$(31) \qquad G_u = \Sigma G_{ii} + \Sigma' A_{ij},$$

where G_{ii} is the principal group of the sub-algebra A_{ii} $(i = 1, \cdots, r)$, and where the symbol Σ' means that $i \neq j$.

Since the linear sets A_{ij} are supplementary in their sum, a term A_{ij} of (31) has no element except zero in common with the sum of the remaining terms, and a term G_{ii} has no element whatever in common with the sum of the remaining terms.

We have therefore expressed any group G_u in terms of the groups G_{ii}, each of which is the principal group of an algebra A_{ii} of signature (p_i); either A_{ii} is a simple algebra (if $E_{ii} = 0$) or its difference algebra $A_{ii} - E_{ii}$ is simple.

25. Let us now consider some of the more striking sub-groups of the entire group G_u. First of all it is clear that $\bar{G}_u = \Sigma G_{ii}$ is a sub-group, and since the sub-algebra ΣA_{ii}, of which $\bar{G}_u$ is the principal group, is the direct sum of $A_{11}, \cdots, A_{rr}$, it follows (§ 19) that G_u is the direct product of the sub-groups

$$(32) \qquad \begin{aligned} G_1 &= G_{11} + u_2 + \cdots + u_r, \\ G_2 &= u_1 + G_{22} + \cdots + u_r, \\ &\cdots\cdots\cdots\cdots \\ G_r &= u_1 + u_2 + \cdots + G_{rr}. \end{aligned}$$

In the special case where $\Sigma' A_{ij} (= \Sigma' E_{ij}) = 0$ and therefore in the still more special case where A_u is semi-simple, the sub-group $\bar{G}_u$ is the entire group G_u.

By (29_2) and the corollary of § 21 we see that

$$u + E_u = \Sigma (u_i + E_{ii}) + \Sigma' E_{ij}$$

is an invariant sub-group of G_u, and that the quotient-group $G_u/(u + E_u)$ is $G'_u = \Sigma G'_{ii}$. By the same corollary, $u_i + E_{ii}$ is an invariant sub-group of G_{ii}. The greatest common sub-group of $\bar{G}_u$ and $u + E_u$ is $J = \Sigma (u_i + E_{ii})$, which is invariant in $\bar{G}_u$, but not usually in $u + E_u$ or in G_u. J, like H, is obviously a direct product of r sub-groups $J_1, \cdots, J_r$, say, unless some of the linear sets E_{ii} are zero. If all of them are zero, so that ΣA_{ii} is semi-simple, then $J = u$.

It is easy to see that G_u is generated by its two sub-groups $\tilde{G}_u$ and $u+E_u$ and that every element x of G_u is expressible, in more than one way unless $J=u$, as a product yz, where y and z are elements of $\tilde{G}_u$ and $u+E_u$, respectively, and also as a product $z'y'$. Hence $\tilde{G}_u$ and $u+E_u$ are permutable * groups.

Other notable sub-groups of G_u are of the form

$$\sum_1^r G_{ii} + \Sigma' A_{kl},$$

where k and l do not run through *all* the values from 1 to r, but only a part of them, e. g., from 1 to s $(s<r)$. Still others are of the form

$$(\Sigma G_{ii} + \Sigma' A_{ij}) + (\Sigma G_{kk} + \Sigma' A_{kl}),$$

where i, j run through a part of the values from 1 to r and k, l the remaining values, e. g., $i, j = 1, \cdots, s$, and $k, l = s+1, \cdots, r$. The latter sub-group is evidently the direct product of the groups

$$\Sigma G_{ii} + \Sigma' A_{ij} + u_{s+1} + \cdots + u_r,$$
$$u_1 + \cdots + u_s + \Sigma G_{kk} + \Sigma' A_{kl}.$$

Similarly there are sub-groups expressible as the sum of three or more terms, each of the form $\Sigma G_{ii} + \Sigma' A_{ij}$.

26. In any field for which the principal theorem on algebras (Dickson, § 78) has been proved, that is, in the present state of our knowledge, in any non-modular field, the structure of the groups contained in an algebra can be determined somewhat more completely than we have been able to do so far.

Let A be an algebra in a non-modular field, and G_u an entire group belonging to A. As in §§ 23, 24, let E_u be the exceptional sub-algebra (if it exists) of $A_u = uAu$, and let $A'_u = A_u - E_u$. Then by the principal theorem on algebras there exists in A_u a semi-simple sub-algebra B_u equivalent to A'_u, and having one and only one element in each class $[x]$ of elements of A_u, modulo E_u, so that $A_u = B_u + E_u$.

Moreover, B_u has the same modulus u as A_u; hence the principal group H_u of B_u is a sub-group of the principal group G_u of A_u. But H_u has one and only one element in each class $[x]$ of elements of G_u, modulo E_u. Hence by the theorem of § 21,

$$G_u = H_u + E_u. \tag{33}$$

---

* Burnside, *Theory of Groups*, § 33. The definition is there given only for finite groups, but is clearly applicable also to infinite groups.

By § 25, G_u has the invariant sub-group $u + E_u$.

27. *Definition:* If a group G has two sub-groups G_1 and G_2, having only the identity in common, and if every element of G is expressible in one and only one way as a product $g_1 g_2$, where g_1 and g_2 are elements of G_1 and G_2, respectively, and also as a product $g'_2 g'_1$, then we shall call G the *product* of G_1 and G_2. The two sub-groups are therefore permutable. Every direct product is a product, but not conversely.

In a similar manner we define the product of more than two groups; each group will then have only the identity in common with the product of all the rest.

We shall now prove that G_u (§ 26) *is the product of its sub-groups H_u and $u + E_u$.*

For by (33) any element of G_u can be written $h + e$, where h and e are elements of H_u and E_u, respectively. Our goal will be reached if we can find unique elements e', e'' in E_u, such that

$$h(u + e') = h + e = (u + e'')h. \tag{34}$$

But (34_1) implies that $he' = e$, and therefore that $e' = h^{-1}e$, which is in E_u, since the latter is invariant in A_u. Conversely, $e' = h^{-1}e$ implies (34_1). Similarly we can find e'' uniquely in E_u.

28. For the case in which G_u is of signature (26), where $r > 1$, let us see what further light is shed by our new result on the analysis of G_u given in §§ 24, 25. In the first place every algebra A_{ii} $(i = 1, \cdots, r)$ will now have a simple sub-algebra B_{ii} equivalent to A'_{ii}, such that $A_{ii} = B_{ii} + E_{ii}$ and $B_u = \Sigma B_{ii}$, which is a direct sum. Hence if H_{ii} is the principal group of B_{ii}, we have

$$H_u = \Sigma H_{ii},$$

and by the theorem of § 19, H_u is the direct product of the r sub-groups

$$\begin{aligned} H_1 &= H_{11} + u_2 + \cdots + u_r, \\ &\cdots\cdots\cdots\cdots \\ H_r &= u_1 + u_2 + \cdots + H_{rr}. \end{aligned} \tag{35}$$

Combining this result with that of § 27, we have the

THEOREM: *Any group G_u of signature $(p_1, \cdots, p_r, 0, \cdots, 0)$, where $r > 1$, that belongs to an algebra A in a non-modular field, can be expressed* *as a product of* $r+1$ *sub-groups* $H_1, \cdots, H_r$, $u+E_u$, *each of which has only the identity* u *in common with the product of all the rest;* $u+E_u$ *is invariant in* G_u *and* H_i $(i=1, \cdots, r)$ *is simply isomorphic with the principal group of a simple algebra* B_{ii}; *the product of* $H_1, \cdots, H_r$ *is their direct product.*

29. The sub-group $\bar{G}_u = \Sigma G_{ii}$ (§ 25) can now be further decomposed. For since $A_{ii} = B_{ii} + E_{ii}$, $G_{ii} = H_{ii} + E_{ii}$, and this group, by § 27, is the product of H_{ii} and $u_i + E_{ii}$. Hence, comparing (35) and (32), we see that G_i is the product of H_i and J_i (§ 25), where

$$
\begin{aligned}
J_1 &= (u_1 + E_{11}) + u_2 + \cdots + u_r, \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
J_r &= u_1 + u_2 + \cdots + (u_r + E_{rr}).
\end{aligned}
$$

Finally $\bar{G}_u$ is the product of H_u and J, where H_u is the direct product of $H_1, \cdots, H_r$ and J is the direct product of $J_1, \cdots, J_r$. That is, $\bar{G}_u$ *is the product of the* $2r$ *sub-groups* H_i, J_i $(i=1, \cdots, r)$.

Another sub-group worth mentioning is $H_1 + E_u = H_{11} + u_2 + \cdots + u_r + \Sigma E_{ij}$, which is evidently the product of H_1 and $u + E_u$.

30. *Example.* Let $A = (u_1, u_2, x, y)$, with the modulus $u = u_1 + u_2$ and the multiplication table

	u_1	u_2	x	y
u_1	u_1	0	x	0
u_2	0	u_2	0	y
x	x	0	0	0
y	y	0	0	0

Its principal group G_u is of signature $(1, 1)$. Since $A_{11} = (u_1, x)$, $A_{12} = 0$, $A_{21} = (y)$, and $A_{22} = (u_2)$, the groups G_{11}, G_{22}, $\bar{G}_u$, and $G_u = G_{11} + A_{12} + A_{21} + G_{22}$ consist of the respective sets of elements $\alpha u_1 + \gamma x$ $(\alpha \neq 0)$, $\beta u_2 (\beta \neq 0)$, $(\alpha u_1 + \gamma x) + \beta u_2 (\alpha\beta \neq 0)$, and $\alpha u_1 + \beta u_2 + \gamma x + \delta y (\alpha\beta \neq 0)$.

A has the exceptional sub-algebra $E_u = (x, y)$ and the semi-simple sub-algebra $B = (u_1, u_2)$. Hence G_u is the product of the three sub-groups H_1, H_2 and $u + E_u$, whose respective sets of elements are $\alpha u_1 + u_2$ $(\alpha \neq 0)$, $u_1 + \beta u_2$ $(\beta \neq 0)$, and $u + \gamma x + \delta y$. By § 21, $u + (x)$ and $u + (y)$ are invariant sub-groups of G_u; indeed their product is $u + E_u$.

A has a variety of other entire groups besides G_u, but we shall not go into that.

31. Returning to the general case of an algebra A in an arbitrary field, let us now examine a group G_u of signature $(p_1, 0, \cdots, 0)$, which is the principal group of a sub-algebra uAu of signature (p_1). The groups G_{ii} (§ 24), in terms of which we found the more complicated groups to be expressible, are all of this type. In order to simplify the notation, we shall confine ourselves to the sub-algebra uAu, denoting it by A and its signature by (p).

It is known (Scorza, II, Art. 266; Dickson, § 56) that A is the direct product of a complete matric sub-algebra M of order p^2 and a sub-algebra B (of order r, say) having a primitive modulus; hence $A = B \times M$ is of order rp^2. A, B and M all have the same modulus u. If E is zero or the exceptional sub-algebra of B, $E \times M$ is zero or the exceptional sub-algebra of A. Since $B - E$ is a division algebra, the only divisors of zero in B (if any) are nilpotent and belong to E. All the other elements of B (except zero) belong to its principal group.

The modulus $u = u_1 + \cdots + u_p$ can be expressed, in many ways, as a sum of p primitive idempotents $u_1, \cdots, u_p$, which are mutually nilfacient. Let the basic units of M be c_{ij} $(i, j = 1, \cdots, p)$, where $c_{ij}c_{jk} = c_{ik}$, $c_{ij}c_{lk} = 0$ $(j \neq l)$, and $c_{ii} = u_i$ $(i = 1, \cdots, p)$. Every element x of A (Scorza, II, p. 245) can then be written

$$x = \Sigma b_{ij}c_{ij},$$

where b_{ij} is an element (not necessarily a basic unit) of B. If we regard c_{ij} as a p-rowed matrix having all its elements zero except the one in the ith row and jth column, which is $= 1$, then

$$x = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pp} \end{pmatrix} = (b_{ij}), \text{ and } u = \Sigma u c_{ii} = (\delta_{ij}u),$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

This new kind of matrix (Dickson, §§ 97, 98) involves a twofold non-commutativity (if B is a non-commutative algebra), first because if $x' = (b'_{ij})$, then $xx' \neq x'x$, and second because if $xx' = x'' = (b''_{ij})$, so that

$$b''_{ik} = \sum_j b_{ij}b'_{jk}, \quad \text{then} \quad b_{ij}b'_{jk} \neq b'_{jk}b_{ij}.$$

32. Let G, H and K be the principal groups* of A, B and M, respectively; H and K are obviously sub-groups of G. In order to obtain a precise determination of the group G, it became necessary to invent a new kind of determinant that will be explained in a later paper.

At present we shall confine ourselves to the special case where B is a commutative algebra; the ordinary kind of determinant will answer our purpose, although its elements are not, as usual, scalars, but elements of B.

THEOREM: *If B is a commutative algebra, an element (b_{ij}) of the algebra $A = B \times M$ will belong to the principal group G of A, if and only if, the determinant $| b_{ij} |$ is neither nilpotent nor zero; that is, if $| b_{ij} |$ belongs to the principal group H of B.*

For, if (b_{ij}) belongs to G, it must have an inverse (b'_{ij}), such that $(b_{ij})(b'_{ij}) = (\delta_{ij}u)$, whence

$$| b_{ij} | \cdot | b'_{ij} | = u^p = u.$$

Therefore $| b_{ij} |$ must belong to H. The converse is evidently true and the theorem is proved.

If B is a division algebra $(E = 0)$, the criterion becomes simply

$$| b_{ij} | \neq 0.$$

33. Even when B is non-commutative, we can easily determine some of the more important sub-groups of G and express their elements as matrices. Take K, for instance. Since the elements of M are of the form $k = (\alpha_{ij}u)$, where α_{ij} is a scalar, the elements k of its principal group K are characterized by the condition $| \alpha_{ij} | \neq 0$.

Then take H. Its elements are evidently the matrices $h = (\delta_{ij}b)$, where b is any element of B that is neither nilpotent nor zero.

Since $hk = kh = (\alpha_{ij}b)$, the sub-groups H and K are permutable and generate a sub-group $\bar{G}$ consisting of the matrices $\bar{g} = (\alpha_{ij}b)$, where $| \alpha_{ij} | \neq 0$ and b is neither nilpotent nor zero. H and K are invariant in $\bar{G}$, but not necessarily in G. The greatest common sub-group J of H and K consists of the elements $(\delta_{ij}\alpha u)$, where $\alpha \neq 0$. J is invariant in G (and therefore in $\bar{G}$), because all its elements are invariant. The quotient-group

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* Of course A has a multitude of other entire groups besides G, any one of which, G_1, is of signature (p_1), where $p_1 < p$. G_1 is itself the principal group of a sub-algebra of order rp_1^2 and can be represented either as a group of p_1-rowed matrices or as a group of "singular" p-rowed matrices, of which the constituents are in either case elements of B. See *American Journal of Mathematics*, Vol. 31 (1909), §§ 10, 11.

$\bar{G}/J$ consists of the classes $[\bar{g}]$, each class $[\bar{g}]$ consisting of all the scalar multiples $\alpha\bar{g}$ of $\bar{g}$ where $\alpha \neq 0$.

Since the analogous quotient-groups H/J and K/J have no elements in common except the identity $[u]$, it follows that $\bar{G}/J$ is the direct product of H/J and K/J. $\bar{G}$ may therefore be called the *quasi-direct product* of H and K. This gives us the

THEOREM: *In an algebra A having a modulus and the signature (p), which is the direct product of the sub-algebras B and M, the principal group of A has a sub-group $\bar{G}$ which is the quasi-direct product of the principal groups of B and M.*

The set of all the possible products of two factors, one from B and the other from M, or in other words the set of all the materices $(\alpha_{ij}b)$, where the α's and b are unrestricted, is a curious affair; for although it is clearly a semi-group, it is not a group, nor even a linear set, much less a sub-algebra.

Another sub-group of G is our old friend $u + EM$ (§ 21, corollary), which is invariant in G and consists of the matrices $(\delta_{ij}u + e_{ij})$, where the elements e_{ij} belong to E. Similarly $u + E$, consisting of the matrices $(\delta_{ij}(u + e))$, is an invariant sub-group of H.

34. In a non-modular field we can, as in §§ 26-29, analyse our group G of signature $(p, 0, \cdots, 0)$ more fully. Our algebra $A = B \times M$ will then contain a simple sub-algebra $A' = B' \times M$, equivalent to $A - EM$, such that $A = A' + EM$ and $B = B' + E$. Let G' and H' be the principal groups of A' and B', respectively; then $G = G' + EM$ and $H = H' + E$. The elements of H' are the matrices $(\delta_{ij}b')$, where b' is any element not zero of the division algebra B'; and the elements of G' are matrices of the form (b'_{ij}), where the elements b'_{ij} belong to B' and are such that if B' is a commutative algebra, $|b'_{ij}| \neq 0$. By § 27, *G is the product of its sub-groups G' and $u + EM$,* and H is the product of H' and $u + E$.

35. The groups studied in this paper are somewhat analogous to certain known groups arising in the theory of numbers. Consider the m classes $[x]$ of integers congruent to x with respect to a given composite modulus m. Since the set A of these classes is closed under addition and multiplication, it has many of the properties of a linear associative (and commutative) algebra. Hence it is not surprising that some of the classes form groups and others are non-group-members.

For example, let $m = p^2q$, where p and q are distinct primes. There are four entire groups G_m, G_{p^2}, G_q, G_0, and A is made up of four pseudo-groups $\bar{G}_m$, $\bar{G}_{p^2}$, $\bar{G}_q$, and $\bar{G}_0$.

(1) If x is prime to m, $[x]$ belongs to the principal group G_m, which is of order $p(p-1)(q-1)$.

(2) If $x \equiv 0 \pmod{q}$ and $\not\equiv 0 \pmod{p}$, $[x]$ belongs to G_{p^2}, of order $p(p-1)$.

(3) If $x \equiv 0 \pmod{p}$ and $\not\equiv 0 \pmod{q}$, $[x]$ belongs to $\bar{G}_q$, of order $p(q-1)$. If in addition $x \equiv 0 \pmod{p^2}$, $[x]$ belongs to G_q, of order $q-1$; while if $x \not\equiv 0 \pmod{p^2}$, $[x]$ is a non-group-member and $[x]^2$ belongs to G_q.

(4) If $x \equiv 0 \pmod{pq}$, $[x]$ belongs to $\bar{G}_0$, of order p. If $x \equiv 0 \pmod{p^2q}$, $[x] = [0] = G_0$; while if $x \not\equiv 0 \pmod{p^2q}$, $[x]$ is nilpotent and $[x]^2 = [0]$.

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Linear Ordinary Self-Adjoint Differential Equations of the Second Order.

By Anna Pell Wheeler.

In 1914 Lichtenstein * made connection, without the intermediary of the theory of integral equations, between the theory of linear differential systems of the second order and the theory of linear equations in infinitely many unknowns. Although there are many points of contact between the method used by Lichtenstein and the one used in this paper, the methods are not identical. The method employed here can be applied with only slight modifications to other systems, for example, singular differential systems,† and partial differential systems, elliptic or hyperbolic. The method can be extended to yield more general results both for the orthogonal and polar cases of ordinary differential systems.

I. *Existence of Characteristic Numbers.*

1. *Orthogonal Case.* $\nu_m \neq 0$. The differential system to be considered is

$$\begin{aligned} &u''(x) + q(x)u(x) + \lambda k(x)u(x) = 0, \\ (1) \qquad &\alpha u(0) + \alpha' u'(0) = 0, \\ &\beta u(\pi) + \beta' u'(\pi) = 0, \end{aligned}$$

where the function $q(x)$ is continuous, the function $k(x)$ is continuous and positive in the interval $(0, \pi)$, and $\alpha, \alpha', \beta, \beta'$ are constants such that

$$|\alpha| + |\alpha'| \neq 0, \quad |\beta| + |\beta'| \neq 0.$$

The characteristic functions $\phi_m(x)$ corresponding to the characteristic numbers $\nu_m{}^2$ of the system

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* L. Lichtenstein, "Zur Analysis der unendlichvielen Variabeln. I. Entwicklungssätze der Theorie gewöhnlicher linearer Differentialgleichungen zweiter Ordnung," *Rendiconti del Circolo Matematico di Palermo*, Vol. 38 (1914), pp. 113-166. See also J. Tamarkine, "Problème du développement d'une fonction arbitraire en séries de Sturm-Liouville," *Comptes Rendus*, Vol. 156 (1913), pp. 1589-91.

† Marion Cameron Gray, "A Boundary Value Problem of Ordinary Self-Adjoint Differential Equations with Singularities." To appear in a later number of the *American Journal of Mathematics*.

$$
\begin{aligned}
&u''(x) + \nu_m^2 u(x) = 0,\\
(2)\qquad &\alpha u(0) + \alpha' u'(0) = 0,\\
&\beta u(\pi) + \beta' u'(\pi) = 0,
\end{aligned}
$$

form an orthogonal, normed and closed system of functions

$$
\begin{aligned}
&\phi_m(x) = c_1 \cos \nu_m x + c_2 \sin \nu_m x,\\
&\alpha c_1 + \alpha' \nu_m c_2 = 0,\\
&(\beta \cos \nu_m \pi - \beta' \nu_m \sin \nu_m \pi) c_1 + (\beta \sin \nu_m \pi + \beta' \nu_m \cos \nu_m \pi) = 0,
\end{aligned}
$$

where[*] $\nu_m = m + \gamma_m$, $\gamma_m \to 0$ if α' and β' are both $\neq 0$, $\nu_m = m + \frac{1}{2} + \gamma_m$, if one, but not both, of α' and $\beta' = 0$, and $\nu_m = 1 + m + \gamma_m$ if $\alpha' = \beta' = 0$. The orthogonal system affords a means of passing from a function space F of functions which have continuous derivatives of the second order and satisfy the boundary conditions of (1), to a space H_1 in the Hilbert space H of infinitely many variables, and also to pass back. The known results for linear equations in the space H thus yield results for equations (1) in the space F.

Multiply the differential equation of (1) by $\phi_m(x)$ and integrate between 0 and π

$$\int u''\phi_m + \int qu\phi_m + \lambda \int ku\phi_m = 0 \qquad (m = 1, 2, \cdots).$$

Integrating twice the first integral by parts, and expanding the other two integrals by means of the system $\{\phi_m(x)\}$ we obtain

$$(3)\qquad -\nu_m^2 \int u\phi_m + \sum_{n=1}^{\infty} q_{mn} \int u\phi_n + \lambda \sum_{n=1}^{\infty} k_{mn} \int u\phi_n = 0 \quad (m = 1, 2, \cdots)$$

where the matrices $(q_{mn} = \int \phi_m q \phi_n)$ and $(k_{mn} = \int \phi_m k \phi_n)$ are limited and symmetric.

In this section we assume that no ν_m is equal to zero. The series $\sum_{m,n} (q^2_{mn}/\nu_m^4)$ and $\sum_{m,n} (k_{mn}/\nu_m^4)$ are convergent, and the matrices (q_{mn}/ν_m^2) and (k_{mn}/ν_m^2) are completely continuous. Therefore if the differential system (1) has a continuous solution $u(x)$, the sequence $x_m = \int u\phi_m$ of finite norm satisfies the system of linear equations

$$(4)\qquad x_m - \sum_n (q_{mn}/\nu_m^2) x_n - \lambda \sum_n (k_{mn}/\nu_m^2) x_n = 0 \qquad (m = 1, 2, \cdots),$$

in which the matrices involved are completely continuous. The system (4) can be put into a form for which results are known.

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[*] M. Bôcher, "Boundary Problems in One Dimension," *International Congress of Mathematicians*, Cambridge, 1912, p. 178.

Consider first the case in which the system

$$(5) \qquad x_m - \sum_n (q_{mn}/\nu_m^2)x_n = 0 \qquad (m = 1, 2, \cdots)$$

has no solutions of finite norm. The matrix $(e_{mn} - (q_{mn}/\nu_m^2))$, where $e_{mn} = 0$ if $m \neq n$ and $= 1$ if $m = n$, has a limited reciprocal (r_{mn}), that is

$$(6) \qquad \sum_p r_{mp}\,[e_{pn} - (q_{pn}/\nu_p^2)] = e_{mn},$$

and the system (4) is equivalent to the system

$$(7) \qquad x_m - \lambda \sum_n [\sum_p (r_{mp}k_{pn}/\nu_p^2]x_n = 0 \qquad (m = 1, 2, \cdots).$$

The matrix (r_{mn}/ν_m^2), which is completely continuous, is symmetric,* for it is the reciprocal of the symmetric matrix $(e_{mn}\nu_m^2 - q_{mn})$. The function $k(x)$ by hypothesis is positive in the interval $(0, \pi)$. It is easy to show that the matrix $(k_{mn}/\nu_m\nu_n)$ is positive definite, and then it can be argued that the matrix (k_{mn}) itself is. Hence † there exist characteristic numbers λ_a, which are necessarily real, and corresponding solutions l_{am} of finite norm of

$$(8) \qquad l_{am} - \lambda_a \sum_n (\sum_p r_{mp}k_{pn}/\nu_p^2)\; l_{an} = 0 \qquad (m = 1, 2, \cdots).$$

The system of equations with the transposed matrix has the solutions $\sum_p k_{mp}l_{ap} = m_{am}$ corresponding to λ_a, for

$$\sum_p k_{mp}l_{ap} - \lambda_a \sum_{p,q} (k_{mp}r_{qp}/\nu_p^2)\;(\sum_n k_{qn}l_{an}) = 0 \qquad (m = 1, 2, \cdots)$$

is obtained by multiplying the system (8) by k_{np} and summing, and using the symmetry of the matrix (r_{mn}/ν_m^2). The solutions l_{am} and m_{am} form a biorthogonal system

$$(9) \qquad \sum_m l_{am}m_{\beta m} = e_{a\beta},$$

and further

$$(10) \qquad \sum_{p,m,n} (r_{mp}k_{pn}/\nu_p^2)x_n y_m = \sum_a \;(\sum_m l_{am}y_m \sum_n m_{an}x_n)/\lambda_a$$

for every $\{x_m\}$ of finite norm, and every $\{y_m = \sum_n k_{mn}x_n\}$, and since the matrix (k_{mn}) has a limited reciprocal the relation (10) holds for every $\{x_m\}$ and $\{y_n\}$ of finite norm.

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* If $\sum_p a_{mp}b_{pn} = e_{mn}$, then $\sum_p b_{pm}(\sum_q a_{pq}b_{qn}) = b_{mn}$, and if the matrix (a_{mn}) is symmetric and the order of summation may be interchanged, the matrix (b_{mn}) is symmetric.

† A. J. Pell, "Linear Equations with Unsymmetric Coefficients," *Transactions of the American Mathematical Society*, Vol. 20 (1919), pp. 23-39.

The characteristic numbers can have a limiting point only at infinity. The system (8) is equivalent to

$$(11) \qquad -\nu_m^2 l_{\alpha m} + \sum_n q_{mn} l_{\alpha n} + \lambda_\alpha \sum_n k_{mn} l_{\alpha n} = 0 \qquad (m = 1, 2, \cdots).$$

The matrix $(m_{\alpha m} = \Sigma_p k_{mp} l_{\alpha p})$ is limited,* and hence the matrix $(l_{\alpha m})$ is also, since the matrix (k_{mn}) has a limited reciprocal. The equations (11) show that the sequence $\{\lambda_\alpha \Sigma_n k_{mn} l_{\alpha n}\}$ is of finite norm for every m and that $\{\nu_m^2 l_{\alpha m}\}$ is of finite norm for every α.

The function $u_\alpha(x)$ defined by the absolutely uniformly convergent series

$$u_\alpha(x) = \sum_m l_{\alpha m}\phi_m(x) = \sum_m \nu_m^2 l_{\alpha m}\,(\phi_m(x)/\nu_m^2),$$

has a continuous derivative, given by the absolutely uniformly convergent series

$$\sum_m l_{\alpha m}\phi'(x) = \sum_m \nu_m^2 l_{\alpha m}\,(\phi'_m(x)/\nu_m^2).$$

The functions $u_\alpha(x)$ and $u_\alpha'(x)$ satisfy the boundary conditions of (1). The function $u_\alpha(x)$ has a continuous second derivative, for, let $f = (q + \lambda_\alpha k)u_\alpha$, then

$$\int f\phi_m = \int\left(\int_0^x \int_0^{x_1} f(x_2)\,dx_2dx_1 + c_1x + c_2\right)\phi_m''(x)\,dx$$

$$= -\nu_m^2 \int\left(\int_0^x \int_0^{x_1} f(x_2)\,dx_2dx_1 + c_1x + c_2\right)\phi_m(x)\,dx = \nu_m^2 \int u_\alpha\phi_m,$$

and

$$-u_\alpha(x) = \int_0^x \int_0^{x_1} f(x_2)\,dx_2dx_1 + c_1x + c_2.$$

Moreover, the function $u_\alpha(x)$ satisfies the system (1) for the characteristic number λ_α.

Consider next the case in which there exists a sequence $\{z_m\}$of finite norm which satisfies (5). Since the matrix in (5) is completely continuous there can be only a finite number of such solutions. Let us assume there is only one; the case in which there is more than one can be carried through in a similar manner. There exists a limited matrix (r_{mn}) such that

$$\sum_p r_{mp}\,[e_{pn} - q_{pn}/\nu_p^2)] = e_{mn} - z_m z_n,$$

where $\Sigma_n z_n^2 = 1$. As before, the matrix (r_{mn}/ν_n^2) is symmetric closed and completely continuous. The system to be considered now is

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* A. J. Pell, *l. c.*, pp. 30-31.

$$x_m - z_m \sum_n z_n x_n + \lambda \sum_n \sum_p (r_{mp} k_{pn}/\nu_p^2) x_n = 0 \qquad (m = 1, 2, \cdots), \tag{12}$$

$$\sum_{m,n} z_m k_{mn} x_n = 0.$$

There exist * characteristic numbers λ_a and corresponding characteristic solutions l_{am} for this system. To the value $\lambda_a = 0$, there corresponds the solution $z_m = l_{am}$. The remainder follows as in the preceding case, and the existence of characteristic numbers for the differential system (1), under conditions specified, has been established by means of the properties of the solutions of the system (11). The system of functions $\{u_a(x)\}$ has the orthogonal property, or the systems $\{u_a(x)\}$ and $\{k(x) u_a(x)\}$ have the biorthogonal property

$$\int k \, u_a u_\beta = e_{a\beta}. \tag{13}$$

2. *Polar Case.* $\nu_m \neq 0$. Let the function $k(x)$ change sign in the interval $(0, \pi)$. The matrix (k_{mn}) is then not positive definite, and in order to conclude the existence of real characteristic numbers for the equations (4) or (7) we assume that the matrix (r_{mn}/ν_n^2) satisfying (6) exists, that it is positive definite, and for the present that there exists no sequence $\{f_n\}$ of finite norm such that $\sum (r_{mn}/\nu_n^2) f_n = 0$. A sufficient condition for this is that the matrix $(e_{mn}\nu_n^2 - q_{mn})$ be positive definite for any sequence $\{x_m\}$ such that $\{\nu_m x_m\}$ is of finite norm. This condition is certainly fulfilled when $q(x) < 0$, and also when the matrix $(e_{mn} - q_{mn}/\nu_m \nu_n)$ is positive definite. The latter is true if the positive † characteristic numbers of $(q_{mn}/\nu_m \nu_n)$ are < 1.

The matrix (r_{mn}/ν_n^2) is limited, symmetric and positive definite, hence there exist real characteristic numbers λ_a and corresponding solutions m_{am} of the system

$$m_{am} - \lambda_a \sum_{n,p} (k_{mp} r_{np}/\nu_p^2) m_{an} = 0 \qquad (m = 1, 2, \cdots), \tag{14}$$

and solutions l_{am} of finite norm of the equations (8), where

$$l_{am} = \sum_n (r_{mn}/\nu_n^2) m_{an} = \sum_n (r_{nm}/\nu_m^2) m_{an}, \tag{15}$$

$$m_{am} = \sum_n (\nu_n^2 e_{mn} - q_{mn}) l_{am} = \lambda_a \sum_n k_{mn} l_{an}. \tag{15'}$$

The two systems $\{l_{am}\}$ and $\{m_{am}\}$ form a biorthogonal system

$$\sum_n l_{am} m_{\beta m} = e_{a\beta},$$

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* A. J. Pell, "Linear Equations with Two Parameters," *Transactions of the American Mathematical Society*, Vol. 23 (1922), pp. 201-203.

† L. Lichtenstein, *l. c.*, p. 129.

and the expansion (10) holds for every $\{y_m\}$ of finite norm and every $\{x_m\}$ such that $\{\Sigma_m m_{am} x_m\}$ is of finite norm, in particular, if $x_m = \Sigma_n (r_{mn}/\nu_n^2) z_n$. On account of (15) the matrix (l_{am}) is limited. The characteristic numbers can have no finite limiting point. The sequence $\{\nu_m^2 l_{am}\}$ is of finite norm for every α, and the sequence $\{\lambda_a m_{am}\}$ is of finite norm for every m. As before we obtain the existence of characteristic numbers λ_a for the differential system (1), and corresponding characteristic functions $u_a(x)$ where

$$l_{am} = \int u_a \phi_m,$$

and

$$m_{am} = \int (u_a'' - q u_a) \phi_m = \lambda_a \int k u_a \phi_m.$$

The property corresponding to the orthogonal or biorthogonal property is

$$\lambda_a \int u_a k u_\beta = = e_{a\beta},$$

which is equivalent to

$$\int \bar{u}_a k \bar{u}_\beta = (|\lambda_a|/\lambda_a) e_{a\beta},$$

where $\bar{u}_a(x) = |\lambda_a|^{1/2} u_a(x)$.

Consider next the case in which there exists a sequence $\{z_m\}$ which satisfies (5). Again we obtain the system (12), in which the matrix (r_{mn}/ν_n^2) may be assumed to be positive definite, and $z_m = \Sigma_n (r_{mn}/\nu_n^2) z_n$. A symmetrizing matrix is $(r^*_{mn}) = (\Sigma\, k_{mp} (r_{pq}/\nu^2_q) k_{qn})$, which is positive definite if $k(x)$ vanishes only at the points of a set of measure zero, and is orthogonal only to sequences $\{f_m\}$ for which $\Sigma_n k_{mn} f_n = 0$. Hence real characteristic numbers λ_a exist. The solutions l_{am} and

$$m_{am} = \sum_n r^*_{mn} l_{an} = \sum_p k_{mp} (l_{ap} - z_p \Sigma_q z_q l_{aq})$$

form a biorthogonal system. The sequence $\{\Sigma_n k_{mn} z_n\}$ is orthogonal to $\{l_{am}\}$, and if $\Sigma\, z_m k_{mn} z_n = 0$, $\{\Sigma_p r^*_{mp} z_p\}$ is also orthogonal to $\{l_{am}\}$. In place of (10) we have, for any $\{x_m\}$ and $\{y_n\}$ of finite norm, if $\Sigma\, z_m k_{mn} z_n \neq 0$,

$$(10') \quad \sum_{m,p,q,n} x_m r^*_{mp} (r_{pq}/\nu_q^2) k_{qn} y_n - \frac{\sum\limits_{m,n} x_m r^*_{mn} z_n \sum\limits_{m,n} z_m r^*_{mn} y_n}{\sum\limits_{m,n} z_m k_{mn} z_n}$$

$$= \sum_a \frac{\sum\limits_m m_{am} x_m \sum\limits_n m_{an} y_n}{\lambda_a},$$

and if $\Sigma\, z_m k_{mn} z_n = 0$,

$$(10'') \quad \sum_{m,p,q,n} x_m r^*_{mp} (r_{pq}/\nu_q^2) k_{qn} y_n -$$

$$\frac{\sum_{m,n} x_m r^*_{m,n} z_n \sum z_m r^*_{mn} Y_n + \sum z_m r^*_{mn} y_n \sum x_m r^*_{mn} Z_n}{\sum_{m,n} z_m r^*_{mn} z_n} = \sum_a \frac{\sum_m m_{am} x_m \sum_n m_{an} y_n}{\lambda_a},$$

where $P_n = \sum (r_{ns}/\nu_s^2) k_{sn} p_n$, and $m_{an} = \sum_n r^*_{mn} z_n$ corresponds to

$$\lambda_a = \frac{\sum_{m,n} z_m r^*_{mn} z_n}{\sum_{m,n} z_m r^*_{mn} Z_n}.$$

The existence of solutions of system (1) follows as before.

3. *Case in which* $\nu_1 = 0$. If $\nu_1 = 0$, the system of equations (3) takes the form

$$-\sum_{n=1}^{\infty} q_{1n} x_n = + \lambda \sum_{n=1}^{\infty} k_{1n} x_n,$$

$$\nu_m^2 x_m - \sum_{n=1}^{\infty} q_{mn} x_n = + \lambda \sum_{n=1}^{\infty} k_{mn} x_n \qquad (m = 2, 3, \cdots).$$

If $q(x) \not\equiv 0$ the matrix on the left has a completely continuous reciprocal (r_{mn}) given by

$$r_{mn} = r^*_{mn} + (1/a) \sum_{p=2}^{\infty} r^*_{mp} q_{p1} \sum_{p=2}^{\infty} r^*_{np} q_{p1} \qquad (m, n = 2, 3, \cdots),$$

$$r_{m1} = r_{1m} = -(1/a) \sum_{p=2}^{\infty} r^*_{mp} q_{p1} \qquad (m = 2, 3, \cdots),$$

$$r_{11} = 1/a,$$

where

$$a = -q_{11} + \sum_{m,n=2}^{\infty} q_{1m} r^*_{mn} q_{n1},$$

and (r^*_{mn}) is the reciprocal of $(\nu_m^2 e_{mn} - q_{mn})$, $(m, n = 2, 3, \cdots)$, assuming that $-\nu_m^2 z_m + \sum_n q_{mn} z_n = 0$ $(m = 2, 3, \cdots)$ has no solution of finite norm. The matrix above is symmetric and positive definite if (r^*_{mn}) is. The remainder follows as before for the orthogonal and for the polar case. And if the system $-\nu_m^2 z_m + \sum_n q_{mn} z_n = 0$ has solutions modifications can be made as in §§ 1 and 2.

If the function $q(x)$ is identically zero the equations (3) become

$$\lambda \sum_1^{\infty} k_{1n} x_n = 0,$$

$$-\nu_m^2 x_m + \lambda \sum_1^{\infty} k_{mn} x_n = 0 \qquad (m = 2, 3, \cdots).$$

It is obvious that $\lambda = 0$ is a characteristic number. For $k_{11} \neq 0$, the system above is equivalent to

$$-\nu_m^2 x_m + \lambda \sum_{2}^{\infty} [k_{mn} - (k_{m1}k_{n1}/k_{11})] x_n = 0 \qquad (m = 2, 3, \cdots),$$

$$k_{11}x_1 = -\sum_{2}^{\infty} k_{1n}x_n.$$

This system is easily handled directly. The system of characteristic functions $u_a(x)$ includes the function $u_1(x) = 1$, and the adjoint system of $u_a(x)$ is $v_1 = k/\int k$, $v_a = -\lambda_a k u_a$ $(\alpha \neq 1)$.

For $k_{11} = 0$, the system is equivalent to

$$-\nu_m^2 x_m + \lambda \sum_{n=2}^{\infty} k_{mn}x_n + \lambda k_{m1}x_1 = 0 \qquad (m = 2, 3, \cdots),$$

$$\sum_{2}^{\infty} k_{1n}x_n = 0,$$

or to

$$-\nu_m^2 x_m + \lambda \sum_{n=2}^{\infty} [k_{mn} - (1/a) \sum_{p=2}^{\infty} \frac{k_{1p}k_{pn}}{\nu_p^2} k_{m1} - (1/a) \sum_{p=2}^{\infty} \frac{k_{1p}k_{pm}}{\nu_p^2} k_{n1}] x_n = 0$$
$$(m = 2, 3, \cdots),$$

$$x_1 = -(1/a) \sum_{p,n=2}^{\infty} (k_{1p}k_{pn}/\nu_p^2) x_n,$$

$$a = \sum_{p=2}^{\infty} k_{1p}^2/\nu_p^2.$$

Again this system can be easily handled directly. The function k is orthogonal to all the characteristic functions including $u_1 = 1$. The adjoint system is $v_1 = 1/\pi$, $v_a = -\lambda_a k u_a$ $(\alpha \neq 1)$.

II. *Expansion of Arbitrary Functions.*

4. *Orthogonal Case.* Since the matrix

$$\left(\sum_{p} (r_{mp}k_{pn}/\nu_p) \right)$$

is closed, there is an infinite number of characteristic numbers λ_a, and both the systems $\{l_{am}\}$ and $\{m_{am} = \Sigma_p k_{mp} l_{ap}\}$ are closed, if the solutions of (5) are included. As a consequence it follows that

$$\sum_{a} l_{am} m_{an} = e_{mn},$$

and this is equivalent to

$$\sum_{a} (\sum_{m} l_{am}x_m)(\sum_{n} m_{an}y_n) = \sum x_m y_m$$

for every $\{x_m\}$ and $\{y_n\}$ of finite norm. Hence if $f(x)$ and $g(x)$ are continuous functions in $(0, \pi)$

$$\sum_a \int u_a g \int k f u_a = \int fg,$$

and if the series

$$\sum_a u_a(x) \int k f u_a$$

converges uniformly, it converges to $f(x)$. We proceed to obtain some sufficient conditions to ensure the uniform convergence of these series.

It follows from equations (11) that

$$-\nu_m^2 \int u_a \phi_m/\lambda_a + \int g u_a \phi_m/\lambda_a + \int k u_a \phi_m = 0, \qquad (m = 1, 2, \cdots),$$

and hence that the matrices $(\nu_m^2 \int u_a \phi_m/\lambda_a)$ and $(\nu_m^2 \int u_a \phi_m/|\lambda_a|)$ are limited, $\lambda_a = 0$ and $\nu_m = 0$ being excluded. The matrix $(\nu_m \int u_a \phi_m/|\lambda_a|^{1/2})$ is limited, for

$$\sum_a \frac{\nu_m \int u_a \phi_m}{|\lambda_a|^{1/2}} \frac{\nu_n \int u_a \phi_n}{|\lambda_a|^{1/2}} = \nu_m \left(\sum_a \int u_a \phi_m \cdot \frac{\nu_m^2 \int u_a \phi_n}{|\lambda_a|} \right) \frac{1}{\nu_n},$$

are the elements of a symmetric matrix, equal to a matrix $(\nu_m a_{mn}/\nu_n)$ where (a_{mn}) is a limited matrix.* The same method of proof applies to show that the matrices

$$\left(\frac{|\nu_m|^t \int u_a \phi_m}{|\lambda_a|^{t/2}} \right), \quad \left(\frac{|\lambda_a|^{(2-t)/2} \int k u_a \phi_m}{|\nu_m|^{2-t}} \right)$$

are limited, where $t = 1/2^s$ and s is any positive integer. A repetition shows that the matrices are limited for $0 < t = 1/2^{s_1} + 1/2^{s_2} + \cdots + 1/2^{s_p} < 2$, where the exponents are all positive. If $f(x)$ is a continuous function

$$-\sum_m (\nu_m/|\lambda_a|^{1/2})^t \int u_a \phi_m \cdot \nu_m^{2-t} \int f \phi_m + \int g u_a f/|\lambda_a|^t + (|\lambda_a|/\lambda_a) |\lambda_a|^{(2-t)/2} \int k f u_a = 0,$$

and hence if $\{\nu_m^{2-t} \int f \phi_m\}$ is of finite norm, so also is $\{|\lambda_a|^{(2-t)/2} \int k f u_a\}$. If $\{\nu_m^r \int f \phi_m\}$, where $r < 2$, is of finite norm, $\{\nu_m^t \int f \phi_m\}$ is of finite norm and the norm is less than or equal to the norm of the first, where $t < r$ and has the form above. The sequences $\{|\lambda_a|^{t/2} \int k f u_a\}$ are of finite norm for these values of t and the norms are less than $M \int f^2$. The sequence $\{|\lambda_a|^{r/2} \int k f u_a\}$ is of finite norm if the sequence $\{\nu_m^r \int f \phi_m\}$ is of finite norm, and conversely.

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* A. J. Pell, "Linear Equations with Unsymmetric Systems of Coefficients," *Transactions of the American Mathematical Society*, Vol. 20 (1919), pp. 35-36.

From the equality

$$\frac{u_a(x)}{|\lambda_a|^{r/2}} = \sum_m \frac{\int u_a\phi_m}{|\lambda_a|^{r/2}} \cdot \nu_m{}^r \frac{\phi_m(x)}{\nu_m{}^r}$$

it is seen that $\{u_a(x)/|\lambda_a|^{r/2}\}$ is of finite norm norm. The converse is also true. Hence *any* $\{\nu_m{}^r \int f\phi_m\}$ *is of finite norm, where* $\{\phi_m(x)/\nu_m{}^r\}$ *i can be expanded into an absolutely convergent ser teristic functions* $u_a(x)$ *of the differential system*

$$f(x) = \sum_a u_a(x) \int kfu_a.$$

A more intimate study of matrices such as $(\nu_m{}^r \int$ to more general expansion theorems.

5. *Polar Case.* If the equations (5) have (15) and (15′) may be given the form

$$\nu_m l_{am} = \sum_p (r_{pm}\nu_p/\nu_m)\, m_{ap}/\nu_p, \qquad m_{an}/\nu_n =$$

Since both the matrices $(r_{mn}\nu_n/\nu_m)$ and $[e_{mn} -$ limited and positive definite, and since $\Sigma_m \nu_m l_{am} m$ $(\nu_m l_{am})$ and (m_{an}/ν_n) are limited. Hence the seq which is given by

$$u_a(x) = \sum_m (\phi_m(x)/\nu_m) \cdot \nu_m$$

is of finite norm, and the sequence $\{\lambda_a \int kgu_a = \Sigma$ for every function $g(x)$ such that $\{\nu_m \int g\phi_m\}$ is o function $g(x)$ the series

(16) $$\sum_a u_a(x)\, \lambda_a \int kgu_a$$

is absolutely uniformly convergent. A consequence $\{l_{am}\}$ is closed, and the only sequence $\{f_m\}$ of finite to $\{m_{an}\}$ satisfies $\Sigma_n k_{mn} f_n = 0$. If we denote by

$$h(x) = g(x) - \sum_a \phi_a(x)\, \lambda_a \int$$

the function $k(x)\, h(x)$ is orthogonal to the functi zero. Therefore *if* $g(x)$ *satisfies the conditions t norm*

$$k(x)\, g(x) = \sum_a k(x) u_a(x)\, \lambda_a \int kgu_a = \sum_a (|\lambda_a$$

and the series converges absolutely uniformly.

If the equations (5) has a solution $\{z_m\}$ of finite norm the relations (15) and (15′) are to be replaced by

$$v_m(l_{am} - z_m \sum_n l_{an} z_n) = \lambda_a \sum_p (r_{pm}/v_m) v_p \cdot m_{ap}/v_p,$$

$$m_{an}/v_n = \sum_p [e_{np} - (q_{np}/v_n v_p)] \, l_{ap} v_p.$$

As above it follows that the matrices $v_m(l_{am} - z_m \Sigma_n l_{an} z_n)$ and (m_{an}/v_n) are limited. Since the sequences $\{v_m z_m\}$ and $\{\Sigma l_{an} z_n\}$ are of finite norm, the matrix $(v_m l_{am})$ is also limited. Again we get the absolutely uniform convergence of (16) under the same conditions as above. From (10′) it follows that if $\{h_m\}$ is such that $\Sigma h_m k_{mn} l_{an} = 0$, $\Sigma h_m k_{mn} z_n = 0$, then

$$\Sigma r^*_{mp} r_{pq}/v_q^2 k_{qn} h_n = c \, \Sigma r^*_{mn} z_n,$$

and hence

$$\Sigma k_{mp} r_{pq}/v_q^2 k_{qn} h_n = c \, \Sigma k_{mn} z_n,$$

From this,

$$\Sigma h_m k_{mp} r_{pq}/v_q^2 k_{qn} h_n = 0 \text{ and } \Sigma k_{mn} h_n = 0.$$

Therefore we have the result that *if* $\Sigma z_m k_{mn} z_n \neq 0$ *and the sequence* $\{v_m \int g\phi_m\}$ *is of finite norm*

$$k(x)\, g(x) = \sum_a k(x)\, u_a(x)\, \lambda_a \int gku_a + k(x)\, z(x) \int zkg / \int zkz,$$

and the series converges absolutely uniformly.

If $\Sigma z_m k_{mn} z_n = 0$, then $\Sigma h_m k_{mn} l_{an} = 0$, $\Sigma h_m k_{mn} z_n = 0$, $\Sigma h_m r^*_{mn} z_n = 0$ implies that $\Sigma k_{mn} h_n = 0$. *In this case, if the function* $g(x)$ *satisfies the same conditions as above*

$$k(x)\, g(x) = \sum_a k(x)\, u_a(x) \lambda_a \int gku_a + k(x)\, z(x) \int gk\psi / \int zk\psi + k(x)\, \psi(x) \int gkz / \int zk\psi,$$

$$k(x) z(x) = \psi''(x) + g(x)\psi(x),$$

$$\alpha\psi'(0) + \beta\psi(0) = 0,$$

$$\alpha'\psi'(\pi) + \beta'\psi(\pi) = 0,$$

and the series converges absolutely uniformly.

The equations arising in I § 3 were reduced to equations of the same general character as in §§ 1, 2, and the theory for the expansions need not be carried through.

If a more general differential equation

$$\frac{d}{dx}\left(p\,\frac{du}{dx}\right)+qu+\lambda ku=0$$

were given, where $p(x)$ is positive in $(0,\pi)$, the work would be similar. In place of the matrix $(\nu_m^2 e_{mn})$ and its completely continuous reciprocal, there would be the matrix $(\int(p\phi_m')'\phi_n)$ with its completely continuous reciprocal.

BRYN MAWR COLLEGE.

The Singularities of a Function Defined by a Dirichlet Series.*

By D. V. Widder.†

1. *Introduction.* It is the purpose of the present paper to give a demonstration of a theorem enunciated by the author in a recent number of the *Comptes Rendus.*‡ The theorem is a generalization of a familiar theorem of J. Hadamard concerning the multiplication of the singularities of two functions defined by their power series developments. Thus, if the series

$$\sum_{n=0}^{\infty} a_n z^n, \qquad \sum_{n=0}^{\infty} b_n z^n \tag{1}$$

define functions with singularities at points a and b respectively, the function defined by

$$\sum_{n=0}^{\infty} a_n b_n z^n$$

has no singularities except perhaps at points ab.

This theorem gives an immediate result for a special class of Dirichlet series. For if we make the transformation $z = e^{-s}$ we see that the function defined by the series

$$\sum_{n=0}^{\infty} a_n b_n e^{-ns}$$

has singularities at most at the points $\alpha + \beta$, the points $\alpha = -\log a$ and $\beta = -\log b$ being the singularities of the functions defined by

$$\sum_{n=0}^{\infty} a_n e^{-ns}, \qquad \sum_{n=0}^{\infty} b_n e^{-ns}$$

respectively.

The following simple example § shows that this result does not hold when these special Dirichlet series (power series) are replaced by general Dirichlet series of the type

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* Preliminary report presented to the American Mathematical Society, April 16, 1927.

† National Research Fellow in Mathematics.

‡ "Un théorème sur les séries de Dirichlet," *Comptes Rendus*, 1927, p. 1038.

§ J. Soula has called attention to this fact, making use of a slightly different example: *Journal de Mathématiques Pures et Appliquées*, Series 9, Vol. 4 (1925), p. 340.

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \qquad 0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \qquad \lim_{n=\infty} \lambda_n = \infty, \tag{2}$$

$$\phi(s) = \sum_{n=1}^{\infty} b_n e^{-l_n s}, \qquad 0 \leqq l_1 < l_2 < l_3 < \cdots, \qquad \lim_{n=\infty} l_n = \infty. \tag{3}$$

Take $a_n = b_n = 1$, $\lambda_n = l_n = \log n$. Then $f(s)$ and $\phi(s)$ are both equal to the function $\zeta(s)$ of Riemann,

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s.$$

This function is known to have no singularities in the finite plane except a pole at the point $s = 1$. But the function

$$\sum_{n=1}^{\infty} a_n b_n \, 1/n^s = \zeta(s)$$

has a singularity not at the point $\alpha + \beta = 2$.

We may, however, state Hadamard's theorem in a slightly different form, which indicates more clearly the nature of the situation for Dirichlet series. The function defined by

$$(1/1 - e^{-s}) \sum_{n=0}^{\infty} a_n e^{-ns} = \sum_{n=0}^{\infty} s_n e^{-ns}, \qquad s_n = \sum_{\nu=0}^{n} a_\nu$$

evidently has possible singularities at the points α and zero, so that the function defined by

$$\sum_{n=0}^{\infty} b_n s_n e^{-ns}$$

has possible singularities at the points $\alpha + \beta$ and β. It might then be supposed that the series

$$\sum_{n=1}^{\infty} b_n s_n e^{-l_n s} \tag{4}$$

defines a function with possible singularities at the points $\alpha + \beta$ and β (α now being a singularity of $f(s)$ and β a singularity of $\phi(s)$). This result is clearly not contradicted by the example $f(s) = \phi(s) = \zeta(s)$ employed above. For, in this case the sum of the series (4) is $\zeta(s - 1)$.

The series which we shall discuss

$$\sum_{n=1}^{\infty} b_n e^{-l_n s} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^{\rho} \tag{5}$$

evidently reduces to (4) for $\rho = 0$. The method of proof employed demands, however, that ρ be always positive. But the position of the singularities of the function defined by (5) is as forecast by the above example. The discussion of the singularities of this function is made by use of the integral

$$(1/2\pi i) \int_C f(s)\phi(z-s)\,(ds/s^{\rho+1}) \tag{6}$$

suggested to the author by S. Mandelbrojt. It will be seen that this integral plays the rôle of the integral of Parseval in the proof of Hadamard's theorem.

2. *Proof of the Theorem.* We shall suppose that the series (2) converges for $\sigma > \sigma_1$, and that (3) converges absolutely for $\sigma > \sigma_2$ $(s = \sigma + it)$. Denote by α_j the singularities of $f(s)$ and by β_j those of $\phi(s)$, $(j = 1, 2, \cdots)$. We assume that the points α_j, β_j and $\gamma_j = \alpha_k + \beta_l$ are isolated. Furthermore, if

$$\alpha_j = \alpha_j' + i\alpha_j'',$$

we suppose that there exists a number r such that

$$| \alpha_k' - \alpha_l' | > r \quad \text{when} \quad \alpha_k' \neq \alpha_l', \tag{7}$$

$$| \alpha_k'' - \alpha_l'' | > r \quad \text{when} \quad \alpha_k'' \neq \alpha_l'', \tag{8}$$

$$| \beta_k - \beta_l | > r, \tag{9}$$

$$| \gamma_k - \gamma_l | > r. \tag{10}$$

Under these conditions, the theorem which we wish to prove may be stated as follows.

If for an arbitrarily small number η, there is a $\mu > 0$ such that

$$f(s) = O(| t |^{\mu})^*$$

uniformly for $\sigma \geqq \pi_1$, $| s - \alpha_k | \geqq \eta$, and a $\nu > 0$ such that

$$\phi(s) = O(| t |^{\nu})$$

uniformly for $\sigma \geqq \pi_2$, $| s - \beta_k | \geqq \eta$, then the function defined † *by the series*

$$\sum_{n=1}^{\infty} b_n e^{-l_n s} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^{\rho}, \qquad \rho > \mu + \nu$$

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* For an explanation of this notation see G. H. Hardy and Marcel Riesz, *The General Theory of Dirichlet Series*, Cambridge (1915).

† If the function defined by (5) is not uniform the theorem applies to that branch of the function defined by the immediate prolongation of (5).

has no singularity, the real part of which is greater than the two numbers $\pi_1 + k_2$ *and* $\pi_2 + k_1$, *except perhaps the points* β_k *and* γ_k. *The quantities* k_1 *and* k_2 *are defined by the inequalities*

$$\alpha_k' \leqq k_1, \quad \beta_k' \leqq k_2, \qquad \beta_j = \beta_j' + i\beta_j''.$$

The demonstration of this theorem is based on a consideration of the integral (6) where the curve C is now to be described. Draw a line $\sigma = c_1$, where $c_1 = \pi_1$, if no point α nor the point zero lies on the line $\sigma = \pi_1$. Otherwise, choose c_1 so that no point α nor zero lies in the strip $\pi_1 < \sigma \leqq c_1$. This is possible by the condition (7). There is then a minimum distance δ from points α to the line $\sigma = c_1$. With each point α as center describe a small circle of radius r' as small as desired but less than δ and $r/2$. If the origin is not a point α, denote its distance to the nearest point α by δ_1. In this case make the radius r' of each of the above circles less than the least of the numbers δ, $r/2$, $\delta_1/_2$, and with the origin as center describe a circle of the same radius. The circles have been taken so small that no over-lapping will occur. Connect each of the circles lying to the right of $\sigma = c_1$ to this line by a regular curve in such a way that none of these lines cuts another nor any of the circles drawn. The curve C is to consist of the set of loops or "lacets" thus formed together with the line $\sigma = c_1$, and the sense of description is to be that of increasing t on the line and in the positive sense on the loops.

Now construct a curve C_β as follows. If s is a point on one of the loops described above, consider the locus of points

$$s + \beta_k \qquad (k = 1, 2, 3, \cdots).$$

The curve C_β is to consist of that part of this locus which lies to the right of the line

$$\sigma = c_1 + k_2 \qquad (\beta_k' \leqq k_2)$$

and of this line. Now construct a simply connected closed region D some points of which lie to the right of the lines $\sigma = \sigma_1 + \sigma_2$, $\sigma = \sigma_2$; all points of which lie to the right of the line $\sigma = k_1 + r' + \pi_2$, and no point of which is a point of C_β. If z lies in D we shall show that the improper integral

$$(11) \qquad F(z) = (1/2\pi i) \int_C f(s)\phi(z - s)\,(ds/s^{\rho+1}). \qquad \rho > \mu + \nu,$$

converges to a holomorphic function $F(z)$.

Denote the minimum distance from D to C_β by δ_2. Choose the number η

of the theorem less than δ_2 and less than r'. Then for s on C and z in D there exist numbers M and T independent of σ and z such that

$$(12) \qquad \begin{array}{ll} |f(s)| < M|t|^{\mu}, & |t| \geqq T, \\ |\phi(s)| < M|t|^{\nu}, & |t| \geqq T, \end{array}$$

for,

$$|s - \alpha_k| \geqq r' > \eta, \quad |z - s - \beta_k| \geqq \delta_2 > \eta \quad \text{and}$$
$$\sigma \geqq \pi_1 + k_2, \quad x - \sigma \geqq \pi_2 + k_1, \qquad (z = x + iy).$$

Consequently the integral will converge uniformly in D if the integral

$$(13) \qquad \int_C \frac{|t|^{\mu+\nu}}{|s|^{\rho+1}} |ds|$$

converges. We consider first the integral over the line $\sigma = c_1$. That this converges may be easily seen by use of the equation

$$\int_{t_1}^{t_2} \frac{dt}{t^{\rho-\mu-\nu+1}} = \frac{1}{\rho - \mu - \nu} \left[\frac{1}{t_1^{\rho-\mu-\nu}} - \frac{1}{t_2^{\rho-\mu-\nu}} \right], \qquad \begin{array}{r} t_1 \neq 0, \quad t_2 \neq 0. \\ \rho > \mu + \nu. \end{array}$$

On account of the condition (8) all the points α will surely lie on a denumerable set of lines $t = r_k$ where

$$r_{k+1} - r_k > r, \quad r_1 > 0, \quad r_{-1} < 0, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Hence for every point s on a loop about a point α which lies on $t = t_k$ we may suppose that $|s| > (|k| - 1)r$, $|t| < |k|r$. If l denotes the maximum number of points α (for which $\alpha' > c_1$) on any line $t = r_k$ and L the maximum length of any loop, we see that the integral (13) extended over all the loops for which $|k| \geqq 2$ is less than

$$\frac{2lL}{r^{\rho-\mu-\nu+1}} \sum_{k=2}^{\infty} \frac{k^{\mu+\nu}}{(k-1)^{\rho+1}}$$

This series evidently converges if $\rho > \mu + \nu$. The uniform convergence of (11) is thus established.

We have only to apply a familiar theorem * concerning functions represented by definite integrals to recognize that $F(z)$ is analytic in D.

We shall now show that for x sufficiently large $F(z)$ may be represented by another integral (differing from (11) only in the path of integration) which can be expanded in a Dirichlet series. Let z be a point of D for which $x > \sigma_1 + \sigma_2$, $x > \sigma_2$ A number c_2 can then be chosen such that $c_2 > 0$,

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* E. Goursat, *Cours d'analyse mathématique*, Paris (1911), Vol. 2, p. 268.

$c_2 > \sigma_1$ and $x - c_2 > \sigma_2$. For such points z i
that

$$F(z) = (1/2\pi i) \int_{c_2-i\infty}^{c_2+i\infty} f(s) \tag{14}$$

Here s and $z - s$ lie in the region of converge
respectively. Hence the proof of the convergen
here. To prove the validity of (14) we m
theorem.

Construct a rectangle whose vertices are th

$$c_1 + ig_k, \quad c_2 + ig_k, \quad c_1 + ih_k,$$

$$g_k = (r_k + r_{k+1}),$$

Without loss of generality it may be assumed t
rectangle have no points in common with C.
posed of the sides of this rectangle and those loo
Then

$$\int_{C_k} f(s)\phi(z - s)\,(ds,$$

For, since $x - c_2 > \sigma_2$, and since all points α
the region bounded by C_k, the integrand is anal
to become infinite, the integrals

$$\int_{c_1+ig_k}^{c_2+ig_k} f(s)\phi(z - s)\,(ds/s^{\rho+1}), \qquad \int_{c_1+}^{c_2}$$

approach zero by virtue of the conditions (
results immediately.

In order to expand the integral (14) we f

$$\sum_{n=1}^{\infty} [f(s)/s^{\rho+1}]\; b_n e$$

is uniformly convergent * on any finite line s
(z being chosen as indicated above). It is co
grate term by term

$$(1/2\pi i) \int_{c_2-iT}^{c_2+iT} f(s)\phi(z - s)\,(ds/s^{\rho+1}) = \sum_{n=1}^{\infty}$$

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* G. H. Hardy and M. Riesz, *loc. cit.*, p. 5.

If this series converges uniformly for $T \leqq T_0$, it will be permissible to allow T to become infinite term by term. This uniform convergence evidently results from the following inequalities:

$$\left| \int_{c_2-iT}^{c_2+iT} f(s)\, e^{l_n s}\, (ds/s^{\rho+1}) \right| \leqq M\, e^{l_n c_2} \int_{c_2-iT}^{c_2+iT} \frac{|\, t\,|^{\mu}}{|\, s\,|^{\rho+1}}\, |\, ds\,|$$

$$< M\, e^{l_n c_2} \int_{c_2-i\infty}^{c_2+i\infty} \frac{|\, t\,|^{\mu}}{|\, s\,|^{\rho+1}}\, |\, ds\,| = K\, e^{l_n c_2}$$

$$\sum_{n=1}^{\infty} b_n e^{-l_n z}\, (1/2\pi i) \int_{c_1-iT}^{c_1+iT} [f(s)/s^{\rho+1}]\, e^{l_n s}\, ds \ll (K/2\pi) \sum_{n=1}^{\infty} |\, b_n\,|\, e^{-l_n(x-c_2)}.$$

The dominant series converges since (3) is assumed to converge absolutely for $\sigma > \sigma_2$.

If now we allow T to become infinite and make use of a familiar formula of J. Hadamard,[*] we see that

$$(15) \qquad F(z) = [1/\Gamma(\rho+1)] \sum_{n=1}^{\infty} b_n e^{-l_n z} \sum_{\lambda_m < l_n} a_m (l_n - \lambda_m)^{\rho}.$$

The conditions of applicability of this formula are evidently satisfied since $c_2 > 0$, $c_2 > \sigma_1$.

This series expansion of $F(z)$ evidently applies to points z sufficiently far to the right in D. The integral expression (11) applies in the whole region and serves as an analytic prolongation of (15). On account of the arbitrary nature of the loops employed and by condition (10)[†] it follows that D may include any point of the plane not a point $\alpha + \beta$, β for which $x > \pi_1 + k_2$, $x > \pi_2 + k_1$. The theorem is thus completely established.

3. *Remarks.* If $\pi_1 = \pi_2 = -\infty$, the theorem evidently applies to the entire plane. That such a case may actually arise may be seen at once by use of power series (1) representing functions regular at infinity. An example of a function not defined by a power series and satisfying these conditions is the function

$$f(s) = \left(\sum_{n=1}^{\infty} 1/2^n n^s \right) \cdot \left(\sum_{n=1}^{\infty} e^{-ns} \right).$$

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[*] "Théorie générale des séries de Dirichlet," *Mémorial des sciences mathématiques*, Fasicule 17, p. 9.

[†] Without this condition the set γ might be dense even though (7), (8) and (9) be satisfied.

The first factor is a series absolutely convergent in all the plane, and the second is absolutely convergent for $\sigma > 0$. The product is consequently a Dirichlet series absolutely convergent for $\sigma > 0$. The function $f(s)$ surely has a singularity at $s = 0$, and we have $\mu = 0$, $\pi_1 = -\infty$.

As an example of the theorem take

$$f(s) = \zeta(s), \qquad \phi(s) = \sum_{n=1}^{\infty} e^{-ns}.$$

Here $\nu = 0$, $\pi_2 = -\infty$. The number π_1 may be taken negative and arbitrarily small. Then μ is any number greater than $\frac{1}{2} - \pi_1$.* It follows that the function defined by

$$\sum_{n=1}^{\infty} e^{-nz} \sum_{\log m < n} (n - \log m)^{\rho}$$

has no singularities to the right of the line $\sigma = \pi_1$ except perhaps the points $\beta_k = 2k\pi i$, $\alpha + \beta_k = 2k\pi i + 1$, $(k = 0, \pm 1, \pm 2, \cdots)$. In particular the series

$$\sum_{n=1}^{\infty} z^n \sum_{\log m < n} (n - \log m)$$

defines a function that has no singularities inside the circle $|z| \leqq 1$ except perhaps the points 1 and $1/e$. The theorem may thus be employed to obtain results about power series.

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* G. H. Hardy and M. Riesz, *loc. cit.*, p. 18.

On Entire Function Interpolation.

By I. M. Sheffer.

Introduction. To every entire function $f(z) = \sum_{0}^{\infty} c_n z^n$ we associate a function $P(q)$, $q > 0$, determined by the quantities $| c_0 |, | c_1 |, | c_2 |, \cdots, | c_n |, \cdots$. The definition of $P(q)$ is given in § 1. In this section some simple properties of $P(q)$ are proved, and application is made to prove that *if* a_n $(n = 0, 1, \cdots)$ is a given set of numbers, there exist infinitely many entire functions $E(z)$ such that $E(n) = a_n$. This theorem is usually proved by means of a theorem due to Mittag-Leffler.[*] The proof given here, however, follows from the properties of $P(q)$ in a very elementary manner.

In § 2 it is shown that the first k derivatives (k arbitrary) of $E(z)$ may also be assigned. The proof is by induction. In § 3 application of the final result of § 1 is made to give a new proof of a theorem due to Borel.[†] We are led to an interesting set of polynomials of which we state some important properties.

1. *The Function $P(q)$, and an Application.* Let $f(z) = \sum_{0}^{\infty} c_n z^n$ be entire. To each $q > 0$ corresponds a positive number $P(q)$ such that

$$(1) \qquad | c_n | \leqq P(q) q^n, \qquad (n = 0, 1, \cdots)$$

with the equality sign holding for at least one value of n. It is clear that the equality sign can hold only for a finite number of values of n.

Definition. The values of n for which $| c_n | = P(q) q^n$ shall be termed *maximum indices* corresponding to q.

We now state some simple properties of maximum indices:

Lemma 1.[‡] 1) If $n = n_1$, n_2 are the smallest maximum indices for $q = q_1$, q_2 respectively, where $q_2 < q_1$, then $n_2 \geqq n_1$.

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[*] Osgood, *Funktionentheorie*, Vol. 1, 4th edition, Leipzig (1923), pp. 539-551.

[†] Borel, *Leçons sur les Séries Divergentes*, Paris (1901), pp. 152-153.

[‡] The proofs of 1) to 8) are of the same nature, so it suffices to prove 1): Consider

$$| c_0 |/q_1^0, \ldots, | c_{n_1} |/q_1^{n_1} \text{ and } | c_0 |/q_2^0, \ldots, | c_{n_1} |/q_2^{n_1}.$$

By hypothesis $| c_{n_1} | / q_1^{n_1} > | c_i |/q_1^i$, $(i = 0, 1, \ldots, n_1 - 1)$. Let $q_2 = q_1 a$. Then $a < 1$, and

$$| c_{n_1} |/q_2^{n_1} = | c_{n_1} |/(aq_1)^{n_1} > (| c_i |/q_1^i)(1/a^{n_1}) > | c_i |/(aq_1)^i = | c_i |/q_2^i, \quad (i = 0, 1, \ldots, n_1 - 1).$$

Hence the first maximum index for $q = q_2$ is $\geqq n_1$.

2) If c_s is the first non-zero Taylor coefficient of $f(z)$, then for all q sufficiently large s is the only maximum index. (If $c_0 \neq 0$, q' exists such that $P(q) = \text{constant}$, $q > q'$.)

3) If to $q = q_0$ there correspond the maximum indices $n_0, n_1, \cdots, n_k$, (in order of increasing magnitude), then:

i) for $q < q_0$ no one of the indices $n_0, n_1, \cdots, n_{k-1}$ is a maximum index; and

ii) for $q > q_0$ no one of the indices $n_1, \cdots, n_k$ is a maximum index.

4) If n_0 is a maximum index for q_0, there exists a new $q = q', \leqq q_0$ such that for $q < q'$, n_0 is no longer a maximum index.

5) If n_0 is the only maximum index for q', then q_0, q_1 exist, $q_0 < q' < q_1$, such that n_0 is the only maximum index for each q in $q_0 < q < q_1$.

6) If n_0, n_k are respectively the first and last maximum indices for q_0, then q_1, q_2 exist, $q_2 < q_0 < q_1$, such that:

i) n_0 is the only maximum index, $q_0 < q < q_1$;
ii) n_k is the only maximum index, $q_2 < q < q_0$.

7) If n_0 is the only maximum index for $q_0 < q < q_1$, then n_0 is a maximum index for $q = q_0, q_1$.

8) $P(q_1) \geqq P(q_2)$, $0 < q_1 < q_2$.

LEMMA 2. The function $P(q)$ is continuous, $0 < q$.

Proof: Case i) n_0 is the only maximum index for $q = q_0$. The lemma then follows from 5), since $P(q) = | c_{n_0} | / q^{n_0}$ in the neighborhood of q_0. Case ii): There is more than one maximum index for q_0. Let n_0 be the first and n_k the last. By 6), we can find $q_1 > q_0$, $q_2 < q_0$ such that n_0 is the only maximum index, $q_0 < q < q_1$, and n_k is the only maximum index, $q_2 < q < q_0$. Therefore $P(q)$ is continuous in $q_0 < q < q_1$, and in $q_2 < q < q_0$. Also, $P(q_0^+) = P(q_0^-) = P(q_0)$. Therefore $P(q)$ is continuous at q_0.

LEMMA 3. $P'(q)$ is continuous at each $q > 0$ for which there is a single maximum index, and $P'(q^+)$, $P'(q^-)$ exist (but are unequal) at each $q > 0$ for which there are more maximum indices than one.

Proof: i) If for q_0 there is a single maximum index, n_0, then in the neighborhood of q_0, $P(q) = | c_{n_0} | / q^{n_0}$; whence $P'(q)$ is continuous at q_0.

ii) Let there be more than one maximum index corresponding to q_0. By 6), $P'(q)$ is continuous in the deleted [*] neighborhood of q_0, and [†]

$$\lim_{q=q_0^+} P'(q) = P'(q_0^+) = -P(q_0)n_0/q_0;$$

$$\lim_{q=q_0^-} P'(q) = P'(q_0^-) = -P(q_0)n_k/q_0.$$

Since the first maximum index never decreases as $q > 0$ *decreases*, we see that $P(q)$ represents a continuous curve having a derivative at every point with the exception of a denumerable infinity of angular points.[‡] An angular point occurs when and only when there corresponds to q more than one maximum index.

Denote the angular points by $q = \xi_0, \xi_1, \cdots, \xi_n, \cdots$, in order of *decreasing* magnitude. Clearly the only cluster point is $q = 0$. Since the definition of $P(q)$ involves only the absolute values of the c_n's, it follows that there are infinitely many entire functions having the same P-function.

Given an arbitrary set of constants ξ_n satisfying the single condition $\xi_0 > \xi_1 > \cdots > \xi_n > \cdots$, $\lim \xi_n = 0$; and an arbitrary set of integers k_n satisfying $0 \leqq k_0 < k_1 < \cdots < k_n < \cdots$. Define a function $Q(q)$, $0 < q$ as follows:

$$
\begin{aligned}
&Q(q) = \alpha/q^{k_0}, \quad \alpha > 0, \quad \xi_0 \leqq q;\\
(2) \qquad &Q(\xi_n) = (\xi_{n-1}/\xi_n)^{k_n} Q(\xi_{n-1});\\
&Q(q) = (\xi_n/q)^{k_n} Q(\xi_n), \qquad \xi_n < q < \xi_{n-1}.
\end{aligned}
$$

We see that $Q(q)$ is positive and continuous, and that $Q'(q)$ is continuous at every $q \neq \xi_0, \xi_1, \cdots$, while $Q'(q^+)$, $Q'(q^-)$ exist (and are unequal) at every point $q = \xi_0, \xi_1, \cdots$. Furthermore, $Q(q_1) > Q(q_2)$, $0 < q_1 < q_2 \leqq \xi_0$.

LEMMA 4. There exists an entire function (and consequently an infinite number of them) whose P-function has the points $q = \xi_0, \xi_1, \cdots$ (and only those) as angular points, and such that in the interval $\xi_n < q < \xi_{n-1}$, k_n is the only maximum index.[§] Furthermore, the P-function for this entire function is precisely $Q(q)$.

Proof: $Q(q)q^{k_n} = \xi^{k_n}_{n-1} Q(\xi_{n-1}) =$ a positive constant, $\xi_n \leqq q \leqq \xi_{n-1}$;

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[*] That is, $q = q_0$ itself is excepted.

[†] n_0, n_k are respectively the first and last maximum indices for $q = q_0$.

[‡] If $f(z)$ is a polynomial there are only a finite number of angular points.

[§] For $q > \xi_0$, k_0 is the only maximum index.

denote this constant by[*] A_{k_n}. Consider the function $f(z) = \sum_{n=0}^{\infty} A_{k_n} z^{k_n}$. We have

$$\frac{Ak_{n+1}}{Ak_n} = \xi_n^{k_{n+1}-k_n}(\xi_n/\xi_{n-1})^{k_n} \frac{Q(\xi_n)}{Q(\xi_{n-1})} = \xi_n^{k_{n+1}-k_n}(\xi_n/\xi_{n-1})^{k_n} \quad (\xi_{n-1}/\xi_n)^{k_n} = \xi_n^{k_{n+1}-}$$

Therefore $\lim\limits_{n=\infty} \dfrac{Ak_{n+1}z^{k_{n+1}}}{Ak_n\, z^{k_n}} = \lim\limits_{n=\infty} \; (z\,\xi_n)^{k_{n+1}-k_n} = 0$, since $k_{n+1} - k_n \geqq 1$ and $\xi_n \to 0$. Hence $f(z)$ is an entire function.[†] We wish to show that its P-function is $Q(q)$.

Let q be any point of $\xi_n < q < \xi_{n-1}$, and consider

$$\{\quad\} = \{(Ak_p/q^{k_p})/(Ak_n/q^{k_n})\} = q^{k_n-k_p}\frac{Ak_p}{Ak_n}, \qquad p \neq n.$$

Now

$$\frac{Ak_{n+1}}{Ak_n} = \xi_n^{k_{n+1}-k_n}, \qquad\qquad \frac{Ak_{n+2}}{Ak_n} = \xi_{n+1}^{k_{n+2}-k_{n+1}} \cdot \xi_n^{k_{n+1}-k_n},$$

and in general,

$$\frac{Ak_{n+s}}{Ak_n} = \xi_{n+s-1}^{k_{n+s}-k_{n+s-1}} \cdot \xi_{n+s-2}^{k_{n+s-1}-k_{n+s-2}} \cdot \cdot \cdot \; \xi_n^{k_{n+1}-k_n}.$$

Similarly,

$$\frac{Ak_{n-s}}{Ak_n} = [\,\xi_{n-1}^{k_n-k_{n-1}} \cdot \xi_{n-2}^{k_{n-1}-k_{n-2}} \; \cdot \cdot \cdot \; \xi_{n-s}^{k_{n-s+1}-k_{n-s}}\,]^{-1}.$$

Hence if $p > n$: $p = n + s$, then

$$\{\quad\} = q^{k_n-k_p}\; \xi_{n+s-1}^{k_{n+s}-k_{n+s-1}} \cdot \cdot \cdot \xi_n^{k_{n+1}-k_n} \leqq q^{k_n-k_p}\, \xi_n^{[(k_{n+s}-k_{n+s-1})+\, \cdot\cdot\cdot\, +(k_{n+1}-k_n)]}$$
$$= q^{k_n-k_p}\; \xi_n^{k_p-k_n} = (\xi_n/q)^{k_p-k_n} < 1$$

since $q > \xi_n$ and $k_p > k_n$.

And if $p < n$: $p = n - s$, then

$$\{\quad\} = q^{k_n-k_p}\,[\xi_{n-1}^{k_n-k_{n-1}} \cdot \cdot \cdot \xi_{n-s}^{k_{n-s+1}-k_{n-s}}]^{-1} \leqq q^{k_n-k_p}\, \xi_{n-1}^{k_{n-s}-k_n} = (q/\xi_{n-1})^{k_n-k_p} < 1.$$

Consequently k_n is the only maximum index for $\xi_n < q < \xi_{n-1}$. From 7), then, k_n is a maximum index for $q = \xi_{n-1}, \xi_n$. Therefore k_{n+1}, k_n are maximum indices for $q = \xi_n$, and k_n, k_{n-1} for $q = \xi_{n-1}$. This shows that the angular points of $P(q)$ are precisely $q = \xi_0, \xi_1, \cdot \cdot \cdot$. There remains to identify P with Q.

By definition of a maximum index we have $A_{k_n} = P(q)\,q^{k_n}$, and this holds for $\xi_n \leqq q \leqq \xi_{n-1}$, $n \geqq 1$. Comparing this with the value of $Q(q)$ we have

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[*] $A_{k_0} = Q(\xi_0)\,\xi_0^{k_0} = a$.

[†] $f(z)$ is not a polynomial.

$P(q) = Q(q)$, $\xi_n \leqq q \leqq \xi_{n-1}$, $n \geqq 1$. That is, $P(q) = Q(q)$, $q \leqq \xi_0$. From the property of the P-function, k_0 (A_{k_0} being the first non-zero coefficient of $f(z)$) is the only maximum index for $q > \xi_0$. Therefore $P(q) = A_{k_0}/q^{k_0}$, $q \geqq \xi_0$. That is, $P(q) = [\xi_0^{k_0} Q(\xi_0)]/q^{k_0} = \alpha/q^{k_0} = Q(q)$, $q \geqq \xi_0$. Then, $P(q) = Q(q)$, $0 < q$. The proof is now complete.

Define $F(r)$ by

$$F(r) = \sum_0^\infty | c_n | r^n. \tag{3}$$

LEMMA 5. We have

$$F(r) < P(q) \sum_0^\infty (rq)^n = P(q)/(1 - rq), \quad rq < 1. \tag{4}$$

The proof is immediate.

LEMMA 6. We have

$$F(r) > P(1/r), \quad r > 0. \tag{5}$$

Proof: From (3), $F(r) > | c_n | r^n$, $n = 0, 1, \cdots$. Let n_0 be a maximum index for any value q: $| c_{n_0} | = P(q) q^{n_0}$. Then $F(r) > P(q)(rq)^{n_0}$. (5) now follows on setting $q = 1/r$.

If in (4) we set $q = 1/(r+1)$ we obtain the following corollary:

Corollary. The Function P satisfies the inequality

$$P(1/r) < F(r) < (r+1)P(1/r+1), \quad r > 0. \tag{6}$$

LEMMA 7. Let α_n be an arbitrary set of numbers, $n = 0, 1, \cdots$. There exists an entire function $K(z)$ (and consequently an infinite number) such that $K(n) > | \alpha_n |$.

Proof: Set* $A_n = 1 + | \alpha_{n+1} |$, and choose $\xi_{n-1} = 1/n$, $n = 1, 2, \cdots$. Finally, choose the set of integers k_n: $0 \leqq k_0 < k_1 < \cdots$ such that†

$$\begin{aligned} B_1 &= (\xi_0/\xi_1)^{k_1} \geqq A_1 \\ B_2 &= (\xi_1/\xi_2)^{k_2} A_1 \geqq A_2 \\ &\cdots\cdots\cdots\cdots \\ B_n &= (\xi_{n-1}/\xi_n)^{k_n} A_{n-1} \geqq A_n \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{7}$$

By Lemma 4, there exists an entire function $f(z) = \sum_0^\infty c_n z^n$, $c_n \geqq 0$, corresponding to the ξ_n's and k_n's. Consider the P-function corresponding to $f(z)$:

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* We deal with A_n rather than with α_n because some of the α_n's may be zero.

† This choice can be made, since $\xi_n < \xi_{n-1}$.

$$P(q) = \xi^{k_n}_{n-1}\, P(\xi_{n-1})\, q^{-k_n}, \quad \xi_n \leqq q \leqq \xi_{n-1}.$$

Setting $q = \xi_n$: $P(\xi_n) = (\xi_{n-1}/\xi_n)^{k_n} P(\xi_{n-1})$, $n \geqq 1$, and $P(\xi_0) = \alpha/\xi_0^{k_0}$, $= 1$ by proper choice of α. Then

$$\begin{aligned} &P(\xi_1) = (\xi_0/\xi_1)^{k_1} = B_1 \geqq A_1 \\ (8) \qquad &P(\xi_2) = (\xi_1/\xi_2)^{k_2} P(\xi_1) \geqq (\xi_1/\xi_2)^{k_2} A_1 = B_2 \geqq A_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &P(\xi_n) = (\xi_{n-1}/\xi_n)^{k_n} P(\xi_{n-1}) \geqq (\xi_{n-1}/\xi_n)^{k_n} A_{n-1} = B_n \geqq A_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Now $\xi_n = 1/n + 1$. Consequently

$$P(1/n + 1) \geqq A_n > |\, \alpha_{n+1} \,|, \quad n = 1, 2, \cdots.$$

Using Lemma 6, $f(n+1) > P(1/n+1) > |\, \alpha_{n+1} \,|$, $n = 1, 2, \cdots$. Then $K(z) = |\, \alpha_0 \,| + |\, \alpha_1 \,| + f(z)$ is entire, and $K(n) > |\, \alpha_n \,|$, $n = 0, 1, \cdots$.

LEMMA 8. If $|\, c_n \,| \leqq 1/(n+1)^2$, $n = 0, 1, \cdots$, then

$$(9) \qquad \omega(z) = \sum_0^\infty c_n \frac{\sin \pi(z-n)}{\pi(z-n)}$$

is an entire function, and $\omega(n) = c_n$.

Proof: Consider the region $|\, z \,| \leqq R$, R arbitrary but fixed. Neglect those terms (say $n = 0, 1, \cdots, n_0 - 1$) for which $n - R < 1$. Then, writing $z = x + iy$,

$$\sum_{n=n_0}^\infty \left|\, c_n \frac{\sin \pi(z-n)}{\pi(z-n)} \,\right| \leqq \sum_{n_0}^\infty \frac{1}{\pi(n+1)^2} \left|\, \frac{e^{i\pi(z-n)} - e^{-i\pi(z-n)}}{2i} \,\right|$$
$$\leqq \sum_{n_0}^\infty \frac{1}{2\pi(n+1)^2}(e^{\pi y} + e^{-\pi y}) \leqq \sum_{n_0}^\infty \frac{e^{\pi R}}{\pi(n+1)^2}.$$

That is, $\omega(z)$ is analytic in $|\, z \,| \leqq R$, R arbitrary. Hence $\omega(z)$ is entire. Setting $z = n$: $\omega(n) = c_n$.

Corollary. The function

$$(10) \qquad V(z) = (z+1)^2 \sum_0^\infty \frac{1}{(n+1)^2} \frac{\sin \pi(z-n)}{\pi(z-n)}$$

is entire, and* $V(n) = 1$, $n = 0, 1, \cdots$.

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* But $V(z) \not\equiv 1$.

We now arrive at the desired theorem:

THEOREM 1. *Given an arbitrary set of numbers* a_n, $n=0, 1, \cdots$. *There exist infinitely many entire functions* $E(z)$ *for which* $E(n)=a_n$.

Proof: It suffices to find one such function, since we can then multiply it by any positive power of $V(z)$ of equation (10) to get further such functions. By Lemma 7, $K(z)$, entire, exists such that $b_n=K(n)>(n+1)^2|a_n|$. Set $c_n=a_n/b_n$. Then $|c_n|\leqq 1/(n+1)^2$, whence from Lemma 8, an entire function $\omega(z)$ exists such that $\omega(n)=c_n$. Therefore $E(z)=K(z)\omega(z)$ is entire, and $E(n)=a_n$.

Before taking up the work of § 2, we remark that the problem just solved is completely equivalent * to the problem of finding a solution of the system of linear equations

$$\sum_{j=0}^{\infty} i^j x_j = a_i, \quad i=0, 1, \cdots. \tag{11}$$

2. $E(n), E'(n), \cdots, E^{(k)}(n)$ *Assigned*, $n=0, 1, \cdots$. In this section we shall be concerned with the proof of

THEOREM 2. *Given* $(k+1)$ sets of numbers ${}_0c_n, {}_1c_n, \cdots, {}_kc_n$, $n=0, 1, \cdots$. *Infinitely many entire functions* $E(z)$ *exist for which* $E^{(r)}(n)={}_rc_n$, $r=0, 1, \cdots, k$.

The proof will be carried through by induction:

LEMMA 9. Let c_n be any set of numbers. Infinitely many entire functions $G(z)$ exist such that $G(n)=c_n$, $G'(n)=0$, $n=0, 1, \cdots$.

Proof: By Theorem 1 there exists $F(z)$, entire, such that $F(n)=c_n$. Consider the entire function $\zeta(z)=z-1+\cos 2\pi z-(1/2\pi)\sin 2\pi z$. We have $\zeta(n)=n$, $\zeta'(n)=0$, $n=0, 1, \cdots$. Set † $G(z)=F(\zeta(z))$. $G(z)$ is entire, and $G'(z)=\dfrac{dF(\zeta)}{d\zeta}\dfrac{d\zeta}{dz}$. Therefore $G(n)=c_n$, $G'(n)=0$. $G(z)$ is then a solution.

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* That is, every solution $E(z)=\sum_{j=0}^{\infty} x_j z^j$ of § 1 gives a solution (x_j) of (11), and conversely.

† There are clearly an infinity of such functions $G(z)$, since this is true of $F(z)$. This remark applies to the later lemmas also.

LEMMA 10. Let c_n, e_n be any two sets of numbers. There exist infinitely many entire functions $G(z)$ such that $G(n) = c_n$, $G'(n) = e_n$, $n = 0, 1, \cdots$.

Proof: By Theorem 1 and Lemma 9, entire functions $A(z)$, $B(z)$, $C(z)$ exist * such that

$$A(n) = c_n, \quad A'(n) = 0; \quad B'(n) = e_n; \quad C(n) = B(n), \quad C'(n) = 0.$$

Then $G(z) = A(z) + B(z) - C(z)$ satisfies the conditions of the lemma.

This proves Theorem 2 for the case $k = 1$. We now turn to the general case. *Assume* the theorem to hold for $k - 1$. We shall then prove it for k.

LEMMA 11. Given c_n; infinitely many entire functions $G(z)$ exist such that $G(n) = c_n$, $G^{(r)}(n) = 0$, $r = 1, 2, \cdots, k$.

Proof: By hypothesis, $F(z)$ exists such that $F(n) = c_n$, $F^{(r)}(n) = 0$, $r = 1, \cdots, k - 1$. Let $\theta(z)$ be an entire function, which we shall presently determine, and consider the function

$$G(z) = F(\theta(z)). \tag{12}$$

We have

$$\begin{aligned} G(z) &= F(\theta) \\ G'(z) &= \frac{dF(\theta)}{d\theta} \frac{d\theta}{dz} \\ G''(z) &= \frac{dF(\theta)}{d\theta} \frac{d^2\theta}{dz^2} + \frac{d^2 F(\theta)}{d\theta^2} \left(\frac{d\theta}{dz} \right)^2 \qquad (13) \\ G'''(z) &= \frac{dF(\theta)}{d\theta} \frac{d^3\theta}{dz^3} + \frac{d^3F(\theta)}{d\theta^3} \left(\frac{d\theta}{dz} \right)^3 + 3 \frac{d^2F(\theta)}{d\theta^2} \frac{d\theta}{dz} \frac{d^2\theta}{dz^2} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

In general, $G^{(r)}(z)$ contains a term $\frac{dF(\theta)}{d\theta} \frac{d^r\theta}{dz^r}$, and a term $\frac{d^rF(\theta)}{d\theta^r} \left(\frac{d\theta}{dz} \right)^r$ all the other terms † contain some derivative of $F(\theta)$ with respect to θ, of order s, $2 \leqq s \leqq r - 1$. Choose

---

* If $H(z)$, entire, is such that $H(n) = e_n$, then $B(z) = \int_0^z H(z)\,dz$ is entire and its derivative takes on the values e_n.

† This is true for $r = 1, 2, 3$. Assume it true up to $r - 1$: $G^{(r-1)}(z) = \frac{dF(\theta)}{d\theta} \frac{d^{(r-1)}\theta}{dz^{r-1}} + \frac{d^{r-1}F(\theta)}{d\theta^{r-1}} \left(\frac{d\theta}{dz} \right)^{r-1}$ + terms involving derivatives of F with respect to θ of degree $\leqq r - 2$. On differentiating with respect to z, we get the result stated.

$$\theta(z)=\zeta(z)+\omega_1(z)+\cdots+\omega_{k-2}(z), \tag{14}$$

where $\zeta(z)=z-1+\cos 2\pi z-(1/2\pi)\sin 2\pi z$, and $\omega_i(z)$, entire, is defined as follows:*

$\omega_1(z)=A_1\ (\sin 2\pi z)^3$, choosing A_1 so that
$\omega_1(n)=\omega_1'(n)=\omega_1''(n)=0,\quad \omega_1^{(3)}(n)=-\zeta^{(3)}(n).$
$\omega_2(z)=A_2\ (\sin 2\pi z)^4$, choosing A_2 so that
$\omega_2(n)=\omega_2'(n)=\omega_2''(n)=\omega_2^{(3)}(n)=0,$
$\omega_2^{(4)}(n)=-\ [\zeta^{(4)}(n)+\omega_1^{(4)}(n)].$

. . . .

$\omega_{k-2}(z)=A_{k-2}\ (\sin 2\pi z)^k$, choosing A_{k-2} so that
$\omega_{k-2}(n)=\omega'_{k-2}(n)=\cdots=\omega_{k-2}^{(k-1)}(n)=0,$
$\omega_{k-2}^{(k)}(n)=-[\zeta^{(k)}(n)+\omega_1^{(k)}(n)+\cdots\omega_{k-3}^{(k)}(n)].$

Such a choice of $A_1,\cdots,A_{k-2}$ can be made since $\zeta^{(r)}(n)$ and $\omega_i^{(r)}(n)$ $(i=1,\cdots,r-3)$ are independent of n. Then $\theta(z)$ is entire and $\theta(n)=\zeta(n)=n$, $\theta'(n)=0$, $\theta''(n)=-4\pi^2$, $\theta^{(r)}(n)=0$, $r=3,4,\cdots,k$.

Now $\left.\dfrac{d^rF(z)}{dz^r}\right|_{z=n}=0$, $r=1,\cdots,k-1$, and since $\theta(n)=n$, it follows that $\left.\dfrac{d^rF(\theta)}{d\theta^r}\right|_{z=n}=0$, $r=1,\cdots,k-1$. Hence, returning to (13):

$$G(n)=F(\theta(n))=c_n.$$

$$G'(n)=\left.\frac{dF(\theta)}{d\theta}\right|_{z=n}\cdot\theta'(n)=0$$

. . . .

$$G^{(r)}(n)=\left.\frac{dF(\theta)}{d\theta}\right|_{z=n}\cdot\overset{(r)}{\theta}(n)+\left.\frac{d^rF(\theta)}{d\theta^r}\right|_{z=n}\cdot[\theta'(n)]^r+$$

[terms involving derivatives of $F(\theta)$ with respect to θ of order $\leqq r-1$]

$$=0,\quad r=2,3,\cdots,k.$$

The lemma is now proved.

---

* In this definition we make use of the fact that at the points $z=n$, $(\sin 2\pi z)^r$ and its first $r-1$ derivatives are zero, the rth derivative is not zero, and all derivatives after the $(r-1)$st are independent of n.

Now let ${}_0c_n, \cdots, {}_kc_n$ be assigned. By Lemma 11 and our assumption for $k-1$, we can find $A(z)$, $B(z)$, $C(z)$ such that

$$A(n) = {}_0c_n,\ A'(n) = {}_1c_n, \cdots, A^{(k-1)}(n) = {}_{k-1}c_n;$$
$$^* B'(n) = 0, \cdots, B^{(k-1)}(n) = 0,\ B^{(k)}(n) = {}_kc_n - A^{(k)}(n);$$
$$C(n) = B(n),\ C'(n) = 0, \cdots, C^{(k-1)}(n) = 0,\ C^{(k)}(n) = 0.$$

Then $E(z) = A(z) + B(z) - C(z)$ is entire and $E^{(r)}(n) = {}_rc_n$, $r = 0, 1, \cdots, k$. There are infinitely many such functions $E(z)$.† The proof of Theorem 2 is now complete.

Remarks. 1) Theorem 2 is equivalent to the theorem:

THEOREM. *There are infinitely many solutions* (x_i) *of the* $(k+1)$ *simultaneous systems*

$$\sum_{j=0}^{\infty} i^j x_j = {}_0c_i, \quad \sum_{j=0}^{\infty} j i^{j-1} x_j = {}_1c_i, \cdots, \quad \sum_{j=0}^{\infty} j(j-1)\cdots(j-k+1) i^{j-k} x_j = {}_kc_i$$
$$i = 0, 1, \cdots.$$

2) If ${}_0c_n, \cdots, {}_kc_n$ are real, then infinitely many of the solutions $E(z)$ are real; i. e., if $E(z) = \sum_0^\infty g_n z^n$, then g_n is real.

3) There are infinitely many solutions $E(z) = \sum_0^\infty g_n z^n$ for which $g_0, g_1, \cdots, g_{r-1}$ (r arbitrary) can be preassigned, provided we do not give ${}_0c_n, {}_1c_n, \cdots, {}_kc_n$ for $n = 0$.

Proof: 1) is immediate; 2) follows from consideration of the proof of Theorems 1 and 2; 3) follows from Theorem 2 by choosing‡ ${}_ic_0 = i!\, g_i$, $i = 0, 1, \cdots, r-1$.

3. *Proof of a Theorem of Borel.* Consider the function

$$(1) \qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots,$$

analytic about the origin. An entire function

$$(2) \qquad \zeta(z) = b_0 + b_1 z + \cdots + b_n z^n + \cdots$$

exists such that $\zeta(n) = a_n$, $n = 0, 1, \cdots$. That is,

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[*] See footnote (*), page 336.

[†] See footnote (†), page 335.

[‡] If $r-1 > k$, we can assign ${}_ic_n$ at pleasure ($n \geqq 1$), ($i = k+1, \ldots, r-1$), and get a solution.

$$a_n = b_0 + nb_1 + n^2b_2 + \cdots + n^kb_k + \cdots. \tag{3}$$

Therefore

$$f(z) = b_0 + z\,(\sum_0^\infty b_k 1^k) + z^2\,(\sum_0^\infty b_k 2^k) + \cdots. \tag{4}$$

If we combine *formally* the coefficients of b_k and sum from $n = 0$ to ∞ we obtain

$$f(z) \sim b_0u_0(z) + b_1u_1(z) + \cdots + b_nu_n(z) + \cdots, \tag{5}$$

where

$$\begin{aligned} u_0(z) &= 1 + z + z^2 + \cdots + z^k + \cdots; \\ u_n(z) &= z + 2^nz^2 + \cdots + k^nz^k + \cdots, \quad n > 0. \end{aligned} \tag{6}$$

We proceed to discuss the functions $u_n(z)$. Clearly

$$\begin{aligned} u_0(z) &= 1/(1-z), \\ u_n(z) &= zu'_{n-1}(z). \end{aligned} \tag{7}$$

The first few $u_n(z)$'s are

$$u_0(z) = 1/(1-z), \quad u_1(z) = z/(1-z)^2, \quad u_2(z) = z(1+z)/(1-z)^3.$$

This suggests that we can write $u_n(z)$ in the form

$$u_n(z) = P_n(z)/(1-z)^{n+1}, \tag{8}$$

where $P_n(z)$ is a polynomial of degree precisely n. To prove this we *define* $P_n(z)$ by (8) and substitute into (7). On simplifying we see that

$$\begin{aligned} P_0(z) &= 1, \\ P_n(z) &= z(1-z)P'_{n-1}(z) + nzP_{n-1}(z). \end{aligned} \tag{9}$$

From (9) we obtain (by induction) the following properties of $P_n(z)$:

1) $P_n(z)$ is a real polynomial of degree n.
2) $P_n(0) = 0,\ n > 0;\ P_n(1) = n!$
3) The coefficients of $P_n(z)$ are positive * integers.
4) $P_n(x) > 0,\ x > 0;\ |P_n(z)| \leqq n!,\ |z| \leqq 1.$

Further properties are:

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* If we set $P_n(z) = a_{n,n}z^n + \ldots + a_{n,0}$ and substitute into (9), the relations defining $a_{n,i}$ in terms of the $a_{n-1,k}$'s are linear with *positive* coefficients. Of course $P_n(z)$ has no constant term, $n > 0$.

5) $P_n(x_0) > x_0^n n!$, $0 < x_0 < 1$.

Proof: $P_n(x_0) > 0$, $P_{n-1}(x_0) > 0$, $P'_{n-1}(x_0) > 0$,

$$x_0(1-x_0)P'_{n-1}(x_0) > 0.$$

Therefore

$$P_n(x_0) > nx_0P_{n-1}(x_0) > nx_0 [(n-1)x_0P_{n-2}(x_0)] > \cdots > n(n-1)\cdots 2x_0^{n-1}P_1(x_0) = n!x_0^n.$$

6) $P_n(x_0) < x_0^n n!$, $x_0 > 1$.

The proof is like that of 5).

Corollary. We have $|P_n(z)| \leqq n!|z|^n$, $|z| > 1$.

LEMMA 12. If $\sum_0^\infty |c_n u_n(x_0)|$ converges, where x_0 is real and satisfies $(1/2) \leqq x_0 < 1$, then M exists such that $|c_n| \leqq M/n!$ for all n.

Proof: We have $u_n(x_0) = P_n(x_0)/(1-x_0)^{n+1} > 0$, and $P_n(x_0) > n!x_0^n$. Therefore

$$\sum_0^\infty |c_n| u_n(x_0) \gg [1/(1-x_0)] \sum_0^\infty |c_n| r^n n!,$$

where $r = x_0/(1-x_0) \geqq 1$. By hypothesis the series on the right must converge, and from this fact the lemma easily follows.

Corollary. Under the same hypothesis, $|c_n| \leqq (M/n!)[(1-x_0)/x_0]^n$.

LEMMA 13. If $|c_n| \leqq (M/n!)$, then $f(z) = \sum_0^\infty c_n u_n(z)$ converges absolutely for every (finite) z in the region* T: $R(z) < 1/2$, $|1-z| > 1$; and uniformly in every finite (closed) region in T. Consequently $f(z)$ is analytic in T.

Proof: Let z be any point in T.

i) $|z| \geqq 1$. Then $|P_n(z)| \leqq n!|z|^n$, from which follows that $\sum_0^\infty |c_n u_n(z)| \leqq (M/|1-z|) \sum_0^\infty |z/(1-z)|^n$, which converges since $|z/(1-z)| < 1$ in T.

---

* $R(z)$ = real part of z.

ii) $|z| \leqq 1$. Then $|P_n(z)| \leqq n!$, and $\sum_0^\infty |c_n u_n(z)| \leqq M/|1-z| \sum_0^\infty 1/|1-z|^n$, which converges.

This proves absolute convergence. Now let S be any (finite) closed region in T. Divide S into two closed parts S_1, S_2 (if necessary) by the circle $|z| = 1$. Uniform convergence in S_1 and in S_2 follow at once, and therefore in S.

Let us now return to the function $f(z)$.

THEOREM 3. *If* $f(z) = \sum a_n z^n$ *is analytic about the origin, and if an entire function* $\zeta(z) = \sum_0^\infty b_n z^n$ *exists with* $|b_n| \leqq M/n!$, *such that* $\zeta(n) = a_n$, *then* $f(z)$ *is analytic at every (finite) point in* T.

Proof: It is easily * seen that equation (5) of § 3 is valid in the neighborhood of the origin. Now apply Lemma 13.

Corollary 1. If $\zeta(z) = \sum_0^\infty b_n z^n$ is entire with $|b_n| \leqq M/n!$, then $F(z) = \sum_0^\infty \zeta(n) z^n$ has no (finite) singularity in T.

Corollary 2. If $f(z) = \sum_0^\infty a_n z^n$, analytic about the origin, has a (finite) singularity in T, then no entire function $\zeta(z) = \sum_0^\infty b_n z^n$ exists with $|b_n| \leqq M/n!$ such that $\zeta(n) = a_n$.

THEOREM 4. *Given the set of numbers* a_n, $n = 0, 1, \cdots$. *A necessary condition that* $\zeta(z) = \sum_0^\infty b_n z^n$, *entire, exist with* $|b_n| \leqq M/n!$ *for some* M, *such that* $\zeta(n) = a_n$, *is that* $|a_n| \leqq Me^n$.

Proof: We are given that $\zeta(z)$ exists. Therefore $|\zeta(0)| \leqq M$, $|\zeta(k)| \leqq M \sum_0^\infty (k^n/n!) = Me^k$. This proves the theorem since $a_k = \zeta(k)$.

Corollary. If $\zeta(z) = \sum_0^\infty b_n z^n$ is entire with $|b_n| \leqq M/n!$ and such that either $\zeta(2n) = 0$ or $\zeta(2n+1) = 0$, $n = 0, 1, \cdots$, then $F(z) = \sum \zeta(n) z^n$ is entire.

Proof: $F(z)$ is analytic in T. But also $F(z)$ is either an odd function or an even function, and is therefore analytic in T', the image of T in the imaginary axis. $T + T'$ gives every point of the z-plane except the origin. But by Theorem 4, $F(z)$ is analytic at $z = 0$. Hence $F(z)$ is entire.

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* In fact, the series converges in $|z| < 1/e$ at least.

We come now to Borel's theorem:

THEOREM. *If* $\zeta(z) = \sum_0^\infty b_n z^n$ *is an entire function of genus zero, then* $f(z) = \sum_0^\infty \zeta(n) z^n$ *is analytic everywhere (in the finite plane) save possibly at* $z = 1$.

Proof: We use the following property* of a function of genus zero:

$n!\,|\,b_n\,|\,c^n \rightarrow 0$ for every c however large (but fixed).

Now $f(z) = \sum_0^\infty b_n u_n(z) = \sum_0^\infty b_n P_n(z)/(1-z)^{n+1}$; and for $z \neq 1$,

$$\sum_0^\infty \left| \, b_n \frac{P_n(z)}{(1-z)^{n+1}} \right| \leqq \frac{M}{|1-z|} \sum_0^\infty \left(\frac{l}{c\,|\,1-z\,|} \right).\dagger$$

If we suppose c chosen so that $l/c\,|\,1-z\,| < 1$, we have convergence. Uniform convergence in any closed (finite) region not including $z = 1$ follows at once. This proves the theorem.

We remark that the possibility of $z = 1$ being a singular point is not illusory. Consider for example $\zeta(z) = z$. Here $\zeta(n) = n$, so that $f(z) = \sum_0^\infty n z^n = z/(1-z)^2$.

Related to the polynomials $P_n(z)$ is another set $Q_n(z)$ defined by $Q_0(z) = 1$, $zQ_n(z) = P_{n+1}(z)$, $n > 0$, which has some interesting properties. We state a few of these properties:

1) $Q_n(z)$ is a real polynomial of degree n, with positive integral coefficients.

2) $Q_n(z)$ satisfies the equation

$$(10) \qquad Q_n(z) = z(1-z)Q'_{n-1}(z) + (nz+1)Q_{n-1}(z), \; n > 0.$$

3) $Q_n(z)$ is a reciprocal polynomial. (That is, for every zero ξ there is a zero $1/\xi$.)

3′) $Q_{2n+1}(-1) = 0$.

4) The zeros of $Q_n(z)$ are real, simple, and (except for the zero $z = -1$ when n is odd) irrational; they separate the zeros of $Q_{n-1}(z)$; and they all lie in the interval $-\infty < z < 0$.

THE UNIVERSITY OF MICHIGAN.

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* Due to Poincaré. Cf. Borel, *Fonctions Entières*, pp. 53-55.

† l is the larger of $|\,z\,|$ and 1.

Optics in Space of Constant Non-Vanishing Curvature.

By James Pierpont

1. *Introduction.* In a paper presented at the last meeting of the Society (Dec. 1926)* I showed that in such a space central optical collineation gives only an image congruent with the object. As is well known the theory of collineation enables one to establish a body of theorems of great importance in e-optics † and the fact that this theory is not available in E- and H-spaces would prove a great bar to our study of optics in such spaces if some other general method were not at hand. It is the purpose of this paper to outline a method which is partially successful, in the hope that others, more competent, may greatly improve thereon. In any case the method as here developed reaches results that can not be obtained by other means as far as known to the author.

The essential difference between this and the foregoing paper lies in this: In the former, light was regarded from the standpoint of rays, here it is regarded as a wave phenomenon.

I was led to take this view by a very remarkable memoir ‡ by C. S. Hastings "On Certain New Methods and Results in Optics" and this present paper must be regarded as an attempt to extend Hastings' methods with more or less success to space of constant non-vanishing curvature. I shall however for the sake of clearness speak only of H-space but it will be seen that the methods hold for E-space as well.

2. *Hyperbolic Geometry.* As one can define the metric of H-geometry in a variety of ways we will begin by recalling the particular form adopted here and then develop a few results needed in the optical part of this paper. Let x, y, z be three real variables which we regard as defining a point. Let $R > 0$ a constant

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* "Optics in Hyperbolic Space." The method there used can be applied without trouble to elliptic space.

† For *e*-, *H*-, *E*- read euclidean, hyperbolic, elliptic.

‡ *Memoirs of the National Academy of Sciences*, Vol. 6 (1893), p. 37. Reprinted with slight changes in his book *Light*, Yale Bicentennial Publications. Scribners, New York, 1901.

$$r^2 = x^2 + y^2 + z^2, \quad \lambda = 4R^2 - r^2 > 0, \quad ds^2 = dx^2 + dy^2 + dz^2.$$

The metric is then defined by

$$(1) \qquad d\sigma = 4R^2 ds/\lambda.$$

This is Riemann's definition. To form a concrete picture of abstract H-geometry we may regard x, y, z as ordinary cartesian coordinates of a point. Then the points of H-space are represented in the model by the points lying within the e-sphere $\lambda = 0$. For points near the origin 0 we see $d\sigma = ds$ nearly, a fact we shall employ.

H-planes are represented in the model by e-spheres cutting $\lambda = 0$ orthogonally. The intersection of two H-planes is an H-straight. In particular H-planes through O are in the model also e-planes. H-spheres are e-spheres in the model whose centers are not those of the H-spheres unless at the origin. Angles in H-geometry are the same as the corresponding angles in the model. The model is thus conformal, a fact we largely employ in the sequel. Bodies in H-space may be moved about freely such that the distance $d\sigma$ between adjacent points remains unaltered. This fact is most useful as it enables us to move a certain point to the origin O when H-planes and straights through it become e-planes and straights while angles remain unaltered. From (1) we find that the H-length σ of a straight segment OP, $P = (xyz)$ is given by

$$(2) \qquad s = 2R \tanh (\sigma/2R).$$

If s receives the increment ds

$$(3) \qquad ds = \operatorname{sech}^2 (\sigma/2R) \cdot d\sigma.$$

The H-length of an arc ϕ of an H-circle K, center A, of radius ρ is

$$(4) \qquad \sigma = R\phi \sinh (\rho/R).$$

Its curvature

$$(5) \qquad \Gamma = d\phi/d\sigma = 1/R \sinh (\rho/R).$$

K regarded as an e-circle has its center A_1 on the straight OA, its radius call r. Let this straight cut K in B and C. Let

$$\begin{aligned} OA_1 &= a, & OB &= b, & OC &= c, \text{ in e-measure,} \\ &= \alpha, & &= \beta, & &= \gamma, \text{ in H-measure.} \end{aligned}$$

Then by (2)

$$2r = c - b = 2R\,\{\tanh(\gamma/2R) - \tanh(\beta/2R)\}, \qquad \gamma - \beta = 2\rho.$$

Hence
$$r = \frac{R \sinh(\rho/R)}{\cosh(\gamma/2R)\cosh(\beta/2R)}\;.$$

If K passes through O, $\beta = 0$, $\gamma = 2R$, whence

$$r = R \tanh(\rho/R). \tag{6}$$

In the following we shall deal with H-spherical light waves and spherical lenses which pass through or near O, the relation (6) is thus true if we neglect small quantities of order greater than 1 which we do throughout and which is also done in this kind of work in e-optics.

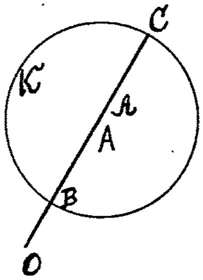

FIG. 1.

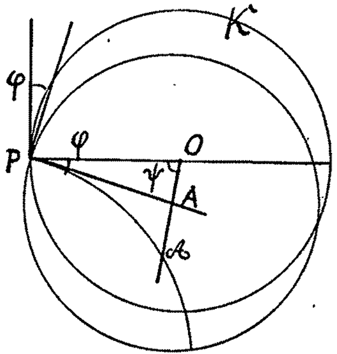

FIG. 2.

In figure 2 an H-circle K of radius $\rho(r)$* and center O is turned through the angle ϕ at P. Let the new circle have e-radius r'. We need to know how much r' differs from r where ϕ is small.

Let $POA = \psi$, $\psi' = 90 - \psi$, $PA = s$. Then
$$\tan\psi' = \cosh(\rho/R)\tan(\phi/2)$$

$$s = \frac{r\cos\psi'}{\cos(\phi - \psi')} = \frac{r}{\cos\phi + \sin\phi\tan\psi'}\;.$$

Hence
$$s = \frac{r}{\cos\phi + \sin\phi\tan(\phi/2)\cosh(\rho/R)}\;.$$

If ϕ is small

$$s = \frac{r}{1 + (\phi^2/2)\cosh(\rho/R)}\;. \tag{7}$$

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* Means, radius is ρ in H-measure and r in e-measure.

Hence for small enough ϕ we may take $s = r$.
In figure 3, K is an H-circle whose center is O and radius $\rho(r)$. We turn K about P through an angle ϕ getting K' whose e-center Ω has coordinates $(-\alpha, -\beta)$. The straight OQ makes the small angle θ with OP and cuts K in Q_1; the segment QQ_1, has length $\delta(d)$. The H-length of PQ is σ. We wish to establish the important relation

$$(8) \qquad \delta = \sigma \tan \phi,$$

neglecting small quantities of order > 1.

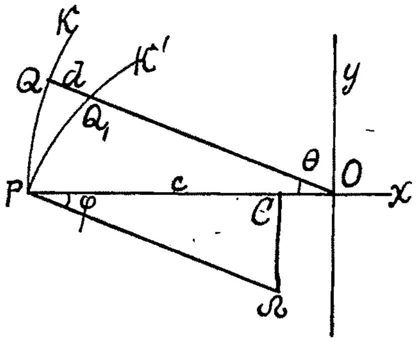

FIG. 3.

To this end regard K' as an e-circle obtained by rotating K through the angle ϕ. Then $PQ = P\Omega = r$. Let $PC = c$. The equation of K' is

$$(x + \alpha)^2 + (y + \beta)^2 = r^2.$$

The equation of OQ is $y = -x \tan \theta = -x\theta$.

Thus the x coordinate of Q_1 in numerical value is

$$r_1 = r - \beta\theta(r/c) = r - r\theta \tan \phi.$$

The projection of QQ_1 on the x-axis is $d \cos \theta = r - r_1 = r\theta \tan \phi$. Thus neglecting θ^2, etc.,

$$(9) \qquad d = r\theta \tan \phi.$$

We obtain (8) from (9) as follows. Suppose $P\Omega = r$, nearly, as it certainly is if ϕ is small.

By (3) $\qquad d = \delta \operatorname{sech}^2 (\rho/2R)$.

By (4) $\qquad \theta = \sigma/R \sinh (\rho/R)$, $\qquad$ by (2) $\quad r = 2R \tanh (\rho/2R)$.

These in (9) give (8) at once.

3. *Relations between Curvatures.* In figure 4, LML' is a cross-section of a spherical lense of *H*-curvature Λ. A spherical wave meets the lense and for small angles of incidence, is bent into another spherical wave. Let Γ, Γ' be the H-curvatures of the wave fronts LWL', $LW'L'$. Thus while the incident wave front travels from M to W the refracted wave travels from M to W'. We have then the relation

$$(10) \qquad MW' = nMW, \quad n \text{ index of refraction.}$$

But $MW' = MN - NW'$, $MW = MN - NW$. Hence

$$(11) \qquad MN - NW' = n(MN - NW).$$

We may regard these H-spheres or rather H-circles as e-circles whose e-curvatures are L, C, C' respectively. Then setting $LN = p$ in e-measure

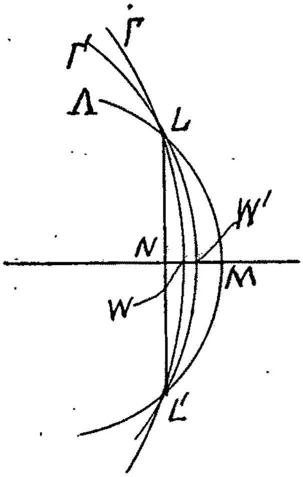

Fig. 4.

$$NM = \tfrac{1}{2}p^2L, \quad NW = \tfrac{1}{2}p^2C, \quad NW' = \tfrac{1}{2}p^2C'.$$

These in (11) give

$$(12) \qquad L - C' = n(L - C) \text{ or } C' = mL + nC, \quad (m = 1 - n).$$

From (12) we have denoting the radii of these circles by r', l, r respectively,

$$1/r' = m/l + n/r.$$

Let us now pass back to H-space. If we take our origin O at or near M we have using (6)

$$\coth(\rho'/R) = m\coth(\lambda/R) + n\coth(\rho/R) \tag{13}$$

where ρ', λ, ρ are the radii of the above circles regarded as H-circles. By giving C, C', L in (12) appropriate signs the relation (12) is valid for all cases. For example, a convex lense with n such that C, C' have opposite curvatures we have $C' = mL - nC$, whence

$$\coth(\rho'/R) = m\coth(\lambda/R) - n\coth(\rho/R). \tag{14}$$

Let us compare this result with that given in my December paper mentioned above for a particular case:

$$R = 1000, \quad a = 600, \quad r = 10, \quad n = 2/3, \quad \phi = 30'.$$

We had

$$\cos\chi = \cos(\alpha - \beta)\cos\theta + \sin(\alpha - \beta)\sin\theta\cosh(\sigma/R)$$

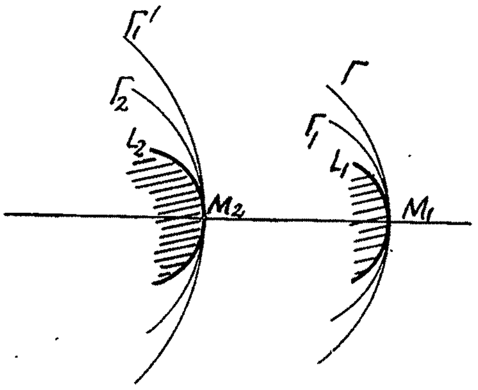

FIG. 5.

or setting $\cosh(\sigma/R) = 1 + C'$, and noting $\alpha - \beta - \theta = \phi - \beta$

$$\cos\chi = \cos(\phi - \beta) + C'\sin\theta\cdot\sin(\alpha - \beta) = \cos(\phi - \beta) + E,$$

$$\sinh(\eta/R) = \frac{\sin(\alpha - \beta)}{\sin\chi}\cdot\sinh(\sigma/R).$$

We find:

$$\theta = .0001479073, \quad \alpha = \theta + \phi = .008874553, \quad \beta = .0059163688,$$
$$\delta = \phi - \beta = .002810277, \quad \alpha - \beta = .002958184,$$
$$\sigma = 608{,}06, \quad C' = .1906, \quad E = 8{,}3.10^{-8},$$
$$\chi = .002780641, \quad \eta = 642.37, \quad \rho' = \eta - \sigma = 34{,}3.$$

If we use (14) we get $\rho' = 34{,}3$ in perfect agreement.

In figure 5 the spherical wave front of curvature Γ strikes the interface of a lense L_1 and is bent into the spherical wave front of radius ρ_1, and curvature Γ_1. It then strikes the interface L_2. If η_1 is the H-distance M_1M_2 the radius of the wave front at L_2 is $\rho_1 - \eta_1$ hence its curvature is

$$\Gamma_1' = \frac{1}{R \sinh[(\rho_1 - \eta_1)/R]} = \frac{1}{\sinh(\rho_1/R) - (\eta_1/R)\cosh(\rho_1/R)},$$

since η_1/R is small. Hence

$$(15) \qquad \Gamma_1' = \frac{\Gamma_1}{1 - \eta_1\Gamma_1 \cosh(\rho_1/R)} = g_2\Gamma_1; \quad 1/g_2 = 1 - \eta_1\Gamma_1 \cos(\rho_1/R).$$

The lense interface L_2 gives Γ_1' the curvature Γ_2 etc.

To get relations that are easily handled we may regard these wave fronts as e-spheres whose curvatures are C, C_1, C_1', C_2. Let $M_1M_2 = h_1$ in e-measure. The radius of the first refracted wave front at M_1 is $1/C_1$, at M_2 its radius is $(1/C_1) - h_1 = C_1/(1 - h_1C_1)$. Thus its curvature at M_2 is

$$C_1' = \mu_2 C_1, \qquad \mu_2 = 1/(1 - h_1C_1).$$

We have then analogous to (12) for a set of lenses

$$(16) \qquad \begin{aligned} C_1 &= m_1L_1 + n_1\mu_1C \\ C_2 &= m_2L_2 + n_2\mu_2C_1 \\ C_3 &= m_3L_3 + n_3\mu_3C_2, \text{ etc.} \end{aligned}$$

Here $\mu_1 = 1$ is set in for symmetry, n_k is the index at the kth interface L_k $m_k = 1 - n_k$, $1/\mu_k = 1 - h_{k-1}C_{k-1}$, h_k is the e-distance between the faces $L_{k-1}L_k$, L_k is the e-curvature of the length L_k. It is understood that the curvatures are taken with their proper sign. The relations (16) are the (a) of Hastings' paper. They are fundamental in his theory.

As an example, let us consider a thin convex lense in air. The lense being thin we may take $h_1 = 0$. Then at the first interface L_1,

$$C_1 = n_1C - m_1L_1.$$

At the second interface L_2,

$$C_2 = m_2L_2 - n_2C_1.$$

Set $n_1 = n$, then $n_2 = 1/n$, $m_2 = (n-1)/n$, hence

$$C_1 + C_2 = (n-1)/n(L_1 + L_2), \; 1/r_1 + 1/r_2 = [(n-1)/n](1/l_1 + 1/l_2)$$

or by (6)

$$(17) \quad \coth(\rho_1/R) + \coth(\rho_2/R) = [(n-1)/n][\coth(\lambda_1/R) + \coth(\lambda_2/R)].$$

Here λ_1, λ_2 are the radii of the two faces of the lense; if ρ_1 is the distance of the source of light, ρ_2 is the distance of the image, all in H-measure. The extension of this result to a system of thin lenses in contact is obvious. It is hard to see how such results can be obtained by the trigonometric calculations of rays as in my first paper.

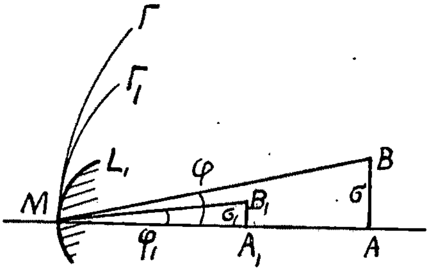

FIG. 6.

4. *Magnification. The Sine Law.* In figure 6, the small object AB of length σ becomes A_1B_1 of length σ_1 after refraction. The wave front of radius ρ and curvature Γ, has radius ρ_1 and curvature Γ_1 after refraction. We have from the right H-triangle

$$\sinh(\sigma/R) = \sin\phi \sinh(\rho/R) \quad \text{or} \quad \sigma/R = \phi \sinh(\rho/R).$$

Similarly $\sigma_1/R = \phi_1 \sinh(\rho_1/R)$. Hence

$$\sigma_1/\sigma = \phi_1/\phi \cdot \sinh(\rho_1/R)/\sinh(\rho/R).$$

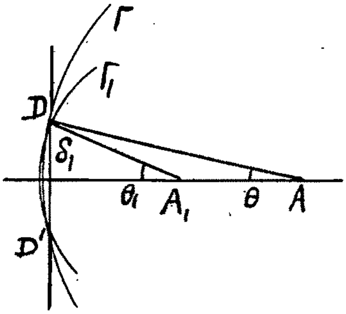

FIG. 7.

Now $\theta_1/\theta = n$ the index of refraction, hence by (15)

$$\sigma_1/\sigma = n_1(\Gamma/\Gamma_1) = n_1 g_1(\Gamma/\Gamma_1), \quad g_1 = 1,$$

g_1 being introduced for symmetry. After a 2nd refraction we have similarly

$$\sigma_2/\sigma_1 = n_2 g_2 (\Gamma_1/\Gamma_2),$$

and after p refractions

$$\sigma_p/\sigma_{p-1} = n_p g_p (\Gamma_{p-1}/\Gamma_p).$$

Here $g_2, g_3, \cdots$ are as in (15). Multiplying these, gives as magnification

$$M = \sigma_p/\sigma = GN(\Gamma/\Gamma_p) \qquad (G = g_1 g_2 \cdots g_p,\ N = n_1 n_2 \cdots n_p). \tag{18}$$

For a thin lense system we may take $G = 1$. Suppose we have a small diaphragm DD' of diameter $2\delta_1$ at the first lense as in figure 7. Then if $A_1D = \rho_1$, $AD = \rho$.

$$\sin\theta = \sinh(\delta_1/R)/\sinh(\rho/R) = \delta_1\Gamma$$

$$\sin\theta_1 = \delta_1\Gamma_1.$$

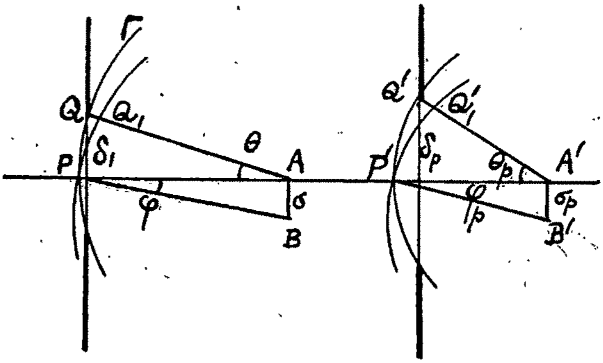

FIG. 8.

Hence

$$\sin\theta_1/\sin\theta = \Gamma/\Gamma_1 = g_1(\Gamma/\Gamma_1), \quad g_1 = 1.$$

Similarly diaphragms $\delta_2 \cdots \delta_p$ at the other interfaces give

$$\sin\theta_1/\sin\theta_2 = g_2(\Gamma_1/\Gamma_2), \cdots, \quad \sin\theta_{p-1}/\sin\theta_p = g_p(\Gamma_{p-1}/\Gamma_p).$$

Multiplying these, gives

$$\sin\theta/\sin\theta_p = G(\Gamma/\Gamma_p).$$

This with (18) gives

$$\sigma_p/\sigma = N \sin\theta/\sin\theta_p = N\theta/\theta_p, \quad \theta,\ \theta_p \text{ small.} \tag{19}$$

This is the celebrated sine law of Lagrange.

Hastings by an ingenious consideration shows that (19) holds when θ, θ_p are not small, provided the incident and emergent wave fronts are spherical, i. e., the instrument is free from spherical aberration. In figure 8, $P\,Q$ is the incident wave front issuing from A and $P\,Q_1$ that issuing from B the other extremity of a small object σ.

In the other half of the figure $P'Q'$, $P'Q'_1$ have a similar meaning. The incident wave front is limited by a diaphragm of semi-diameter δ_1. The diaphragm δ_p is the optical image of δ_1. Then Q_1, Q'_1 are corresponding points and hence setting $QQ_1 = \epsilon$, $Q'Q'_1 = \epsilon_p$ in H-measure,

$$(20) \qquad \epsilon_p = N\epsilon$$

since the velocity of light in the last medium is N times that in the first. To prove (20) let us refer to figure 3 which we suppose first is euclidean. Let $e = QQ_1$ in e-measure. The equation of the e-circle K' is as before, while

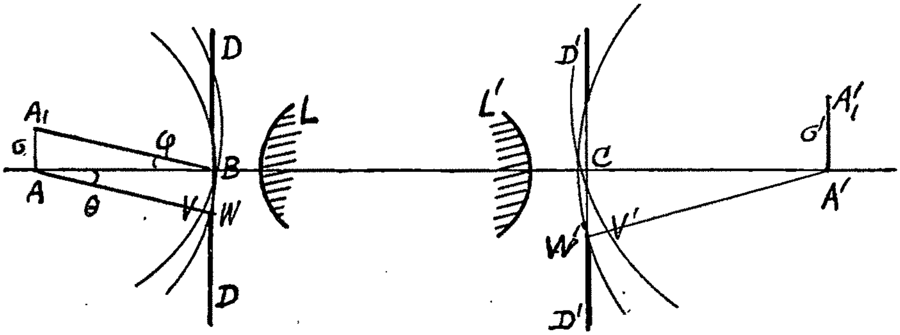

FIG 9.

the equation of the straight OQ is $y = -x\tan\theta$ where now θ is not small. The x coordinate of Q_1 is now in numerical value $r_1 = r\cos - \frac{1}{2}r\phi\sin 2\theta$. The projection of QQ_1 on the x axis is

$$e\cos\theta = r\cos\theta - r_1 = \tfrac{1}{2}r\phi\ \sin 2\theta.$$

Hence $$e = r\phi\sin\theta.$$

Using as before (2), (3), (4), $$\epsilon = \sigma\sin\theta.$$

Similarly $$\epsilon_p = \sigma_p\sin\theta_p.$$

These with (20) give

$$\epsilon_p/\epsilon = N = \sigma_p\sin\theta_p/\sigma\sin\theta.$$

Hence

$$\sigma_p/\sigma = N \sin\theta/\sin\theta_p, \tag{21}$$

which is (19) with this difference that θ, θ_p are not restricted to small values.

5. *Telescopes, Microscopes.* In figure 9, L and L' are object glass and ocular of a telescope or microscope. In front of L is a small diaphragm of semi-diameter δ, behind L' is another of semi-diameter δ', so placed that D' is the optical image of D. The extremities of $A\,A_1$ of an object of length σ send out light waves which at B are inclined at an angle ϕ. We are now under the condition of figure 3. Thus by (8)

$$VW = BW \tan\phi = \delta \tan\phi.$$

Similarly on the right we have

$$V'W' = B'W' \tan\phi' = \delta' \tan\phi'.$$

Now N being the ratio of the velocity of light at the two ends $V'W' = N \cdot VW$. Thus

$$\tan\phi'/\tan\phi = N(\delta/\delta'). \tag{22}$$

Let us now suppose figure 9 represents a telescope. Let $AB = \alpha$, $BC = \tau$, $\alpha + \tau = \beta$. Let σ subtend an angle ϕ at B, an angle ψ at C, and an angle ϕ' as seen through the instrument. We define the magnification afforded by the instrument by

$$M = \phi'/\psi. \tag{23}$$

Now from figure 9

$$\sigma = R\phi \sinh(\alpha/R) = R\psi \sinh(\beta/R).$$

Thus using (22)

$$M = \frac{\phi'}{\phi} \cdot \frac{\phi}{\psi} = N \frac{\delta}{\delta'} \, \frac{\sinh(\beta/R)}{\sinh(\alpha/R)}. \tag{24}$$

For a telescope $N = 1$ also τ/R is small, and hence we can take $\alpha = \beta$. Thus

$$M = \delta/\delta'.$$

Let us take the diaphragm δ to be the edge of the object glass, and δ' its ocular image. Hence Ramsden's rule

$$M = \frac{\text{diameter of objective}}{\text{diameter of its ocular image}} \tag{25}$$

holds for E- and H-spaces.

For a microscope in e-space the magnification is taken as

$$M = \frac{\text{size as seen through instrument}}{\text{size as seen at distance 10 inches}}$$

As we know nothing about the eyes of beings in other spaces than our own, let us suppose ω is the distance of easy vision and Σ is the apparent size of the object as seen through the instrument, coming from the distance ω. We may take as magnification

$$M = \frac{\tanh(\Sigma/R)}{\tanh(\sigma/R)} \tag{26}$$

But $\tanh(\Sigma/R) = \sinh(\omega/R)\tan\phi'$, $\tanh(\sigma/R) = \sinh(\omega/R)\tan\psi$.

Thus

$$M = \frac{\tan\phi'}{\tan\psi} = \frac{\tan\phi'}{\tan\phi}\cdot\frac{\tan\phi}{\tan\psi} = N\cdot\frac{\delta}{\delta'}\cdot\frac{\tan\phi}{\tan\psi}$$

Now

$$\frac{\tan\phi}{\tan\psi} = \frac{\sigma}{\alpha}\cdot\frac{\omega}{\sigma}\text{, approximately.}$$

Hence

$$M = N\,\frac{\delta}{\delta'}\cdot\frac{\omega}{\alpha}\text{, for a microscope.} \tag{27}$$

There are other results which may be obtained from Hastings' memoir under certain restrictions, we will however stop at this point.

On an Imprimitive Group of Order 5184.*

By J. R. Musselman.

In the study of collineation groups certain types of groups which, in the plane, have a fixed triangle and, in space, have a fixed tetrahedron are classed as imprimitive. The existence of such groups is clear and their group structure is easily ascertained. Thus, from the group standpoint, the real problem consists of the determination of the primitive collineation groups. As a result the configurations determined by primitive groups have been studied rather completely and those determined by imprimitive groups have been rather neglected. The object of this paper is to discuss a configuration of 108 lines, 216 points and 216 planes with which is associated an imprimitive group of order 5184. Connected with this configuration are eight cubic surfaces, eighteen quadric surfaces, and fifty-four tetrahedra, whose geometrical properties are described, as well as the group of order 5184 which leaves the configuration invariant.

1. *The Bilinear Invariant of Two Cubics.* Two binary cubics

$$a_0x_0^3 + 3a_1x_0^2x_1 + 3a_2x_0x_1^2 + a_3x_1^3$$

and

$$a_0'x_0^3 + 3a_1'x_0^2x_1 + 3a_2'x_0x_1^2 + a_3x_1^3$$

have the bilinear invariant

$$a_0a_3' - 3a_1a_2' + 3a_2a_1' - a_3a_0'. \tag{1}$$

This invariant can be expressed in terms of the symmetric functions of the roots of the above two cubics as

$$3s_3' - s_2's_1 + s_1's_2 - 3s_3 \tag{2}$$

where $\qquad s_3' = a_3'/a_0', \qquad s_3 = a_3/a_0,$ etc.

If we designate the roots of the two cubics as α_0, α_1, α_2 and β_0, β_1, β_2 respectively it is well known that the invariant (1) can be expressed in terms of the differences of the roots as

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* Presented to the American Mathematical Society, Dec. 29, 1924 and May 7, 1927.

$$\sum^{6} (\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2).$$

It is unnecessary, however, to use all six terms of the above sum of differences to express the function (2). For it is evident that the function

$$\gamma \equiv (\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) + (\alpha_0 - \beta_1)(\alpha_1 - \beta_2)(\alpha_2 - \beta_0)$$
$$+ (\alpha_0 - \beta_2)(\alpha_0 - \beta_0)(\alpha_2 - \beta_1)$$

when expressed in terms of the symmetric functions of the α_i and β_i ($i = 0, 1, 2$) is precisely (2) itself.

This fact suggests two questions—given two n-ics how many terms of differences of their roots it is necessary to use to form the bilinear invariant when expressed in symmetric functions of the roots of the n-ics; and, secondly, given one term of differences, how can we select the others that must go with it to form the required function?

A little investigation will show that in the general case of two binary n-ics the number of terms of differences of the roots necessary to form the function (2) is the least common multiple of the binomial coefficients of the n-ic. For the case $n = 2$, it is necessary to use both terms; for the case $n = 3$, the number of terms of differences of the roots necessary to form (2) is only three.

The answer to the second question will be discussed elsewhere for the general n-ics. For the case $n = 3$, given any term as $(\alpha_0 - \beta_0)(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)$ to obtain the other terms which together with it produce the function (2) we must keep the α_i fixed and permute the β_i cyclically. Also if we had kept the β_i fixed and permuted the α_i cyclically we likewise should have obtained the function γ. *Hence given any term of differences, to obtain the two terms to go with it, carry out on the given term, the even permutations of the symmetric* $G_{3!}$ *of either set of the roots.*

The three remaining terms of differences

$$\gamma' \equiv (\alpha_0 - \beta_0)(\alpha_1 - \beta_2)(\alpha_2 - \beta_1) + (\alpha_0 - \beta_2)(\alpha_1 - \beta_1)(\alpha_2 - \beta_0)$$
$$+ (\alpha_0 - \beta_1)(\alpha_1 - \beta_0)(\alpha_2 - \beta_2)$$

will likewise produce the function (2). It is evident that the two functions γ and γ' are formally multiply isomorphic with the $G_{3!}$ on either set of roots; since both γ and γ' are unaltered by $\{1, (\alpha_0\alpha_1\alpha_2), (\alpha_0\alpha_2\alpha_1)\}$; and are interchanged by the permutations $(\alpha_0\alpha_1)$, $(\alpha_0\alpha_2)$, $(\alpha_1\alpha_2)$. We have here an illustration of a $3 - 1$ isomorphism with the symmetric $G_{3!}$

2. *The Cubic Surface and a* G_{36}. The purpose of this paper is to study the equation $\gamma = 0$ or $\gamma' = 0$. Let us write $\gamma = 0$ in the more familiar form

$$(x_1 - x_4)(x_2 - x_5)(x_3 - x_6) + (x_1 - x_5)(x_2 - x_6)(x_3 - x_4) \\ + (x_1 - x_6)(x_2 - x_4)(x_3 - x_5) = 0.$$

In projective coordinates, this represents a cubic manifold in space of five dimensions, with in evidence, 27 planes on it, such as $x_1 - x_4 = x_2 - x_6 = x_3 - x_5 = 0$, which planes all pass through the point (1, 1, 1, 1, 1, 1). If we take a section of the manifold by the S_3 $x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0$ we obtain thereby a cubic surface in S_3 which is invariant at least under a G_{72}, generated by the $G_{3!}$ on x_1, x_2, x_3; the $G_{3!}$ on x_4, x_5, x_6; and the G_2 which interchanges x_1, x_4; x_2, x_5; and x_3, x_6.

The equation of the cubic surface in S_3 will now take the form

$$\text{(3)} \qquad \begin{cases} x_1x_2x_3 = x_4x_5x_6 \\ x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0. \end{cases}$$

For our purpose it will be sufficient to find the fixed points, fixed lines and fixed planes of the G_{36}, generated by the two symmetric $G_{3!}$'s. In this G_{36} are 15 collineations of the type $(\beta_1\beta_2)$ which leave invariant 6 points and 9 lines, such that 3 lines are on each point. They also leave invariant 6 planes and 9 other lines, such that 3 lines are in each plane. The 6 points and the 6 planes are on two other lines. The coordinates of the 6 points are as follows:

p_0: $(0, 1, -1, 0, 0, 0)$ $\qquad$ q_0: $(0, 0, 0, 0, 1, -1)$
p_1: $(1, 0, -1, 0, 0, 0)$ $\qquad$ q_1: $(0, 0, 0, 1, 0, -1)$
p_2: $(1, -1, 0, 0, 0, 0)$ $\qquad$ q_2: $(0, 0, 0, 1, -1, 0)$

To write the equations of the lines and planes compactly, let us write x_0, x_1, x_2 for x_1, x_2, x_3 and y_0, y_1, y_2 for x_4, x_5, x_6 and take subscripts modulo 3. Then the 6 planes can be written

$$P_i: x_{i+1} - x_{i-1} = 0 \qquad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$$
$$Q_i: y_{i+1} - y_{i-1} = 0 \qquad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$$

The 9 lines on the 6 points can be written

$$\text{(4)} \quad \overline{p_iq_k}: x_{i+1} + x_{i-1} = y_{k+1} + y_{k-1} = 0 \quad x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = 0.$$

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The 9 lines on the 6 planes can be written

$$(5)\quad \overline{P_iQ_k}:\ x_{i+1}-x_{i-1}=y_{i+1}-y_{i-1}=0,\ x_0+x_1+x_2=y_0+y_1+y_2=0.$$

We shall denote the first set of 9 lines by a, the second by a'.

In the G_{36} are 4 collineations of the type $(\beta_1\beta_2\beta_3)$ which leave invariant two lines and four points.

$$L_1:\ x_0-x_1=x_1-x_2=x_2-x_0=0\quad x_0+x_1+x_2=y_0+y_1+y_2=0.$$

$$L_2:\ y_0-y_1=y_1-y_2=y_2-y_0=0\quad x_0+x_1+x_2=y_0+y_1+y_2=0.$$

$$r:\ (0,0,0,\omega,\omega^2,1)\qquad t:\ (\omega,\omega^2,1,0,0,0)$$

$$s:\ (0,0,0,\omega^2,\omega,1)\qquad u:\ (\omega^2,\omega,1,0,0,0).$$

On L_1 are the points q_i, the planes P_i and the points r and s are the Hessian points of the set q_i; similarly on L_2 are the points p_i, the planes Q_i and the points t and u are the Hessian points of the set p_i.

In the G_{36} are 4 collineations of the type $(\beta_1\beta_2\beta_3)(\alpha_1\alpha_2\alpha_3)$, which have as fixed lines the four lines $\overline{rt}$, $\overline{ru}$, $\overline{st}$ and $\overline{su}$. The remaining 12 collineations of the type $(\beta_1\beta_2\beta_3)(\alpha_1\alpha_2)$ have as fixed points, in addition to the ten already named, six others lying on L_1 and L_2; which points are the cubicovariant points of the sets of p_i and q_i respectively. Moreover, the 9 lines a are the joins of the p_i and q_i; the 9 lines a' are the joins of the cubicovariant sets of p_i and q_i.

3. *The 27 Lines of the Surface.* It is evident that the 9 lines a are contained in the surface (3). The equations of the remaining 18 lines of the surface can be written down without difficulty. There is a second set of 9 lines which we shall call b whose equations are

$$(6)\qquad x_i-\omega^k y_i=x_{i+1}-\omega^k y_{i-1}=x_{i-1}-\omega^k y_{i+1}=0.\quad \Sigma\, x=\Sigma\, y=0$$
$$(i,k=0,1,2);\quad \omega^3=1.$$

These 9 lines meet by threes on 6 points which lie on two lines

$$L_3:\ x_0-\omega x_1=y_0-\omega y_2=0\qquad \Sigma\, x=\Sigma\, y=0$$

$$L_4:\ x_0-\omega x_2=y_1-\omega y_2=0\qquad \Sigma\, x=\Sigma\, y=0.$$

On L_3 are the points

$$(1,\omega^2,\omega,\omega,\omega^2,1);\ (\omega,1,\omega^2,\omega,\omega^2,1);\ (\omega^2,\omega,1,\omega,\omega^2,1).$$

On L_4 are the points

$$(1, \omega, \omega^2, \omega^2, \omega, 1); \quad (\omega, \omega^2, 1, \omega^2, \omega, 1); \quad (\omega^2, 1, \omega, \omega^2, \omega, 1).$$

Moreover the points r, and u are the Hessian points of the set on L_3; similarly s and t for the set on L_4.

The third set of 9 lines which we shall call c have for their equations

$$(7) \qquad x_0 - \omega^k y_{0+i} = x_1 - \omega^k y_{1+i} = x_2 - \omega^k y_{2+i} = 0 \qquad \Sigma x = \Sigma y = 0$$
$$(i, k = 0, 1, 2); \quad \omega^3 = 1.$$

These 9 lines meet by threes on 6 points which lie on two lines

$$L_5: \ x_0 - \omega x_2 = y_0 - \omega y_2 = 0 \qquad \Sigma x = \Sigma y = 0$$
$$L_6: \ x_0 - \omega x_1 = y_1 - \omega y_2 = 0 \qquad \Sigma x = \Sigma y = 0.$$

On L_5 are the points

$$(1, \omega, \omega^2, \omega, \omega^2, 1); \quad (\omega, \omega^2, 1, \omega, \omega^2, 1); \quad (\omega^2, 1, \omega, \omega, \omega^2, 1).$$

On L_6 are the points

$$(1, \omega^2, \omega, \omega^2, \omega, 1); \quad (\omega, 1, \omega^2, \omega^2, \omega, 1); \quad (\omega^2, \omega, 1, \omega^2, \omega, 1).$$

Moreover the points r and t are the Hessian points of the set on L_5; similarly s and u for the set on L_6.

Now the lines L_3, L_4, L_5 and L_6 together with L_1 and L_2 are the edges of a tetrahedron of which the points r, s, t, u are the vertices. On each edge is a set of three points of which the vertices on that edge are the Hessian points. The 27 lines of the surface (3) are the joins of the sets of three points on the opposite edges. Since the 27 lines of the surface meet by threes in a plane in 18 points, this surface must be an Eckardt surface.*

On the edges L_1 and L_2 of the tetrahedron we have also the cubicovariant sets. Complete the configuration by finding the cubicovariant points of the sets on L_3, L_4, L_5 and L_6. Join the cubicovariant sets on L_3 and L_4 by lines, also those on L_5 and L_6. We obtain in this way two sets of 9 lines each which we shall denote by b' and c' respectively. Their equations are

$$(8) \qquad b': \ x_i + \omega^k y_i = x_{i+1} + \omega^k y_{i-1} = x_{i-1} + \omega^k y_{i+1} = 0 \quad \Sigma x = \Sigma y = 0$$
$$c': \ x_0 + \omega^k y_{0+i} = x_1 + \omega^k y_{1+i} = x_2 + \omega^k y_{2+i} = 0 \quad \Sigma x = \Sigma y = 0$$
$$(i, k = 0, 1, 2); \quad \omega^3 = 1.$$

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* Eckardt, *Mathematische Annalen*, Vol. 10 (1876), p. 265.

We thus have 54 lines meeting by threes in a plane at 36 points on the edges of a tetrahedron. Any line is also met by two others not in a plane, six times; the points of intersection falling into two sets of three each, of which the two points on the edge of the tetrahedron are their Hessian points. On each line lie 6 points and also 6 planes giving altogether a set of $54 \cdot 6/3 = 108$ points and 108 planes. On any given plane lie 15 points and on any point are 15 planes.

From the 54 lines we can select 27 in four different ways, which 27 lines lie on a surface of the Eckardt type. Thus the lines

a, b, c are on $x_0x_1x_2 = y_0y_1y_2$

a, b', c' are on $x_0x_1x_2 = - y_0y_1y_2$

a', b, c' are on $(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)$
$$= - (y_0 - y_1)(y_1 - y_2)(y_2 - y_0)$$

a', b', c are on $(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)$
$$= (y_0 - y_1)(y_1 - y_2)(y_2 - y_0).$$

4. *The Configuration.* Let us start with a tetrahedron and determine the configuration from it. As the surface contained no constants, we have just 15 constants at our disposal; we shall use 12 of them by selecting the reference tetrahedron as our base. Select on any two opposite edges a set of three points such that the vertices are their Hessian points. This leaves one constant at our disposal and we shall select a similar set of three points on some other edge say L_3. The configuration must now be determined from the above conditions.

We have on L_1 points p_0: $(1, 1, 0, 0)$; p_1: $(\omega, 1, 0, 0)$; p_2: $(\omega^2, 1, 0, 0)$;

and on L_2 points q_0: $(0, 0, 1, 1)$; q_1: $(0, 0, \omega, 1)$; q_2: $(0, 0, \omega^2, 1)$.

The equations of the 9 lines joining the points on L_1 to those on L_2 are

$$a_{ik} = \overline{p_iq_k}: \quad x_1 - \omega^i x_2 = x_3 - \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2). \tag{10}$$

The cubicovariant set of points on L_1 are

p'_0: $(-1, 1, 0, 0)$; p'_1: $(-\omega, 1, 0, 0)$; p'_2: $(-\omega^2, 1, 0, 0)$

and the set on L_2 is

q'_0: $(0, 0, -1, 1)$; q'_1: $(0, 0, -\omega, 1)$; q'_2: $(0, 0, -\omega^2, 1)$.

The equations of the 9 lines joining these points are

$$(11) \qquad a'_{ik} = \overline{p'_i q'_k}: \quad x_1 + \omega^i x_2 = x_3 + \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

The set of points on L_3 are $(1, 0, 0, 1)$; $(\omega, 0, 0, 1)$; $(\omega^2, 0, 0, 1)$.

The pencil of quadrics having L_1 and L_2 as one set of generators and L_3 and L_4 as opposite generators can be written

$$x_1 x_3 = \lambda\, x_2 x_4.$$

The three quadrics of this pencil which contain as generators a_{00}, a_{01}, a_{02} are cut on L_4 by the opposite generator through $(1, 0, 0, 1)$ on L_3 in the points $(0, 1, 1, 0)$; $(0, \omega, 1, 0,)$; $(0, \omega^2, 1, 0)$ respectively. The joins of the sets on L_3 and L_4 give the 9 lines

$$(12) \qquad c_{ik}: \quad x_1 - \omega^i x_4 = x_2 - \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).$$

If we join the cubicovariant sets of the above points on L_3 and L_4 we obtain the 9 lines

$$(13) \qquad c'_{ik}: \quad x_1 + \omega^i x_4 = x_2 + \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).$$

Since the lines meet by threes through the intersection of a_{00} and c_{00} passes a line L_{00}. The quadric passing through this point of intersection $(1, 1, 1, 1,)$ and having L_1, L_2 and L_5, L_6 as opposite sets of generators is $x_1 x_4 = x_2 x_3$. The generator through $(1, 1, 1, 1)$ cuts L_5 in $(1, 0, 1, 0)$ and L_6 in $(0, 1, 0, 1)$. Similarly generators through the intersections of a_{00} and c_{11}, of a_{00} and c_{22} cut on L_5 the points $(\omega, 0, 1, 0)$; $(\omega^2, 0, 1, 0)$; on L_6 the points $(0, \omega, 0, 1)$; $(0, \omega^2, 0, 1)$ respectively. The joins of the sets determined thus on L_5 and L_3 can be written as

$$(14) \qquad b_{ik}: \quad x_1 - \omega^i x_3 = x_2 - \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

If we join the cubicovariant sets of these points on L_5 and L_6 we obtain the 9 lines

$$(15) \qquad b'_{ik}: \quad x_1 + \omega^i x_3 = x_2 + \omega^k x_4 = 0 \qquad (i, k = 0, 1, 2).$$

Now the 27 lines a'_{ik}, b'_{ik}, c'_{ik} are lines of the cubic surface

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

Similarly the sets of 27 lines

$$a'_{ik}, b_{ik}, c_{ik}; \quad a_{ik}, b'_{ik}, c_{ik}; \quad a_{ik}, b_{ik}, c'_{ik}$$

are lines of the surfaces

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = 0; \qquad x_1^3 - x_2^3 + x_3^3 - x_4^3 = 0;$$
$$\text{and} \quad x_1^3 - x_2^3 - x_3^3 + x_4^3 = 0 \text{ respectively.}$$

Further, if we join the points p_i on L_1 to the cubicovariant points q'_i on L_2, and similarly for the other 5 sets, we shall obtain a second set of 54 lines, 108 points and 108 planes possessing the same properties as the first set. We shall write the equations of the 6 sets of 9 lines each as

$$\begin{aligned}
A_{ik}&: \quad x_1 - \omega^i x_2 = x_3 + \omega^k x_4 = 0\\
A'_{ik}&: \quad x_1 + \omega\, x_2 = x_3 - \omega^k x_4 = 0\\
B_{ik}&: \quad x_1 - \omega^i x_3 = x_2 + \omega^k x_4 = 0\\
B'_{ik}&: \quad x_1 + \omega^i x_3 = x_2 - \omega^k x_4 = 0\\
C_{ik}&: \quad x_1 - \omega^i x_4 = x_2 + \omega^k x_3 = 0\\
C'_{ik}&: \quad x_1 + \omega^i x_4 = x_2 - \omega^k x_3 = 0 \qquad (i, k = 0, 1, 2).
\end{aligned}$$

The sets of 27 lines A_{ik}, B_{ik}, C_{ik}; A_{ik}, B'_{ik}, C'_{ik}; A'_{ik}, B_{ik}, C'_{ik}; and A'_{ik}, B'_{ik}, C_{ik} are lines of the surfaces

$$-x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \quad x_1^3 - x_2^3 + x_3^3 + x_4^3 = 0,$$
$$x_1^3 + x_2^3 - x_3^3 + x_4^3 = 0, \quad \text{and } x_1^3 + x_2^3 + x_3^3 - x_4^3 = 0$$

respectively.

5. *Quadric Surfaces on the Sets of* 54 *Lines.* There are nine quadric surfaces associated with the configuration such that each quadric contains 12 of the 54 lines a_{ik}, b_{ik}, c_{ik}, a'_{ik}, b'_{ik}, c'_{ik} $(i, k = 0, 1, 2)$. Starting with any line as a_{00}, we choose two other a_{ik} which do not meet it; we can choose either a_{11}, a_{22} or a_{12}, a_{21}. The other lines are then determined for with a_{00}, a_{11}, a_{22} are associated a'_{00}, a'_{11}, a'_{22}, b_{00}, b_{11}, b_{22}, b'_{00}, b'_{11}, b'_{22} which lie on the quadric surface $x_1x_4 = x_2x_3$; with a_{00}, a_{12}, a_{21}, are associated a'_{00}, a'_{12}, a'_{21}, c_{00}, c_{11}, c_{22}, c'_{00}, c'_{11}, c'_{22} which lie on the quadric $x_1x_3 = x_2x_4$. The number of quadrics will be $54 \cdot 2/12 = 9$. They are

$$(16) \qquad x_1x_4 = \omega^i x_2x_3, \quad x_1x_3 = \omega^i x_2x_4, \quad x_1x_2 = \omega^i x_3x_4 \qquad (i = 0, 1, 2).$$

Similarly the 54 lines A_{ik}, B_{ik}, C_{ik}, A'_{ik}, B'_{ik}, C'_{ik} $(i, k = 0, 1, 2)$ have associated with them nine quadric surfaces such that each surface contains 12 of the 54 lines as generators. The equations of the nine surfaces are

(17) $x_1x_4 = -\omega^i x_2x_3, \quad x_1x_3 = -\omega^i x_2x_4, \quad x_1x_2 = -\omega^i x_3x_4 \quad (i = 0, 1, 2).$

These eighteen quadrics are members of three pencils; the parameters of the pencil giving the equations of the quadrics in each case are the cube roots of positive and negative unity.

6. *Desmic Tetrahedra.* The whole configuration of 108 lines, 216 points and 216 planes can best be pictured by noting that it forms 54 tetrahedra, the lines being the edges, the points being the vertices and the planes being the faces. The 54 tetrahedra fall naturally into 27 sets of two each, such that each set of two together with the reference tetrahedron forms a desmic set, i. e.: they are four-fold perspective. Thus the 24 points of the configuration on the four lines a_{00}, a'_{00}, A_{00}, A'_{00} fall into three sets of eight, such that each set with the reference tetraherdon is a desmic set; moreover the 24 planes of the configuration on the four lines are the faces of the tetrahedra. Thus on the 36 lines a_{ik}, a'_{ik}, A_{ik}, A'_{ik} $(i, k = 0, 1, 2)$ are all 216 points and 216 planes forming the 27 sets of desmic tetrahedra, and the remaining 72 lines from the other edges of the tetrahedra. Similarly on the 36 lines b_{ik}, b'_{ik}, B_{ik}, B'_{ik} and on the 36 lines c_{ik}, c'_{ik}, C_{ik}, C'_{ik} $(i, k = 0, 1, 2)$ are all 216 points and 216 planes giving the same 27 sets of desmic tetrahedra.

The triple tangent planes of an Eckardt surface consist of 18 planes in which the three lines of the surface meet in a point, and 27 planes in which the lines do not meet in a point. The 216 planes of the configuration are the triple tangent planes of the latter type of the eight Eckardt surfaces.

Any set of eight points which together with the reference tetrahedron forms a desmic set bears a simple relationship to the eight Eckardt surfaces. As the eight points separate into two tetrahedra, the eight surfaces fall into two sets of four; each set associated with one tetrahedron. Through each vertex of the tetrahedron pass three of the four surfaces in the set, and each vertex with its opposite face are pole and polar plane as to the one surface of the set not passing through that vertex.

7. *The Group of the Configuration.* An Eckardt surface is unaltered by a collineation group of order 648; each collineation leaves the reference tetrahedron invariant. The surface is determined when the reference tetrahedron is given and one triple tangent plane in which the lines of the surface do not meet in a point.* The G_{648} consists of the product of a G_{27}, which

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* Bobek, *Monatshefte für Mathematik*, Vol. 10 (1889), p. 156.

sends the given triple tangent plane into each of the 27 planes, and a G_{24}, which permutes the reference tetrahedron in all possible ways. If we add to this the G_8

$$(18) \qquad x_1' = x_1; \quad x_2' = \pm x_2; \quad x_3' = \pm x_3; \quad x_4' = \pm x_4,$$

which sends any one Eckardt surface into each of the eight surfaces we have the group of the configuration, an imprimitive group of order 5184.

The G_{5184} consists then of the product of a G_{24}, which permutes the reference tetrahedron in all possible ways, and a G_{216} which sends any one point (or plane) into the set of 216 points (or planes). The 18 quadric surfacs are conjugate under the group, each being invariant under a G_{288}. The 27 sets of desmic tetrahedra are conjugate under the G_{5184}; each is unaltered by a G_{192}. The permutative group on the 27 sets of desmic tetrahedra is

$$x_1' = x_1; \quad x_2' = \omega^i x_2; \quad x_3' = \omega^j x_3; \quad x_4' = \omega^k x_4;$$
$$(i, j, k = 0, 1, 2).$$

8. *The Point-Plane Duality.* On each edge of the reference tetrahedron are six points. The six planes, one through each point and the opposite edge, contain all 216 points of the configuration, 36 in each plane. There are thus six systems, of six planes each, containing the 216 points. It can be seen easily that the 36 points in any plane lie by six on 18 lines, the lines passing by six through the two vertices of the tetrahedron in the plane, and through the given point on the opposite edge. The 36 points fall into four sets of nine each, such that each set together with the above mentioned three tetrahedron points are the vertices of the flex triangles of a pencil of cubic curves. The configuration in such a plane is invariant under an imprimitive ternary G_{216}.*

Similarly all 216 planes of the configuration are on the six points on any edge of the tetrahedron, 36 on each point. There are thus six sets, of six points each, which have on them all 216 planes. It can be seen also that the 36 planes on any point lie by six on 18 lines, the lines lying by six in three planes,—the two faces of the tetrahedron on the point and the plane on the point through the opposite edge.

9. *The Imprimitive Ternary* G_{216}. On any two sides of the reference triangle in the plane choose a set of points such that the vertices of the triangle on those sides are the Hessian points of the sets. Construct on both

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* This G_{216} is discussed in the following section.

sides also the cubicovariant points of each given set. Join now the six points on one side with the six points on the second side in all possible ways. These 36 lines will cut the third side of the triangle in six points having the same relation to it as the originally chosen sets have to the sides on which they are located. We have thus a set of 18 points and 36 lines such that three points are on each line and six lines on each point. The 36 lines will fall naturally into four sets of nine each, such that each set together with the reference triangle are the flex triangles of a pencil of cubic curves.

If we designate by a_i the points $0, \omega^i, 1$; by b_i the points $\omega^i, 0, 1$; by c_i the points $\omega^i, 1, 0$; $(i = 0, 1, 2)$ and the cubicovariant sets of the above respectively as a_i', b_i' and c_i' then

$$a_i', b_i', c_i' \text{ are the flexes of } x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0$$

$$a_i', b_i, c_i \text{ are the flexes of } -x_1^3 + x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$$

$$a_i, b_i', c_i \text{ are the flexes of } x_1^3 - x_2^3 + x_3^3 - 6mx_1x_2x_3 = 0$$

$$a_i, b_i, c_i' \text{ are the flexes of } x_1^3 + x_2^3 - x_3^3 - 6mx_1x_2x_3 = 0.$$

These 36 lines intersect in 36 points, not including the original points on the sides of the reference triangle. They are the vertices of the flex triangles of the four pencils of cubics. As such they must also lie on the harmonic polars of the cubics. We obtain thus a set of 36 points and 18 lines, such that three lines are on each point and six points on each line, the lines passing by six through the vertices of the reference triangle. This is the dual configuration of the 18 points and 36 lines.

The configuration consists of four Hesse configurations with a common triangle—the reference triangle which is invariant under the group. The G_4 $x_1' = x_1$; $x_2' = \pm x_2$; $x_3' = \pm x_3$ will send any one of the four Hesse configurations into each other so they are conjugate under the group. A Hesse configuration has associated with it a Hesse G_{216},* which contains a G_{54} leaving one flex triangle invariant. Hence the group of this configuration is the product of this G_{54} and the above G_4 which gives a G_{216}, obviously imprimitive.

A Hesse configuration is determined when one flex triangle, say the reference triangle, is given, and one point say $(1, 1, 1)$ of another flex triangle. The above configuration consequently is determined when the reference triangle is given as the common flex triangle and the four points $(1, \pm 1, \pm 1)$. Now the imprimitive ternary G_{216} consists of the product of a G_9: $x_1' = x_1$; $x_2' = \omega^i x_2$; $x_3' = \omega^j x_3$ $(i, j = 0, 1, 2)$ which leaves each Hesse configuration invariant; and a G_{24} which permutes the four points or the four configurations in all possible ways.

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* A detailed analysis of this group has been recently given by Winger, *University of Washington Publications*, June 1926.

If we take the general case of n Hesse configurations with a common triangle and all conjugate under a collineation group of order n generated by $x_1' = x_1$; $x_2' = \epsilon^i x_2$; $x_3' = \epsilon^j x_3$ where ϵ is an nth root of unity, the permutation group of these n Hesse configurations will have its maximum possible order when $n = 4$. Hence the above configuration is the one of most interest.

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Rational Involutorial Transformations in S_4 which Leave Invariant ∞^4 Quadric Varieties.

BY HOWARD CONWAY SHAUB.

1. *Introduction.* An involutorial transformation I, of a four dimensional space (x), is a birational transformation of period two. By I the points of (x) are associated in pairs P_1 and P_2, such that each is the image of the other in the transformation. If a linear space (x') can be found such that to a point P' of (x') corresponds the two associated points P_1 and P_2 of (x), then I is said to be rational. To a hyperplane in (x') corresponds a variety of the same dimensions in (x) which will be invariant under the involution. We shall consider the simplest case, namely when these varieties are quadrics. The methods employed are the natural extensions to S_4 of those developed in the classical memoirs of De Paolis for S_2 * and S_3.†

2. *Equivalence and Postulation of a Curve.* A point in (x') is determined as the intersection of four hyperplanes. The four corresponding quadric varieties must determine the two corresponding points in the involution, hence any four quadric varieties can have only two variable intersections. Four general quadric varieties intersect in 16 points. It follows that the system of quadric varieties corresponding to the hyperplanes of (x') must be of dimension four, and must have in common basis elements which absorb 14 of the 16 intersections. Since the dimension of the system of all the quadric varieties in S_4 is 14, the basis elements can not impose more than 10 conditions on the system.

The maximum number of simple basis points is 10, in which case there are 6 variable intersections. Therefore the basis system can not consist of points alone. A double point or a double line is also impossible. There must be, then, at least a basis curve C_m. If it is of order m and genus p its *postulation* for quadric varieties is

$$P_m = 2m - p + 1,$$

as in the case of ordinary space.

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* R. De Paolis, "Le trasformazione piane doppie," *Atti delle Reale Accademia de Lincei*, ser. 3, Vol. 1 (1877), pp. 511-544.

† R. De Paolis, "Le trasformazione doppie dello spazio," *Lincei Memorie*, ser. 4, Vol 1 (1885), pp. 576-609.

The equivalence of C_m may be derived from the following theorem due to Caporali:[*]

THEOREM. *If three varieties of orders n_1, n_2, n_3, respectively, pass through the curve C_m, and if the residual curve of intersection of the 3 varieties meets C_m in t points, then $m(n_1+n_2+n_3-3)=r+t$, where $r=2m+2p-2$ is the rank of C_m.*

Hence in our case $t=m-2p+2$. The order of the residual curve is $8-m$. The residual curve meets a fourth quadric variety of the system in $16-2m-t$ points not on C_m. Hence the equivalence of C_m is

$$E_m=2m+t=3m-2p+2.$$

In finding possible sets of basis elements it will be convenient to denote by S the number of simple basis points; that is

$$S=14-E_m=12-3m+2p.$$

And we shall denote the number of linearly independent quadric varieties through the fixed basis elements by N. Then

$$N=15-P_m-S=14-2m+p-S=2+m-p.$$

Any set of values of m, p, and S which satisfies the inequality $N \geqq 5$, and which leads to no geometrical contradiction, furnishes a set of basis elements for I. By varying m and p we obtain the following sets:

m	p	S	N
2	-1	4	5
3	0	3	5
3	-1	1	6
4	1	2	5
4	0	0	6
5	2	1	5
6	3	0	5

It will now be shown from geometrical considerations that the last set does not give a (1, 2) correspondence. A C_6, $p=3$, lies in an S_3 and not on a quadric surface. The 5 linearly independent quadric varieties through C_6 consist therefore of the S_3 and 5 linearly independent hyperplanes. Thus these quadric varieties define a linear correspondence instead of a (1, 2) correspondence.

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[*] E. Caporali, *Memorie di Geometria*, Naples (1888), p. 320.

3. *Equivalence and Postulation of a Plane.* Let

$$\pi \equiv x_1 = x_2 = 0,$$

be a fundamental plane. The most general form of a quadric through π is

$$V_2 \equiv b + cx_3 + dx_4 + ex_5 = 0,$$

where c, d, and e are linear in x_1 and x_2, and b is quadratic in the same variables. The left hand member contains 9 homogeneous constants, therefore the postulation of a plane for a quadric variety is 6. To find the residual intersection of 3 quadric varieties through π, take a variable hyperplane $x_1 = \lambda x_2$ through π. This hyperplane meets V_2 in a plane given by

$$\begin{aligned} &x_1 = \lambda x_2, \\ &H \equiv b(\lambda)x_2 + c(\lambda)x_3 + d(\lambda)x_4 + e(\lambda)x_5 = 0, \end{aligned}$$

where c, d, and e are linear in λ, and b is qudaratic in λ. The residual curve of intersection of 3 quadric varieties through π is the locus of intersections of the 4 hyperplanes

$$\begin{aligned} &x_1 = \lambda x_2, \\ &H_i = b_i(\lambda)x_2 + c_i(\lambda)x_3 + d_i(\lambda)x_4 + e_i(\lambda)x_5, \qquad (i = 1, 2, 3). \end{aligned}$$

From the matrix of the coefficients

$$\left\| \begin{array}{ccccc} -1 & \lambda & 0 & 0 & 0 \\ 0 & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & e_1(\lambda) \\ 0 & b_2(\lambda) & c_2(\lambda) & d_2(\lambda) & e_2(\lambda) \\ 0 & b_3(\lambda) & c_3(\lambda) & d_3(\lambda) & e_3(\lambda) \end{array} \right\|$$

we have the parametric equations of the residual curve in the form

$$x_1 = \lambda\theta_3(\lambda), \quad x_2 = \theta_3(\lambda), \quad x_3 = \phi_4(\lambda), \quad x_4 = \psi_4(\lambda), \quad x_5 = \chi_4(\lambda);$$

where, on the right, the suffix denotes the degree in λ. Thus we see that the residual intersection of 3 quadric varieties through π is a rational C_4, which meets π in the points corresponding to the 3 roots of $\theta_3(\lambda) = 0$. A fourth quadric variety meets the residual C_4 in 8 points. Since 3 of these points are on π, only 5 of them are variable, showing that the equivalence of a plane is 11. Hence for this case

$$S = 3, \quad N = 6.$$

Consider next the case of a fundamental plane and a line meeting the plane. Let the plane be

$$\pi \equiv x_1 = x_2 = 0;$$

and let the line be

$$l \equiv x_1 = x_3 = x_4 = 0.$$

The most general quadric variety through π and l has an equation of the form

$$V_2 \equiv ax_1 + cx_3 + dx_4 + ex_1x_5 = 0,$$

where a, c, and d are linear in x_1 and x_2, and e is a constant. There are 7 homogeneous constants, hence the postulation of π and l together is 8. The equation of the residual curve of intersection of 3 quadric varieties through π and l is obtained as in the preceding case. The coordinates of points on this curve, when expressed in terms of the elements of the matrix

$$\left\|\begin{matrix} -1 & \lambda & 0 & 0 & 0 \\ 0 & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & e_1 \\ 0 & b_2(\lambda) & c_2(\lambda) & d_2(\lambda) & e_2 \\ 0 & b_3(\lambda) & c_3(\lambda) & d_3(\lambda) & e_3 \end{matrix}\right\|$$

are of the form

$$x_1 = \lambda\theta_2(\lambda), \quad x_2 = \theta_2(\lambda), \quad x_3 = \lambda\phi_2(\lambda), \quad x_4 = \lambda\psi_2(\lambda), \quad x_5 = \chi_2(\lambda).$$

The residual curve is therefore a cubic which meets π in two points corresponding to the roots of $\theta_2(\lambda) = 0$, and also meets l in one point given by $\lambda = 0$. A fourth quadric variety through π and l meets C_3 in 6 points, two of which are on π and one on l, leaving 3 variable intersections. Thus the combined equivalence of π and l is 13, and we have

$$S = 1, \quad N = 6.$$

Consider next the case where l does not meet π. Let

$$\pi \equiv x_1 = x_2 = 0;$$

and

$$l \equiv x_3 = x_4 = x_5 = 0.$$

Then the most general quadric variety through π and l has an equation of the form

$$cx_3 + dx_4 + ex_5 = 0,$$

where c, d, and e are linear and binary in x_1 and x_2. There are 6 homogeneous constants, hence the postulation for π and l together is 9. As in the previous case, we cut 3 quadric varieties through π and l by a hyperplane $x_1 = \lambda x_2$. The 3 resulting hyperplanes have the equations

$$c_i(\lambda)x_3 + d_i(\lambda)x_4 + e_i(\lambda)x_5 = 0, \qquad (i = 1, 2, 3),$$

which have solutions for the 3 values of λ given by

$$\Delta_3(\lambda) \equiv \begin{vmatrix} c_1(\lambda) & d_1(\lambda) & e_1(\lambda) \\ c_2(\lambda) & d_2(\lambda) & e_2(\lambda) \\ c_3(\lambda) & d_3(\lambda) & e_3(\lambda) \end{vmatrix} = 0.$$

For each of these values λ_i, $(i = 1, 2, 3)$, the 3 hyperplanes have a plane through l in common which meets the hyperplane $x_1 = \lambda x_2$ in a line. Therefore the residual C_3 consists of 3 lines meeting π and l. A fourth quadric variety through π and l meets these three lines only on π and l. There are, therefore, no variable intersections; that is the equivalence of π and l is 16. Hence a plane and a line not meeting the plane do not furnish a set of basis elements for a $(1, 2)$ correspondence.

4. *Equivalence and Postulation of a Quadric Surface.* Let

$$\Gamma_2 \equiv f_2(x_1, x_2, x_3, x_4) = x_5 = 0,$$

be a fundamental quadric surface. The most general form of a quadric variety through Γ_2 is

$$Af_2(x_1, x_2, x_3, x_4) + x_5(ax_1 + bx_2 + cx_3 + dx_4 + ex_5) = 0.$$

The equation contains 6 homogeneous constants, hence the postulation of Γ_2 is 9. Three quadratic varieties

$$Af_2 + u_i x_5 = 0. \qquad (i = 1, 2, 3),$$

through Γ_2 have as a residual intersection a conic C_2 determined by the plane $\pi \equiv u_1 = u_2 = u_3 = 0$ and the quadric cone $Af_2 + u_1 x_5 = 0$. The plane π meets Γ_2 in 2 points A, B, which must lie on C_2 since C_2 contains all the points common to π and $Af_2 + u_1 x_5 = 0$. Hence the residual intersection of 3 quadric varieties through Γ_2 is a conic meeting Γ_2 in two points. A fourth quadric variety through Γ_2 meets C_2 in 4 points, 2 of which must be A and B. The remaining two intersections P_1, P_2, vary with the set of four quadrics and are the images of the intersection P' of the 4 corresponding hyperplanes in (x'). Hence the equivalence of Γ_2 is 14, giving the values

$$S = 0, \quad N = 6.$$

The line P_1P_2 passes through a fixed point O common to all the hyperplanes which together with $H \equiv x_5 = 0$ constitute composite quadrics of the system; the corresponding hyperplanes in (x') have a point O' in common.

As a special case of a fundamental quadric surface we have two planes meeting in a line. But if the two planes meet only in a point the postulation is 11; consequently these two planes cannot make up a fundamental system for a $(1, 2)$ correspondence.

A cubic surface σ_3 will not serve for a basis element. For if σ_3 lies in an S_3 it does not lie on a quadric variety because the section of a quadric variety by an S_3 is a quadric surface. Again if σ_3 is the residual intersection of two quadric varieties through a plane then it does not lie in an S_3. Hence the number of linearly independent quadric varieties through σ_3 is the same as the number of linearly independent quadrics through a space cubic, namely 3. Since N must be at least 5, the quadric varieties through σ_3 do not furnish a $(1, 2)$ correspondence.

5. *Sets of Basis Elements.* We have found, then, the following sets of basis elements for a $(1, 2)$ correspondence:

a). C_2, $p = -1$, and 4 points,
b). C_3, $p = 0$, and 3 points,
c). C_3, $p = -1$, and 1 point,
d). C_4, $p = 1$, and 2 points,
c₂). C_4; $p = 0$,
c₃). C_5, $p = 2$, and 1 point,
c₄). A plane, and 3 points not on the plane,
c₅). A plane, a line meeting the plane, and a point not on the line or the plane,
e). A quadric surface.

Case c₃) is equivalent to case e). Since $p = 2$, C_5 lies in a hyperplane and hence must lie on all the quadrics which are the sections of the quadric varieties of the system by this hyperplane. But since the postulation of a C_5 for a quadric is 9, there is just one quadric which contains C_5. Let this quadric be

$$Q \equiv x_5 = \phi_2(x_1, x_2, x_3, x_4) = 0.$$

The general equation of a quadric variety has the form

$$f_2(x_1, x_2, x_3, x_4) + x_5 \sum_1^5 a_i x_i = 0.$$

But since Q is the only quadric containing C_5, the above equation must reduce to the form

$$\phi_2(x_1, x_2, x_3, x_4) + x_5 \sum_1^5 a_i x_i = 0.$$

Thus if the quadric variety contains C_5 it must also contain the quadric surface on which C_5 lies, that is we have a fundamental quadric surface.

Cases c_i), ($i = 2, 4, 5$) can be reduced by a birational transformation to case e). For in the set of ∞^5 quadric varieties there are ∞^4 through an additional point O, which determine a birational transformation T. Of the ∞^4 quadric varieties used in defining the $(1, 2)$ correspondence, ∞^3 pass through O and are transformed by T into quadric varieties composed of a fixed hyperplane,—the image of O,—and a variable hyperplane. Any quadric variety of the system, not through O, is transformed into another quadric variety. It follows that the ∞^4 quadric varieties used in defining the $(1, 2)$ correspondence are transformed into quadric varieties through a fixed quadric surface, namely the intersection of the hyperplane which is the image of O and the image of any quadric variety of the system not through O. Hence the cases c_i), ($i = 2, 4, 5$) are transformable into case e).

There are therefore, 4 independent types of systems of quadric varieties, Σ, which determine $(1, 2)$ correspondences, namely those with the following basis elements:

I. Quartic curve β_4, $p = 1$, and 2 simple points P_1, P_2.

II. Cubic curve β_3, $p = 0$, and 3 simple points P_1, P_2, P_3.*

III. Two skew lines α_1, β_1, and 4 simple points P_1, P_2, P_3, P_4.

IV. Quadric surface Γ_2.

We proceed to determine the involution I associated with each of these types.

6. *Type I. Basis system a quartic curve* β_4, $p = 1$, *and* 2 *simple points* P_1, P_2. The image of a hyperplane V'_1 in (x') is a quadric variety V_2 in (x), passing through β_4, P_1, P_2. A plane π' in (x'), being the intersection of 2 hyperplanes, has for its image in (x) a quartic surface which is the intersection of the 2 quadric varieties corresponding to the 2 hyperplanes. A line l' in (x'),—the intersection of 3 hyperplanes,—has for its image a quartic

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* G. Aprile, "Di una trasformazione doppia nello spazio a quattro dimensioni," *Giornale di Matematiche* (3), Vol. 56 (1918), pp. 90-140 and Vol. 57 (1919), pp. 129-149.

curve C_4, $p=1$, the residual intersection of the 3 quadric varieties corresponding to the 3 hyperplanes. The curve C_4 passes through P_1, P_2, and meets β_4 in 4 points.

The order of the image in (x') of a hyperplane V_1, in (x), is the number of points in which this image meets a line. To each of these points corresponds, in (x), an intersection of a C_4,—the image of the line,—with V_1. The image of V_1 is therefore a quartic variety in (x'). The order of the image in (x') of a plane π, in (x), is the number of points in which this image meets an arbitrary plane π'. To each of these points corresponds, in (x), an intersection of π with a quartic surface,—the image of π'. The image of π is therefore a quartic surface in (x'). The order of the image in (x') of a line l, in (x), is the number of points in which this image meets an arbitrary hyperplane V_1'. To each of these points there corresponds, in (x), an intersection of a quadric variety,—the image of V_1',—with l. The image of l is therefore a conic in (x').

The hyperplanes in (x') whose images are composed of the quadric varieties of Σ which contain the line λ through P_1, P_2, form a web, and therefore have in common a point O' which is the image of λ.

The hyperplanes in (x') whose images are composed of the hyperplane H in which β_4 lies and a hyperplane through λ, form a net (a subset of the system Σ), and therefore have in common a line λ' through O', which is the image of H. Among the quadric varieties of Σ there is a pencil which does not contain the hyperplane H. A general hyperplane in (x') meets λ' in a point to which corresponds in (x) the intersection of H with one of the quadric varieties of the pencil. The image of a point on λ' is therefore a quadric surface lying in H.

Since the net of hyperplanes through λ' has for its image in (x) the net of hyperplanes through λ and the common hyperplane H, a plane through λ' has for its image a plane through λ. A general hyperplane V_1' meets the plane through λ' in a line whose image is the conic through P_1 and P_2 in which the image of V_1' meets the plane through λ. A general hyperplane V_1, in (x), meets this conic in 2 points; hence a line meeting λ' cuts V_4', the quartic image of V_1, in 2 points not on λ'. A line meeting λ' must therefore meet V_4' in 2 points on λ'; hence λ' is double on V_4'. The complete image of V_4' is a V_8: β_4^4, P_1^2, P_2^2, having as components the original V_1 and H taken twice. The conjugate in I of an arbitrary V_1 is therefore a V_5: β_4^2, P_1^4, P_2^4.

We have seen that the image of a point P' on λ' is a quadric surface of the pencil in H through β_4. Through a point R_1 of H passes a surface Q

of the pencil (the image of a point R' on λ') and a definite plane π through λ. To the plane π corresponds a plane π' through λ'. The image of R_1 is a direction through R' in π'. The image of R' is R_1 and R_2, the residual intersection of Q and π. Hence the image in I of R_1 is R_2, and H is invariant under I.

Corresponding to each of the 4 points in which β_4 meets a C_4,—the image of a line l',— is an intersection of l' with the image of β_4; the image in (x') of β_4 is therefore a quartic variety V_4. Since a general plane thru λ does not meet β_4, a line meeting λ' does not meet V_4 except on λ'; hence λ' is fourfold on V_4'. The image of a plane π through λ and a point P of β_4 is a plane π' through λ'. The images of the lines of intersection with π' of two hyperplanes through an arbitrary point P' of π' are two conics in π through P_1, P_2, P, and Q, a fourth point of π. Thus the images in (x) of P' are P and Q, and hence the image of π' is π and P. As P moves along β_4, π generates a cone K_4 having the line λ for vertex, and π' traces out a similar cone K_4' with the line vertex λ'. That is, the image of β_4 is K_4', but the image of K'_4 is K_4 and β_4. Hence in I the image of β_4 is K_4, which obviously contains β_4.

In general, to a point P' of (x') there correspond two points P_1 and P_2 of (x); but for certain positions of P' the points P_1 and P_2 coincide. The locus of such points P' is the variety of branch points, L'; the locus of the points P_1, P_2, is the variety of coincidences K; that is, K taken twice, is the complete image of L'. The jacobian of the system Σ is the locus of the vertices of the quadric varieties of Σ which are cones. The hyperplanes through a point of L' have for images quadric varieties in (x) with a common tangent line at the corresponding point of K. Hence among them is a cone with this point as vertex, that is K is part of the jacobian of Σ.

If we apply Zeuthen's * formula for the number of coincidences in a multiple correspondence to the (1, 2) correspondence between the points of the line C_1' in (x') and its image C_4, $p=1$ in (x), we find that there are 4 points on C_1' such that their 2 images in (x) coincide. Thus L' is of order 4. Applying the same formula to a line C' which meets λ' and has therefore a conic for image in (x), we find that there are 2 points on such a line, apart from points on λ', such that their images in (x) coincide. Thus λ' must be double on L_4'. Since the image of λ' is H, and the image of L_4' is K^2, then K must be of order 3, must contain the fundamental curve β_4, and have P_1

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* H. G. Zeuthen, *Mathematische Annalen*, Vol. 3 (1871), pp. 150-156.

and P_2 for double points. The jacobian is of order 5, has β_4 for a triple curve, and has K_3 for one component. The residual is therefore a quadric variety having β_4 for a double curve and hence must consist of H taken twice, since β_4 is simple on any noncomposite quadric variety containing it.

To the ∞^3 directions thru the fundamental point P_1 in (x) correspond the ∞^3 points of some variety V', in (x'). The order of V' is the number of points in which it is met by a line l'. But since the C_4 in (x) which is the image of l', passes through P_1 in a single direction, then l' meets V' in just one point; that is the image of P_1 is a hyperplane H_1'. To a point on H_1' there corresponds in (x) a direction through P_1 and a point on the quadric variety of Σ which is the image of H_1'. If the point on H_1' is also on L_4' then both the images of the point are directions at P_1 and lie on the tangent cone to K_3 at P_1. Therefore the image of P_1 in I is the quadric cone of Σ which has P_1 for a vertex. Similarly the image of the point P_2 in I is the quadric cone of Σ which has P_2 for a vertex.

Collecting our results, we have an involution I in which

$$\begin{aligned}
&V_1 \sim V_5:\ \beta_4^2,\ P_1^4,\ P_2^4,\\
&\beta_4 \sim K_4:\ \beta_4,\ P_1^4,\ P_2^4,\\
&P_1 \sim V_2:\ \beta_4,\ P_1^2,\ P_2,\\
&P_2 \sim V_2:\ \beta_4,\ P_1,\ P_2^2,\\
&H \sim H.\\
&K_3:\ \beta_4,\ P_1^2,\ P_2^2.
\end{aligned}$$

The $(1,2)$ correspondence between the spaces (x') and (x) can be expressed algebraically in the form

$$\begin{aligned}
&x_1' = x_5x_1, \quad x_2' = x_5x_2, \quad x_3' = x_5x_3,\\
&x_4' = u_1x_4 + u_2, \quad x_5' = x_4^2 - x_5^2 + v_1x_4 + v_2,
\end{aligned}$$

where the suffixes of u and v indicate the degree in x_1, x_2, x_3; and from these equations all the results obtained can be deduced algebraically. The fundamental elements have the following equations:

$$\begin{aligned}
&H:\ x_5 = 0; \quad \lambda:\ x_1 = x_2 = x_3 = 0; \quad \lambda':\ x_1' = x_2' = x_3' = 0;\\
&\beta_4:\ x_5 = 0; \quad u_1x_4 + u_2 = 0; \quad x_4^2 + v_1x_4 + v_2 = 0;\\
&K_4:\ u_2^2 - u_1u_2v + u_1^2v_2 = 0.
\end{aligned}$$

7. *Type II. Basis system a cubic curve β_3, $p = 0$, and 3 simple points P_1, P_2, P_3.* As in the preceding case, the image of a hyperplane V_1', in (x'),

is a quadric variety V_2, in (x), passing through β_3, P_1, P_2, and P_3. The image of a plane π' is a quartic surface in (x) also passing through β_3, P_1, P_2, and P_3. A line l_1' has for its image a quintic curve, $p=1$, passing through P_1, P_2, and P_3, and meeting β_3 in 5 points.

To each of the points in which the image of a V_1 meets an arbitrary line of (x') there corresponds an intersection of a C_5 with V_1. Hence the image of V_1 is a quintic variety V'_5. To each of the points in which the image of a plane π of (x) meets an arbitrary plane of (x') there corresponds an intersection of a quartic surface with π. Hence the image of π is a quartic surface in (x'). To each of the points in which the image of a line l of (x) meets an arbitrary V_1' there corresponds an intersection of a V_2 of (x) with l. Hence the image of l is a conic in (x').

The hyperplanes of (x') whose images are composed of the space H in which β_3 lies and a hyperplane through the plane π determined by P_1, P_2, and P_3, form a pencil and hence have a plane π' in common which is the image of H and π. The system of hyperplanes Σ' meets π' in a net of lines. A V_1' has for image a V_2 which meets H in a quadric Q_2 through β_3, the image of the line in which V_1' meets π'. The V_2 meets π in a conic through P_1, P_2, and P_3, which is conjugate to Q_2 in the involution I. Two lines of π' meet in a point P' whose images in (x) are the residual intersection P of the 2 corresponding conics of π, and the bisecant p of β_3 which is the residual intersection of the 2 corresponding quadrics of H. As the point P' traces out π', the bisecant of β_3 describes H, and P describes π. An arbitrary V_1 meets p in one point, hence P' is a simple point on V_5', the image of V_1, and the plane π' is simple on V_5'. The complete image of V_5' is a $V_{10}: \beta_3^5, P_1^5, P_2^5, P_3^5$, having the original V_1 and H as components. The conjugate of an arbitrary V_1 in I is therefore a $V_8: \beta_3^4, P_1^5, P_2^5, P_3^5, \pi$.

Since the image of an arbitrary line l' is a C_5 passing simply through P_1, the image of P_1 must be a variety meeting l' in only one point, that is a hyperplane. Hence the images of the fundamental points P_1, P_2, P_3, are hyperplanes H_1', H_2', H_3' respectively. By an argument similar to that used for P_1 and P_2 in Case I the images of H_1', H_2', and H_3' can be shown to be the quadric cones of Σ whose vertices are P_1, P_2, and P_3 respectively; that is, these cones are the images in I of the fundamental points P_1, P_2, and P_3 respectively.

Corresponding to each of the 5 points in which β_3 meets a C_5, the image of an arbitrary line l' of (x'), is an intersection of l' with the image of β_3. Hence the image in (x') of β_3 is a quintic variety B_5'. But the image of a line l' meeting π' in a point P' is composite, consisting of a bisecant of β_3,—

the image of P',—and a C_4 meeting β_3 in 3 points. So l' meets B_5' in 3 points not on π', and hence π' is double on B_5'. The image in (x) of an arbitrary V_5' is a V_{10}: $\beta_3^5, P_1^5, P_2^5, P_3^5$. But since π' is double on B_5', the V_{10} has H as a double component, so that the conjugate of β_3 in I is a variety of order eight, B_8: P_1^5, P_2^5, P_3^5. The image of an arbitrary line in (x) is a conic in (x') meeting π'. The image of a line l meeting β_3 but not lying in H is composite, consisting of 2 lines meeting in a point P. One of these lines lies in B_5'; the other meets B_5' in 4 points apart from P which are the images of the points, not on β_3, in which l meets B_8. Hence β_3 is fourfold on B_8.

On the jacobian of Σ, which is of order 5, β_3 is triple and the points P_1, P_2, and P_3 are double. The hyperplane H, being fundamental, is part of the jacobian. The residual is the variety of coincidences K_4: $\beta_1^2, P_1^2, P_2^2, P_3^2$. Applying Zeuthen's formula to the $(1, 2)$ correspondence between points on the line l' and its image C_5, $p = 1$, we find that there are 4 points on l' such that their images in (x) coincide. Hence the variety of branch points L_4' is of order 4. A line l' meeting π' in P' has for image a C_4 meeting H in 3 points on β_3 and passing through P_1, P_2, and P_3 on π, and hence meeting K_4 in 4 points not on the images of π'. Therefore L_4' meets l' in 4 points apart from π'; that is L_4' does not contain π'.

Collecting our results, we have an involution I in which

$$V_1 \sim V_8\ \beta_3^4,\ P_1^5,\ P_2^5,\ P_3^5,\ \pi.$$

$$\beta_3 \sim V_8:\ \beta_3^4,\ P_1^5,\ P_2^5,\ P_3^5.$$

$$P_1 \sim V_2:\ \beta_3,\ P_1^2,\ P_2,\ P_3.$$

$$P_2 \sim V_2:\ \beta_3,\ P_1,\ P_2^2,\ P_3.$$

$$P_3 \sim V_2:\ \beta_3,\ P_1,\ P_2,\ P_3^2.$$

$$\pi \sim H.$$

$$K_4:\ \beta_3^2,\ P_1^2,\ P_2^2,\ P_3^2.$$

8. *Type III. Basis system 2 straight lines* α_1, β_1, *and* 4 *simple points* P_1, P_2, P_3, P_4. As in types I and II the image of a hyperplane V_1' in (x') is a quadric variety V_2 in (x) passing through $\alpha_1, \beta_1, P_1, P_2, P_3, P_4$. The image of a plane π' is a quartic surface in (x) also passing through $\alpha_1, \beta_1, P_1, P_2, P_3, P_4$. A line l' has for its image in (x) a sextic curve C_6, $p = 1$, passing through P_1, P_2, P_3, P_4, and meeting α_1 and β_1 each in 3 points.

To each of the points in which the image of a V_1 meets an arbitrary line of (x') there corresponds an intersection of a C_6 with V_1. Hence the image of a V_1 is a sextic variety V_6'. To each of the points in which the image of a plane π meets an arbitrary plane of (x') there corresponds in (x) the intersection of a quartic surface with π. Hence the image of π is a quartic surface in (x'). To each of the points in which the image of a line l of (x) meets an arbitrary V_1' there corresponds, in (x), the intersections of a V_2 with l. Therefore the image of l is a conic in (x').

The hyperplane H determined by the lines α_1 and β_1 contains ∞^2 bisecants of α_1, β_1. Through a bisecant a passes a web of V_2's of Σ since every V_2 of Σ contains 2 points of a. To this web of V_2's corresponds the web of V_1''s of Σ' through a point (a'). Hence the image of a bisecant of α_1, β_1 is a point, and the image of H is a surface, Q'. The order of Q' is the number of points in which it is met by an arbitrary plane π'. The image of π' is a quartic surface which does not lie in H, since if it did π' would lie in Q'. H meets this quartic surface in a C_4 which is the intersection of two quadrics of H through α_1 and β_1. The intersection of two such quadrics is a curve of the second order meeting α_1, β_1 each in two points. But when the curve of intersection of 2 surfaces breaks up into 2 curves of the same order the curves also have the same genus. Therefore the residual intersection of the 2 quadrics is two bisecants of α_1, β_1. These bisecants correspond to two intersections of π' and Q', hence Q' is a quadric surface.

Let H' be the hyperplane which has for its image the composite V_2 of Σ consisting of H, and $\overline{H}$ (the hyperplane determined by the four points P_1, P_2, P_3, P_4). Obviously H' contains Q_2', which is the intersection by H' of any V_2' through Q_2'. The image of this V_2' consists of H, the image of Q_2', and a residual V_3. The image of H' is H and $\overline{H}$, and therefore the image of Q_2' is H and the cubic surface σ_3 in which $\overline{H}$ meets V_3. Thus the image of H in I is the cubic surface σ_3.

Since an arbitrary V_1 of (x) meets each bisecant of α_1, β_1 in a point, the V_6' which is the image of V_1 contains Q_2' simply. The image of V_6' is then a V_{12}: α_1^6, β_1^6, P_1^6, P_2^6, P_3^6, P_4^6, σ_3, having the original V_1 and H as components. The conjugate of an arbitrary V_1 in I is therefore a V_{10}: α_1^5, β_1^5, P_1^6, P_2^6, P_3^6, P_4^6, σ_3.

The image in (x) of a line in (x') is a C_6, $p=1$, meeting α_1 and β_1 each in 3 points. Hence the image in (x') of α_1 is a cubic variety A_3', and the image of β_1 is B_3'. The image of a line l' meeting Q' consists of a bisecant of α_1, β_1 and a C_5 meeting α_1, β_1 each in 2 points. Therefore l' meets A_3' and B_3' each in two points apart from Q_2' and so Q_2' must be simple on A_3' and B_3'. The image in (x) of an arbitrary V_3' is a V_6: α_1^3, β_1^3, P_1^3, P_2^3,

P_3^3, P_4^3. But since A_3' contains Q' its image must be a V_5: β_1^2, P_1^3, P_2^3, P_3^3, P_4^3. The image of a line l meeting H in a point of α_1 but not lying in H is a composite conic consisting of a line l_1' of A_3' and a line l_2' meeting l_1'. Then l_2' meets A_3' in 2 points apart from l_1' and therefore A_5 meets l in 2 points not on α_1. Hence α_1 is triple on A_5. We have then, that the image of α_1 in I is A_5: α_1^3, β_1^2, P_1^3, P_2^3, P_3^3, P_4^3 and the image of β_1 is B_5: α_1^2, β_1^3, P_1^3, P_2^3, P_3^3, P_4^3.

Since the image of an arbitrary line l' is a C_5 passing simply through P_1, the image of P_1 must be a variety meeting l' in only one point, that is a hyperplane. Hence the images of the fundamental points P_1, P_2, P_3, P_4 are the hyperplanes H_1', H_2', H_3' H_4' respectively. By an argument similar to that employed in Cases I and II the images of $H_1,'$ H_2', H_3', H_4' can be shown to be quadric cones of Σ whose vertices are P_1, P_2, P_3, P_4 respectively; that is these cones are the images in I of the fundamental points P_1, P_2, P_3, P_4 respectively.

The jacobian of Σ is J_5: α_1^3, β_1^3, P_1^2, P_2^2, P_3^2, P_4^2. It contains H as a component. For let (a) be any point of H. Among the V_2 of Σ which contain the secant of α_1, β_1 through (a) is a cone with (a) as vertex. Hence the residual variety of coincidences is K_4: α_1^2, β_1^2, P_1^2, P_2^2, P_3^2, P_4^2. Applying Zeuthen's formula to the $(1, 2)$ correspondence between points on the line l' and its image C_6, $p = 1$, we find that there are 4 points on l' such that their images on C_6 coincide. Hence the variety of branch points L_4' is of order 4. A line l' meeting Q_2' in a point P' has for image a C_5 meeting α_1 and β_1 each in two points and passing through P_1, P_2, P_3, P_4, and hence meeting K_4 in 4 points not on the image of Q_2'. Therefore l' meets L_4' in 4 points apart from Q_2', and so Q_2' is not on L_4'.

Collecting our results, we have an involution I in which

$V_1 \sim V_{10}$: α_1^5, β_1^5, P_1^6, P_2^6, P_3^6, P_4^6, σ_3.
$\alpha_1 \sim A_5$: α_1^3, β_1^2, P_1^3, P_2^3, P_3^3, P_4^3.
$\beta_1 \sim B_5$: α_1^2, β_1^3, P_1^3, P_2^3, P_3^3, P_4^3,
$P_1 \sim V_2$: α_1, β_1, P_1^2, P_2, P_3, P_4,
$P_2 \sim V_2$: α_1, β_1, P_1, P_2^2, P_3, P_4,
$P_3 \sim V_2$: α_1, β_1, P_1, P_2, P_3^2, P_4,
$P_4 \sim V_2$: α_1, β_1, P_1, P_2, P_3, P_4^2,
$\sigma_3 \sim H$,
K_4: α_1^2, β_1^2, P_1^2, P_2^2, P_3^2, P_4^2.

The $(1, 2)$ correspondences between the spaces (x') and (x) can be expressed algebraically in the form

$$x_1' = x_1x_4, \quad x_2' = x_2x_3, \quad x_3' = x_1(x_3 + ax_5) + bx_3x_5,$$
$$x_4' = x_2(x_4 + cx_5) + dx_4x_5,$$
$$x_5' = (x_1 + x_2 + x_3 + x_4 + x_5)x_5.$$

From these equations the above results can be deduced algebraically. The fundamental elements have the following equations

$$\alpha_1 \equiv x_1 = x_2 = x_5 = 0;$$
$$\beta_1 \equiv x_3 = x_4 = x_5 = 0;$$
$$H\colon\ x_5 = 0,\quad \overline{H}\colon\ x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

and 2 of the points P_i are given by each of the sets of equations,

$$x_1 = x_3 = x_2 + x_4 + x_5 = x_4' = 0;$$
$$x_2 = x_4 = x_1 + x_3 + x_5 = x_3' = 0.$$

9. *Type IV.—Basis system a quadric surface* Γ_2. As in the preceding cases the image of a hyperplane V_2' is a quadric variety V_2, passing through Γ_2. The image of a plane π' is a quadric surface. A line l' has for its image a conic meeting Γ_2 in two points.

To each of the points in which an arbitrary line l' meets the image of a hyperplane V_1 there corresponds in (x) the intersection of a conic and the V_1, hence the image of V_1 is a quadric variety V_2'. To each of the points in which the image of a plane π meets an arbitrary plane of (x') there corresponds an intersection of a quadric surface with π. Hence the image of π is a quadric surface in (x'). To each of the points in which the image of a line l meets an arbitrary V_1' there corresponds an intersection of a V_2 with l.

Let H be the hyperplane which contains Γ_2. It follows from (4) that the image of O' is O and H, and that the image of any line $O'P'$ is the line OP_1P_2 and H. Any hyperplane V_1 in (x) meets H in a plane π. The lines through O meeting π have for images tangent lines to the image V_2' of V_1, hence O' is simple on V_2'. The image of V_2' is a $V_4\colon \Gamma_2^2, O$, having V_1 and H as components. The conjugate in I of V_1 is therefore a $V_2\colon \Gamma_2, O$.

Since the image of a line l' meets Γ_2 in 2 points, the image of Γ_2 is a quadric variety G_2'. Since a general line through O does not meet Γ_2, the corresponding line through O' does not meet G_2' except at O'. Hence G_2' is a quadric hypercone, vertex O'. A line through O meeting Γ_2 has for image a generator of G_2' counted twice, and a generator of G_2' has for image a line through O meeting Γ_2. Hence the image of G_2' is the quadric hypercone G_2, vertex O and base Γ_2.

The jacobian of Σ is $J_5: \Gamma_2^4$. It consists of H counted three times and $K_2: \Gamma_2$. The image of K_2 is G_2' and L_2'.

Collecting results, we have an involution I in which

$$\begin{aligned}&V_1 \sim V_2: \Gamma_2,\ O.\\ &\Gamma_2 \sim G_2: \Gamma_2,\ O,\\ &O \sim H,\\ &K_3: \Gamma_2.\end{aligned}$$

The (1, 2) correspondence between the spaces (x') and (x) can be expressed analytically in the form

$$\begin{aligned}&x_1' = x_1x_5,\ x_2' = x_2x_5,\ x_3' = x_3x_5,\ x_4' = x_4x_5,\\ &x_5' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2,\end{aligned}$$

and from these equations the above results can be deduced algebraically. The fundamental elements have the following equations:

$$\begin{aligned}&\Gamma_2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5 = 0;\\ &H \equiv x_5 = 0,\quad K_2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 = 0,\end{aligned}$$

10. *Conclusion.* This completes the enumeration of the general types of involutions leaving invariant ∞^4 quadric varieties. The special cases in which some of the basis points are coincident or lie on a basis curve are left for later consideration.

CORNELL UNIVERSITY.

A Three-Dimensional Quartic Variety in Four-Space.

By Bing C. Wong.

1. In this paper we shall make use of the involutorial quartic transformation studied by Miss N. Alderton.[*] This transformation is effected by four hyperquadrics. To a point P is made to correspond the point P' of intersection of its polar hyperplanes with respect to the hyperquadrics. P and P' are said to be the images one of the other. By this transformation a curve of order n transforms into a curve of order $4n$, a two-dimensional variety of order n into a two-dimensional variety of $6n$, and a three-dimensional variety of order n into a three-dimensional variety of order $4n$. In particular, the image of a hyperplane S_3 is a three-dimensional quartic variety M_3^4. It is the purpose of this paper to describe a few of its most obvious features. The method is to discover the peculiarities in the hyperplane with reference to the singular manifolds of the transformation here employed and then translate them into the language of the geometry of the three-dimensional quartic variety.

2. This involutorial quartic transformation yields a two-dimensional manifold J_2^{10} of order 10 whose points have for images lines forming a three-dimensional manifold Δ_3^{15} of order 15. The lines are quadri-secants of J_2^{10} and J_2^{10} lies quadruply on Δ_3^{15}. All the three-dimensional quartic varieties which are the images of hyperplanes in four-space have in common J_2^{10}: any two intersect, in addition to J_2^{10}, in a two-dimensional manifold of order 6 which is the image of the plane common to the two corresponding hyperplanes; and any three meet in a curve of order 4 which is the image of the line common to the three corresponding hyperplanes.

3. Consider a hyperplane S_3 and let its corresponding variety be denoted by M_3^4. S_3 meets Δ_3^{15} in a surface D_2^{15} and J_2^{10} in a curve j_1^{10} lying quadruply on a D_2^{15}. Since J_2^{10} is the partial intersection of two three-dimensional quartic varieties, the curve j_1^{10} is also the partial intersection of two quartic surfaces in S_3, the residual intersection being a

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[*] "Involutory Quartic Transformations in Space of Four Dimensions," *University of California Publications in Mathematics*, Vol. 1, pp. 345-358.

sextic curve c_1^6 having 20 points in common with it. Since c_1^6 has 7 apparent double points,[*] j_1^{10} has 25.[†] There are 20 lines meeting j_1^{10} four times [‡] and these lines are rulings of Δ_3^{15} and hence lines of the surface D_2^{15}. The image of such a line is a set of four lines concurrent at a point which is evidently a conical point on M_3^4. Hence M_3^4 has 20 conical points and is therefore of class 68.[§]

4. From the fact that the image of any point on j_1^{10} is a line on M_3^4 we see that M_3^4 has two systems of lines: one, the W-system, is composed of ∞ lines coming from points of j_1^{10} and the other, the T-system, composed of ∞ lines coming from the trisecants of j_1^{10}. The lines of the W-system form a two-dimensional locus which, being part of the manifold of order 60 common to M_3^4 and Δ_3^{15} including J_2^{10} four times, is of order 20. Let it be denoted by W_2^{20}. It passes through the 20 conical points of M_3^4 and, excepting those generators which meet in these points, no two generators meet.

5. Before finding the order of the locus of the lines of the T-system, let us note that the formula expressing the relation between the order n of a surface in S_3 passing through j_1^{10} k times and the order N of the corresponding variety in M_3^4 is

$$N = 6n - k \cdot 20,$$

since the transformed variety includes W_2^{20} k times.

6. The ruled surface formed by the trisecants of j_1^{10} is of order 80.[||] Let it be denoted by $\bar{T}_2^{80}$. It contains j_1^{10} as a 17-fold curve, for through each point of the curve pass 17 lines meeting the curve in two other points, contains 20 quadruple generators coincident with the quadri-secants of j_1^{10}, and also contains a nodal curve of high order. The order of the transform, T_2^{140}, which is the locus of the lines of the T-system in M_3^4, is, according to the formula above, $6 \times 80 - 20 \times 17 = 140$. This variety has 20 quadruple points coincident with the 20 conical points of M_3^4. Each of its lines meets 30 other lines of the same system, for its corresponding trisecant meets 30 other trisecants besides those which it already meets on j_1^{10}; and also

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[*] B. C. Wong, "On the Correspondence between Space Sextic Curves and Plane Quartics in Four-Space," *Bulletin of the American Mathematical Society*, Vol. 32, p. 156.

[†] Salmon, *Analytic Geometry of Three Dimensions*, Art. 345.

[‡] Ibid., Art. 474.

[§] Bertini, *Projektive Geometrie mehrdimensionaler Räume*, 1924, Kapitel VIII, Nr. 23, 24.

[||] Salmon, *Analytical Geometry of Three Dimensions*, Art. 471.

meets 3 lines of the W-system. Each line of the W-system meets 17 lines of T_2^{140}.

7. It is evident that any curve of order n having α points in common with J_2^{10} transforms into a curve of order $4n-\alpha$. In particular, any plane quartic curve through the 10 points in which the plane of the curve meets J_2^{10} transforms into a space sextic curve.* Let a plane containing a line τ of T_2^{140} and a line ω of W_2^{20} meet M_3^4 in those two lines and a conic λ and this conic transforms into a unicursal quintic curve in S_3 with an actual node on j_1^{10} corresponding to the two points in which τ meets λ and having two points in common with the trisecant corresponding to ω. But a plane containing two lines τ and τ' of T_2^{140} meets M_3^4 in those two lines and a conic κ which transforms into a unicursal quartic curve having two points in common with each of the two trisecants of j_1^{10} corresponding to the two lines τ and τ'.

8. We have just seen that there are conics on the variety M_3^4. In fact, there are on it four families of conics. One family, which we shall designate as the B-family, comes from the congruence of bisecants of j_1^{10}. There are in this family ∞^2 members and each member meets two lines of W_2^{20}. Since every bisecant of j_1^{10}, already meeting the surface D_2^{15} in two quadruple points on j_1^{10}, meets it again in 7 points, every conic of this family meets J_2^{10} in 7 points. From the fact that j_1^{10} has 25 apparent double points we see that through every point on M_3^4 pass 25 conics of the family.

9. Any plane meeting M_3^4 in a conic β of the B-family meets it in another conic γ which belongs to another family, which we shall designate as the C-family. There are also ∞^2 conics in this family and through every point of M_3^4 also pass 25 of them. One such conic, γ, meets J_2^{10} in only 3 points, for its companion conic β already meets the variety in 7. These 10 points are the 10 points common to J_2^{10} and the plane of the conics. Every conic γ of the C-family transforms into a unicursal quintic curve in S_3 and since γ meets its companion β in 4 points, this quintic curve has for quadrisecant the bisecant of j_1^{10} of which β is the image.

11. A conic γ' of another family, the C'-family, is obtained by intersecting M_3^4 with a plane containing a line ω of W_2^{20} and a line τ of T_2^{140}. Each conic of this family also meets J_2^{10} in 3 points. It is to this family that the conic λ of Art. 7 belongs. Properly, this family is included in the C-family. It contains ∞^1 conics.

12. If a plane contains two lines of W_2^{20} through a conical point of M_3^4,

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* See B. C. Wong, "On the Correspondence, etc.," above.

it meets M_3^4 in a conic γ'' besides. Since there are 4 lines of W_2^{20} through a conical point of M_3^4 and there are 20 such conical points on M_3^4, there are in all 120 such conics arranged in 20 sets of 6 each. They are included in the C'-family. Such a conic transforms into a unicursal quintic curve with

13. There is another family, the K-family, of ∞^1 conics each of which is obtained by intersecting M_3^4 with a plane containing two lines τ and τ' of the T-system. The transform of a conic κ of this family is already discussed in Art. 7.

14. Now consider the tangent surface A_2^{40} of j_1^{10} in S_3 which is of order 40. Its image $\bar{A}_2^{200}$ in M_3^4 is of order $6 \times 40 - 2 \times 20 = 200$ and contains ∞ conics of the B-family. Each conic meets two consecutive generators of W_2^{20} and meets 36 other conics, for each generator of the developable meets 36 other generators. Any plane σ in S_3 meets A_2^{40} in a plane curve of order 40 with 10 cusps and 720 nodes. The corresponding Σ_2^6 meets $\bar{A}_2^{200}$ in a curve of order 120 having 720 actual nodes. This curve has 10 lines of W_2^{20} for tangent lines. There are in all ∞^3 such curves on M_3^4.

15. The cone K_2^9 with vertex V on j_1^{10} and elements through points of j_1^{10} is of degree 9 and has 17 double elements. The image of K_2^9 is a variety $\bar{K}_2^{34}$ of order 34 in M_3^4 with W_2^{20} discarded. This variety, $\bar{K}_2^{34}$, contains ∞ conics of the B-family all meeting the same generator v corresponding the vertex V of K_2^9 and each meeting another different generator of W_2^{20}. Seventeen of these conics are line-pairs. Hence through any line on M_3^4 which is at the same time a line of J_3^{15} can pass 17 planes which meet M_3^4 each in a conic λ and another line τ.

16. The bisecants of j_1^{10} which meet any assumed line l in S_3 form a ruled surface which has l for 25-fold line and j_1^{10} for 9-fold curve. This surface, R_2^{70}, is of order 70 and has 80 triple generators * and a nodal curve of high order. The transform of R_2^{70} is a variety of $\bar{R}_2^{240}$ of degree $6 \times 70 - 9 \times 20 = 240$ in M_3^4. On this variety, $\bar{R}_2^{240}$, is a 25-fold quartic curve λ_1^4 which is the image of the line l. Through each point of λ_1^4 pass 25 conics of the variety. Each ruling of W_2^{20} meets 9 conics and each conic meets two rulings of W_2^{20} and 28 other conics.† Corresponding to the 80 triple generators of R_2^{70} there are 80 triple lines on $\bar{R}_2^{240}$. In general, any plane section of $\bar{R}_2^{240}$ is a group of 240 points lying on a plane quartic curve, but there

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* See Salmon, *Analytic Geometry of Three Dimensions*, Art. 471.

† For each generator of R_2^{70} meets 28 other generators besides those on the line l and the curve j_1^{10}.

are ∞ planes meeting $\bar{R}_2^{240}$ in conics and 80 of them are line-pairs. Non-consecutive conics meet on a curve corresponding to the nodal curve on R_2^{70}.

17. A plane σ through the line l meets the surface R_2^{70} in a degenerate curve composed of l counted 25 times and 45 bisecants of j_1^{10}. Hence, the corresponding Σ_2^6 passes through the curve λ_1^4 on M_3^4 into which l transforms and the total intersection of Σ_2^6 and $\bar{R}_2^{240}$ is composed of 10 lines of W_2^{20} each counted 9 times, λ_1^4 counted 25 times and 45 conics. Each of the 45 conics meets λ_1^4 once, each of the 10 lines meets 9 conics each once, and each conic meets once 28 other conics.

18. If the line l meets j_1^{10} in the point V, the ruled surface generated by a line meeting l once and j_1^{10} twice is a surface R_2^{61} of degree 61, for the cone K_2^9 of Art. 15 has to be discarded. The transform of R_2^{61} is a variety $\bar{R}_2^{206}$. The cubic surve λ_1^3 into which l transforms is 25-fold on $\bar{R}_2^{206}$ and each line of W_2^{20} meets 8 conics of the variety and each conic meets 21 other conics.

19. There are again 80 planes which meet $\bar{R}_2^{206}$ in line-pairs and 17 of these pass through the line v coming from the point V on j_1^{10}. Any plane σ through l meets R_2^{61} in a degenerate curve composed of l 25 times and 36 other lines. Hence the corresponding Σ_2^6 meets $\bar{R}_2^{206}$ in 36 conics, the cubic curve λ_1^3 counted 25 times, the line v 25 times and 9 other lines of W_2^{20} each counted 8 times.

20. Now suppose the line l is a bisecant of j_1^{10} meeting it in the points V and V'. The bisecants of j_1^{10} which meet l form a ruled surface R_2^{52} excluding the two cones with vertices at V and V'. The transform of R_2^{52} is a variety $\bar{R}_2^{172}$ in M_3^4. Through every point on l pass 24 rulings of R_2^{52}, for l itself is a bisecant. Hence l is 24-fold on the surface and j_1^{10} is now 7-fold. The conic λ_1^2 which l has for image is 24-fold on $\bar{R}_2^{172}$, that is, through every point on it pass 24 other conics of the variety. Each of the generators of W_2^{20} meets 7 conics of $\bar{R}_2^{172}$ and each conic meets 14 others.

21. Among the 80 planes which meet $\bar{R}_2^{172}$ in line-pairs there is one group of 17 all passing through the line v corresponding to the point V and another group of 17 all passing through the line v' corresponding to the point V'. Any plane σ through the line l meets R_2^{52} in a degenerate curve composed of l counted 24 times and 28 other lines. Hence the corresponding Σ_2^6 meets $\bar{R}_2^{172}$ in λ_1^2 counted 24 times, 8 lines of W_2^{20} each counted 7 times, the lines v and v' each counted 24 times and 28 conics each counted once.

22. Let l be a trisecant of j_1^{10} meeting it in the points V, V', V''.

The ruled surface formed by the bisecants of j_1^{10} which meet l is an R_2^{43} and its transform is an $\bar{R}_2^{138}$ in M_3^4. The line l, being a trisecant, is the equivalent of three bisecants; hence it is a 22-fold line on R_2^{43}, and the line λ which is the image of l is a 22-fold line on $\bar{R}_2^{138}$, that is, through each point of λ pass 22 conics of the variety. Each ruling of W_2^{20} meets 6 conics and each conic meets 10 others.

23. Among the 80 planes which meet $\bar{R}_2^{138}$ in line-pairs there are three groups of 17 each, those of one group all passing through the line v corresponding to the point V, those of another all passing through the line v' corresponding to the point V', and those of the third all passing through the line v'' corresponding to the point V''. The Σ_2^6 corresponding to a plane σ through the line l meets $\bar{R}_2^{138}$ in the lines λ, v, v', v'' all counted 22 times, 7 rulings of W_2^{20} each counted 6 times and 21 conics each counted once.

24. If l is a quadri-secant of j_1^{10}, the locus of bisecants of j_1^{10} which meet l is a ruled surface R_2^{34} excluding the four cones with vertices at the points where l meets the curve. Its transform is a variety $\bar{R}_2^{104}$ of order 104 in M_3^4. The line l being a quadri-secant is the union of 6 bisecants, hence from every point on it 19 bisecants can be drawn to j_1^{10}. Now the transform of l is a set of four lines v, v', v'', v''' meeting in a point L which is a singular point of M_3^4 and also a singular point of $\bar{R}_2^{104}$. All the conics of this variety pass through L. Each line of W_2^{20} meets 5 conics and each conic meets 6 others.

25. All of the 80 planes which intersect $\bar{R}_2^{104}$ in line-pairs pass through L, and of these 17 pass through v, 17 through v', 17 through v'', and 17 through v'''. The variety Σ_2^6 corresponding to any plane through l meets $\bar{R}_2^{104}$ in the lines v, v', v'', v''' each counted 19 times, 6 lines of W_2^{20} each counted 5 times, and 15 conics each once.

26. It remains to mention that any quartic surface F_2^4 in S_3 which contains j_1^{10} transforms into a two-dimensional variety $\bar{F}_2^4$ also of order 4 in M_3^4 as is seen from the formula $N = 6n - k \cdot 20$ for $n = 4$, $k = 1$. $\bar{F}_2^4$ is of the same nature as F_2^4. The fact that F_2^4 contains j_1^{10} shows that it is the intersection of S_3 and a three$_4$dimensional quartic variety $\bar{M}_3^4$ which is the image of a hyperplane $\bar{S}_3$, and that $\bar{F}_2^4$ is the intersection of $\bar{S}_3$ with M_3^4. There is one quartic surface, I_2^4, in S_3 which is its own image and that is the surface in which S_3 meets its own transform M_3^4.

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Compound Singularities of the Rational Plane Quartic Curve.

By J. H. Neelley.

1. *Introduction.* This paper is concerned with those types of the rational plane quartic curve which have two or more coincident singularities, known as compound singularities.[*] In the first part certain theorems are obtained and the invariant conditions are found for the existence of such types. In the second part we introduce some special quartic surfaces whose plane sections include all possible types of rational quartics with compound singularities.

We represent the rational plane quartic R^4_2 by the three binary forms

$$x_i = (\alpha_i t)^4 \qquad (i = 0, 1, 2)$$

and then deduce the pencil of apolar quartics

$$(\beta t)^4 + \lambda(\gamma t)^4$$

The combinants of this pencil involve determinants of the type $\begin{vmatrix} \beta_i & \beta_j \\ \gamma_i & \gamma_j \end{vmatrix}$: which are usually designated by p_{ij}. The invariant conditions for compound singularities are expressed in terms of these combinants.

2. *The Ramphoid Cusp.*[†] The ramphoid cusp or cusp of the second kind has been called by Cayley a node-cusp. To find the necessary and sufficient conditions for its existence, let the cusp parameter be $t = 0$ and the parameter value of an inflectional point $t = \infty$. Then taking as a reference scheme the tangents to the curve at the cusp and at the point of inflection and the line joining these points, the fundamental quartics[‡] are

$$(1) \qquad (\beta t)^4 \equiv 12bt^2 + 12ct, \quad (\gamma t)^4 \equiv 4dt^4 + 4et^3$$

Hence $p_{01} = p_{04} = p_{14} = p_{23} = p_{24} = p_{34} = 0$

$$p_{02} = 8bd, \; p_{03} = 12cd, \; p_{12} = 2be, \; p_{13} = 3ce.$$

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[*] Cayley, *Quarterly Journal*, Vol. 7, p. 212.

[†] Salmon, *Higher Plane Curves*, Chapter VI; F. Klein, *Projective Geometry*, p. 108.

[‡] W. Stahl, "Ueber die rationalen ebenen Curve vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 101, p. 305.

The sextic giving the parameter values of the inflectional points[*] becomes

$$(2) \qquad (st)^6 \equiv 2bdt^5 + (3cd + be)t^4 + 2cet^3 = 0.$$

So three points of inflection coincide at the ramphoid cusp. We note that the other two intersections of the quartic and the conic on the inflectional points[†] have their parameter values given by

$$(3) \qquad (2cd - be)t^2 - cet = 0$$

which shows that the conic on the points of inflection cuts out four coincident points of $R_2{}^4$ at a ramphoid cusp.

Upon consideration of the five fundamental invariants[‡] of $R_2{}^4$ we find that

$$(4) \qquad I'_2 - 4I_2 = 0, \quad I_4 = 0, \quad I_6 = 0.$$

The first of these invariants vanishes when the curve has three double lines on a point; I_4 is the undulation invariant and I_6 the cusp invariant.[§] So the curve with a ramphoid cusp satisfies both the cusp and undulation conditions. This follows from the fact that two points of inflection coincide to form either a cusp or an undulation as the discriminant of the sextic giving the parameters of the inflectional points involves both cusp and undulation invariants.[‖] This discriminant, being of degree 10 in the p's, can only differ from the product of these two invariants by a constant factor and we find that

$$(5) \qquad 2^6\Delta = I_4 I_6.$$

The curve may have cusps without affecting the fundamental involution but undulations do affect this involution. The ambiguity on this point which is evident in most references is removed by the following. When the curve has one undulation the fundamental sextic becomes the sixth power of the undulation. And as line sections of the curve are the second polars of the sextic and three independent ones recover the curve this type curve cannot be recovered as only the undulation line can be gotten from the sextic. Then if the curve has two undulations the sextic is not unique but becomes a pencil of sextics which includes the sixth powers of the undulation parameters as members. And again the curve cannot be recovered from the sextic as all second polars are lines of a pencil which includes the two undulation lines. If the curve has three undulations there is a net of sextics which includes the sixth powers of the three undulations as members. Any member of the net which is a linear combination of each of these sixth powers will have a net of second polars which recovers the curve. For instance, if the undulation parameters are 0, ∞, $+1$, the net is

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[*] A. Brill, "Ueber rationale Curve vierter Ordnung," *Mathematische Annalen*, Vol. 12, p. 104; J. E. Rowe, "Bicombinants of Rational Plane Quartic, etc.", *Transactions of the American Mathematical Society*, Vol. 13.

[†] Stahl, "Rationale ebene Curven vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 101, p. 307.

[‡] The Morley invariants as given by Winger: *American Journal of Mathematics*, Vol. 36, p. 55, and Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

[§] Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

[‖] A. Brill, "Ueber rationale Curve vierter Ordnung," *Mathematische Annalen*, Vol. 12, p. 112; Stahl, "Rationale ebene Curven vierter Ordnung," *Crelle's Journal für Mathematik*, Vol. 104, p. 319.

$$\lambda t^6 + \mu T^6 + \nu(t - T)^6 = 0$$

and if λ, μ and ν are not zero the second polars will recover the curve.

The curve may have an undulation or a cusp in addition to a ramphoid cusp, but requiring the curve to have two ramphoid cusps forces it to degenerate into two double contact conics.

The condition that a binary sextic have a triple root is that Salmon's three invariants A, B and C must satisfy the three relations[*]

$$(6) \qquad A^2 - 100B = 0, \quad 4AB - 5C = 0, \quad AC - 80B^2 = 0.$$

We shall now show the equivalence of conditions (4) and (6). Computing all invariants involved and using the general form of the flex sextic with one root 0 and one ∞ we have

$$I_2 = p_{03}p_{23} - p_{02}p_{24} + p_{12}p_{14} + p_{12}p_{23} - p^2_{13}.$$

$$I'_2 = 8p_{04}p_{13} - 12p_{03}p_{23} - 12p_{12}p_{14} + 36p_{12}p_{23} - p^2_{04} - 4p^2_{13}.$$

$$I_4 = p^4_{04} - 16p^3_{04}p_{13} - 72p_{02}p^2_{04}p_{24} + 96p_{03}p^2_{04}p_{23} - 576p^2_{04}p_{12}p_{23} \\ + 96p^2_{04}p_{12}p_{14} + 1296p^2_{02}p^2_{24}.$$

$$I_6 = 36p_{04}p^2_{12}p_{13}p^2_{23} - 42p_{04}p^3_{12}p_{23}p_{24} - 42p^2_{02}p_{14}p^3_{23} + 6p_{02}p^2_{12}p^2_{23}p_{24} \\ - 9p^2_{02}p^2_{13}p^2_{24} + 12p^2_{02}p_{03}p_{23}p^2_{24} - 24p_{03}p^2_{12}p^3_{23} - 24p^3_{12}p_{14}p^2_{23} \\ + 16p^3_{12}p^3_{23} + 12p^2_{02}p_{12}p_{14}p^2_{24} + 48p^2_{02}p_{12}p_{23}p^2_{24} - 16p^3_{02}p^3_{24} \\ + 27p^2_{02}p^4_{23} + 27p^4_{12}p^2_{24}.$$

$$A = -(1/40)\,(p^2_{04} - 8p_{04}p_{13} + 60p_{02}p_{24} - 48p_{03}p_{23} - 48p_{12}p_{14} \\ - 96p_{12}p_{23} + 64p^2_{13}).$$

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[*] Salmon, *Modern Higher Algebra*, 4th edition, p. 263.

$$\begin{aligned}
B = (1/2^4 \cdot 10^4)\ (&p^4_{04} + 10000p^2_{02}p^2_{24} - 4032p^2_{04}p_{12}p_{23} - 5120p_{02}p_{04}p^2_{23} \\
&- 5120p_{04}p^2_{12}p_{24} - 12800p_{02}p_{12}p_{23}p_{24} - 200p_{02}p^2_{04}p_{24} \\
&+ 8704p_{02}p^2_{13}p_{24} - 16p^3_{04}p_{13} - 128p^2_{04}p^2_{13} - 1024p_{04}p^3_{13} \\
&+ 4096p^4_{13} + 224p^2_{04}p_{12}p_{14} + 2048p_{04}p_{12}p_{13}p_{14} + 14336p_{03}p_{12}p^2_{23} \\
&+ 224p_{03}p^2_{04}p_{23} + 2048p^2_{03}p_{13}p_{24} + 14336p^2_{12}p_{13}p_{24} + 1024p^2_{03}p^2_{23} \\
&+ 4096p^2_{12}p^2_{23} + 1024p^2_{12}p^2_{14} - 6144p_{03}p^2_{13}p_{23} - 6144p_{12}p^2_{13}p_{14} \\
&- 12288p_{12}p^2_{13}p_{23}).
\end{aligned}$$

$$\begin{aligned}
C = (1/2^9 \cdot 5^6)\,(&- p^6_{04} + 1000000p^3_{02}p^3_{24} - 262144p^6_{13} \\
&- 67200p_{02}p^2_{04}p_{12}p_{23}p_{24} - 480000p^2_{02}p_{03}p_{23}p^2_{24} \\
&- 480000p_{02}p_{04}p^2_{12}p^2_{24} - 1920000p^2_{02}p_{12}p_{23}p^2_{24} - 30000p^2_{02}p^2_{04}p^2_{24} \\
&- 1920000p^2_{02}p^2_{13}p^2_{24} + 84480p^3_{04}p^2_{12}p_{24} - 521968p^2_{04}p^2_{12}p_{14}p_{23} \\
&+ 1915392p^2_{04}p^2_{12}p^2_{23} - 122880p^2_{02}p_{14}p^3_{23} + 24p^5_{04}p_{13} \\
&+ 96p^4_{04}p^2_{13} - 6656p^3_{04}p^3_{13} + 6144p^2_{04}p^4_{13} + 98304p_{04}p^5_{13} \\
&- 49152p^2_{02}p^4_{14} - 1382400p^2_{02}p^4_{23} + 84480p_{02}p_{03}p^2_{04}p_{23}p_{24} \\
&- 511968p^3_{03}p_{12}p^2_{24} - 122880p_{03}p^3_{12}p^2_{24} - 99072p^3_{04}p_{12}p_{13}p_{23} \\
&- 215040p_{02}p^2_{12}p^2_{14}p_{24} - 215040p_{02}p^2_{03}p^2_{23}p_{24} \\
&- 1536000p_{02}p^2_{12}p^2_{23}p_{24} - 336p^4_{04}p_{12}p_{14} + 9216p^3_{04}p_{12}p_{13}p_{14} \\
&+ 27648p^2_{04}p_{12}p^2_{13}p_{14} - 270336p_{04}p_{12}p^3_{13}p_{14} - 10752p^4_{04}p_{12}p_{23} \\
&+ 473088p^2_{04}p_{12}p^2_{13}p_{23} - 1425408p_{04}p_{12}p^3_{13}p_{23} \\
&+ 921600p_{02}p_{04}p^2_{13}p^2_{23} - 2260992p_{02}p^3_{13}p^2_{23} + 921600p_{04}p^2_{12}p^2_{13}p_{24} \\
&+ 2899968p_{02}p_{12}p^2_{13}p_{23}p_{24} + 208896p_{04}p^2_{12}p_{13}p^2_{14} \\
&+ 1155072p^2_{03}p_{12}p^3_{23} + 2310144p_{03}p^2_{12}p^3_{23} + 300p_{02}p^4_{04}p_{24} \\
&- 49152p^2_{03}p^2_{04}p^2_{23} - 1382400p^4_{12}p^2_{24} + 9216p^3_{03}p_{04}p_{14}p_{24} \\
&+ 27648p^3_{03}p^2_{14}p_{23} + 208896p^3_{03}p_{14}p^2_{23} + 1155072p^3_{12}p^2_{14}p_{23} \\
&+ 2310144p^3_{12}p_{13}p_{23}p_{24} - 32768p^3_{12}p^3_{14} - 32768p^3_{03}p^3_{23} \\
&- 262144p^3_{12}p^3_{23} - 336p_{03}p^4_{04}p_{23} - 270336p^2_{03}p^3_{13}p_{24} \\
&- 2260992p^2_{12}p^3_{13}p_{24} - 319488p^2_{12}p^2_{13}p^2_{14} - 319488p^2_{03}p^2_{13}p^2_{23} \\
&- 1277952p^2_{12}p^2_{13}p^2_{23} + 589824p_{03}p^4_{13}p_{23} + 589824p_{12}p^4_{13}p_{14} \\
&+ 1179648p_{12}p^4_{13}p_{23}).
\end{aligned}$$

From these we obtain the relations

$$(I'_2 - 4I_2)^2 - I_4 \equiv 320(A^2 - 100B) \tag{7}$$

$$3072I_6 - 192I_2I_4 + (I'_2 - 4I_2)^2\,(I'_2 - 20I_2) \equiv -2^4 \cdot 10^4(4AB - \tag{8}$$

$$(9)\qquad 10240I_6(60I_2+I'_2)+5I_4^2+I_4(10640I_2^2+440I_2I'_2-7I'^2_2)$$
$$-2(I'_2-4I_2)^2(560I_2^2+120I_2I'_2-I'^2_2)\equiv-2^6\cdot 10^6(AC-80B^2)$$

So if equations (4) are satisfied the conditions that the flex sextic have a triple root are fulfilled. Thus the conditions (4) are sufficient to make three flexes coincident. To show that they are necessary, suppose the conditions (6) satisfied, whence

$$(10)\qquad I_4=(I'_2-4I_2)^2$$

$$(11)\qquad 3072I_6=(I'_2-4I_2)^2(212I_2-I'_2)$$

$$(12)\qquad (I'_2-4I_2)^2(260I_2-I'_2)\ (60I_2+I'_2)=0$$

Since $(st)^6$ has a triple root its discriminant vanishes, whence

$$(13)\qquad (1/2^6)I_4I_6=0.$$

Therefore either $I_4=0$ or $I_6=0$. If $I_4=0$ then $I'_2-4I_2=0$ and then $I_6=0$ from (10) and (11). If I_6 vanishes there are two cases. Either I'_2-4I_2 or $212I_2-I'_2$ vanishes. If $I'_2-4I_2=0$, I_4 vanishes and conditions (4) are satisfied. But if I'_2-4I_2 does not vanish we have $212I_2-I'_2=0$ and either $260I_2-I'_2=0$ or $60I_2+I'_2=0$. In any case all second order invariants of the curve vanish and consequently all invariants vanish. This form of R_2^4 has been discussed by Winger,* and avoiding this special type of the curve we have the theorem: *The necessary and sufficient condition that the rational plane quartic have a ramphoid cusp is that three points of inflection coincide. This requires that the three invariants* I'_2-4I_2, I_4 *and* I_6 *vanish simultaneously.*

3. *The Tacnode.* The tacnode has been considered as the result of two coincident nodes and as the nodes are associated with the catalectic sets of the fundamental pencil the tacnode condition has been given as the coincidence of two of the catalectic sets.† If the fundamental pencil associated with R_2^4 is

$$(\beta t)^4+\lambda(\gamma t)^4$$

the catalecticant is

$$(1)\qquad \begin{vmatrix} \beta_0+\lambda\gamma_0 & \beta_1+\lambda\gamma_1 & \beta_2+\lambda\gamma_2 \\ \beta_1+\lambda\gamma_1 & \beta_2+\lambda\gamma_2 & \beta_3+\lambda\gamma_3 \\ \beta_2+\lambda\gamma_2 & \beta_3+\lambda\gamma_3 & \beta_4+\lambda\gamma_4 \end{vmatrix}=0$$

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* *American Journal of Mathematics*, Vol. 36, p. 56.

† Rowe, *Transactions of the American Mathematical Society*, Vol. 12, p. 304.

and its discriminant is

$$(2)\qquad -(1/2^6)\{I_2I_4 - I_2(I'_2 - 4I_2)^2 - 64I_6\}$$

which is taken as the tacnodal invariant. It is obvious that the ramphoid cusp is one type of the tacnode as its conditions make (2) vanish.

Consider R_2^4 with a tacnode and an acnode. Call the contacts of the two tangents to the curve from the tacnode 0 and ∞ and the pencil of lines on the tacnode cuts out the involution $t + T = 0$. Naming one tacnodal parameter 1 the other is -1. Then using the real double line and the joints of its contacts t_1 and t_2 to the tacnode as reference lines we have R_2^4 given by

$$(3)\qquad \begin{aligned} x_0 &= (t^2-1)(t^2-t_2^2) \\ x_1 &= (t^2-1)(t^2-t_1^2) \\ x_2 &= (t-t_1)^2(t-t_2)^2 \end{aligned}$$

and

$$(4)\qquad \begin{aligned} (\beta t)^4 &\equiv 4t^3 - 4t_1t_2t \\ (\gamma t)^4 &\equiv t_1t_2(t_1+t_2)t^4 - 2(1+t_1^2+t_2^2+t_1^2t_2^2+4t_1t_2)t^3 \\ &\qquad + 6t_1t_2(t_1+t_2)t^2 + t_1t_2(t_1+t_2) \end{aligned}$$

The parameters of the acnode are

$$(5)\qquad t = \pm(-t_1t_2)^{1/2}$$

Now the line on the tacnode which with the tacnodal line forms a pair apolar to x_0 and x_1 of (3) cuts out the quartic

$$(6)\qquad (t_1^2+t_2^2-2)t^4 - 2(t_1^2t_2^2-1)t^2 + (2t_1^2t_2^2 - t_1^2 - t_2^2)$$

If this line is on the acnode $t_1 = 1/t_2$ and the acnodal parameters are $\pm i$. Then the curve is self-projective. We show this by observing that Salmon's invariant M, the equivalent of the Morley invariant I_9, vanishes.

$$(7)\qquad M^2 = A(AE - I^2)^2 - 8D(I^3 - 9AEI + 54DE)$$

where E = 0 is the tacnodal condition, so

$$(8)\qquad M^2 = I^3(AI - 8D) \text{ and } AI - 8D = 0.$$

However $I \neq 0$. This condition (8) for a self-projective R_2^4 with a tacnode in the Morley invariants is

$$(9)\qquad I_2'^3 + 68I_2'^2I_2 + 432I'_2I_2^2 - 2880I_2^3 - I'_2I_4 - 12I_2I_4 - 1024I_6 = 0.$$

If the curve has a tacnode and a crunode the reference scheme gives the crunodal parameters as $\pm(t_1t_2)^{1/2}$. Now to make this curve self-projective the crunode must lie on the line (6), but in that case the crunode coincides with the tacnode and we have an oscnode.* This type of curve will be discussed in the next section of this paper.

We wish to establish the theorem that a tacnode is the result of two coincident double lines of the curve. Call two inflectional parameters $t=0$ and $t=\infty$ and use the inflectional lines and their join as a triangle of reference. Then $R_2{}^4$ is

$$\begin{aligned} x_0 &= t^4/e + t^3/d \\ x_1 &= \qquad\quad t^3/d + t^2/c + t/b \\ x_2 &= \qquad\qquad\qquad\qquad\quad t/b + 1/a \end{aligned} \tag{10}$$

and

$$(\beta t)^4 \equiv at^4 + 4bt^3 + 6ct^2, \quad (\gamma t)^4 \equiv 6ct^2 + 4dt + e. \tag{11}$$

A double line of the curve is of the form

$$(t^2 + 2\lambda t + \mu)^2 \tag{12}$$

and the quartic (12) must be apolar to the two base quartics (11). So

$$\begin{aligned} a\mu^2 - 4b\lambda\mu + 2c\mu + 4c\lambda^2 &= 0 \\ 4c\lambda^2 - 4d\lambda + 2c\mu + e &= 0. \end{aligned} \tag{13}$$

Eliminating λ from (13) we have

$$\begin{aligned} a^2c\mu^4 - 4b(ad-2bc)\mu^3 + 2(2ad^2 + 2b^2e - 8bcd - ace)\mu^2 \\ - 4d(be-2cd)\mu + ce^2 = 0 \end{aligned} \tag{14}$$

which gives the four values of μ which, together with the four corresponding values of λ, make (12) represent the four double lines of $R_2{}^4$. The discriminant of equation (14) is

$$(1/4)(ad^2 - b^2e)^2 I'_6. \tag{15}$$

Where I'_6 is the tacnodal invariant. So we have two possible conditions for (14) to have a double root. But $ad^2 - b^2e = 0$ does not give a double root for λ and so (12) does not represent two equal forms under this condition, as we have four distinct double lines. Thus a tacnode is the only condition upon which we have two coincident double lines of the curve. If μ were eliminated from equations (13) we would come to the same conclusion from a study of the resulting quartic in λ. Hence the theorem: *The necessary and sufficient condition for a plane rational quartic to have a tacnode is that two double lines be coincident.*† This theorem shows that the tacnode is a self-dual compound singularity, two coincident nodes or two coincident double lines being the necessary and sufficient condition for such.

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* Salmon, *Higher Plane Curves*, 2nd edition, p. 207.

† Wieleitner, *Theorie der algebraischen, höhern Kurven.*

We may distinguish between the tacnode which is an *embrassement* and the one which is an *opposition* by consideration of the quartic (14). This quartic has a double root when the curve has a tacnode and the other two roots form the coefficients of the remaining two distinct double lines. In the case of the *embrassement* the distinct double lines are real and so the distinct roots of (13) must be real. But for the *opposition* the remaining double lines are imaginary and so the distinct roots of the quartic must be imaginary. So we have the *embrassement* or *opposition* according as

$$2Hg_2 - 3\alpha g_3 \lessgtr 0 \tag{16}$$

where α is the leading coefficient of (13) and H, g_2 and g_3 are the regular invariants of the binary quartic.*

4. *The Oscnode.* The three nodes of the rational plane quartic may coincide as consecutive points giving rise, not to a triple point, but to an oscnode which is an osculation of two branches of the curve. We shall now show that this compound singularity is due to three coincident double lines. With a reference scheme similar to that of the previous section the catalectic cubic is

$$\begin{aligned} &c^3\lambda + (3c^3 + b^2e - 2bcd - ace)\lambda^2 \\ &\qquad + (3c^3 + ad^2 - 2bcd - ace)\lambda + c^3 = 0 \end{aligned} \tag{1}$$

and the quartic giving the coefficient of the double lines is

$$\begin{aligned} &16ac^2\lambda^4 - 32(acd - bc^2)\lambda^3 + 8(ace + 2ad^2 - 4bcd)\lambda^2 \\ &\qquad - 8(ade - bce - 2c^2d)\lambda + ae^2 - 4c^2e = 0. \end{aligned} \tag{2}$$

We have an oscnode when (1) is a cube and the condition is the identical vanishing of its Hessian. That is

$$\frac{3c^3}{3c^3 + b^2e - 2bcd - ace} = \frac{3c^3 + b^2e - 2bcd - ace}{3c^3 + ad^2 - 2bcd - ace} = \frac{3c^3 + ad^2 - 2bcd -}{3c^3}$$

Equating these in pairs we have the condition that (1) be a cube. That is

$$A = B = C = 0 \tag{3}$$

where

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* Burnside and Panton, *Theory of Equations*, Vol. 1, Ex. 31, p. 153.

$$A \equiv a^2c^2e^2 + a^2d^4 - 2a^2cd^2e + 6ac^3d^2 - 3ac^4e - 4abcd^3 + 4abc^2de$$
$$+ 4b^2c^2d^2 - 3b^2c^3e - 6bc^4d.$$
$$B \equiv a^2cd^2e - a^2c^2e^2 - ab^2d^2e + ab^2ce^2 + 6ac^4e - 4abc^2de + 2abcd^3$$
$$+ 12bc^4d - 3ac^3d^2 - 3b^2c^3e - 4b^2c^2d^2 + 2b^3cde.$$
$$C \equiv a^2c^2e^2 - 3ac^3d^2 - 3ac^4e - 2ab^2ce^2 + 4abc^2de - 6bc^4d + 6b^2c^3e$$
$$+ 4b^2c^2d^2 - 4b^3cde + b^4e^2.$$

Then the conditions that equation (2) have a triple root are that its invariants g_2 and g_3 vanish. Computing g_2 we find that it is $(3/2^6)A$. Now using the fact that the Hessian of cubic (1) has the same discriminant as the cubic we have

$$(4) \qquad 4AC - B^2 = KI'_6$$

where I'_6 is the tacnodal invariant. But the discriminant of the quartic (2) is

$$g_2{}^3 - 27g_3{}^2 \equiv 2^{10}c^6I'_6$$

whence

$$(5) \qquad g_2{}^3 - 27g_3{}^2 \equiv Kc^6(4AC - B^2).$$

And relation (5) shows that three coincident nodes necessitates three coincident double lines. Then the converse is true when we neglect the factor c. This may be done as $c = 0$ gives a curve with two distinct cusps and so cannot be a type of oscnode.* So we have the theorem: *The necessary and sufficient condition for a rational plane quartic curve to have an oscnode is the coincidence of three double lines of the curve.*†

5. *The Oscnodal Conditions.* Considering an oscnode as formed by sending the third node to the point of a tacnode on the curve and using a reference scheme similar to that proposed for the tacnode we find that the contacts of the real double line are in the relation $t_2 = \pm 1/t_1$, where the sign depends upon the third node being an acnode or a crunode. In both cases the curve may be represented by

$$(1) \qquad \begin{aligned} x_0 &= t^4 - (1 + 1/t_1{}^2)t^2 + 1/t_1{}^2 \\ x_1 &= t^4 - (1 + t_1{}^2)t^2 + t_1{}^2 \\ x_2 &= t^4 - 2(t_1 - 1/t_1)t^3 + (t_1{}^2 - 4 + 1/t_1{}^2)t^2 + 2(t_1 - 1/t_1)t + 1 \end{aligned}$$

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* If $6c$ and $6c'$ are the coefficients of t^2 in the members of the fundamental pencil we do not have two cusps if either c or c' vanishes but the curve has distinct double lines.

† Wieleitner, *l. c.*

and

$$(2)\qquad \begin{aligned}(\beta t)^4 &\equiv t^3 + t\\ (\gamma t)^4 &\equiv t^4 + 2(t_1 - 1/t_1)t^3 + 6t^2 + 1.\end{aligned}$$

The catalecticant is $\lambda^3 = 0$ and consideration of the Morley invariants shows that together with the tacnode invariant there is an

$$\bar{I}_4 \equiv (I'_2 - 4I_2)(I'_2 + 60I_2) - I_4$$

which vanishes. When this $\bar{I}_4$ is calculated for a curve with a tacnode, it vanishes if the third node is at the tacnode. But $\bar{I}_4 \equiv 0$ for the ramphoid cusp so to distinguish between the two types of curve it is necessary that $I_6 \neq 0$. Hence the theorem: *The necessary and sufficient conditions for an oscnode are*

$$I'_6 = 0, \quad \bar{I}_4 = 0, \quad I_6 \neq 0.$$

If there is a cusp at the oscnode we have the same curve which would arise if the ramphoid cusp type had the third node coincident with the cusp. In addition to the ramphoid cusp invariants the type requires that each of the self-apolar members of the fundamental pencil be apolar to the Hessian of the other.

Using for reference lines the line on the oscnode which with the oscnodal line forms a pair apolar to x_0 and x_1 of (1), the oscnodal line and the other double line, the curve is given by

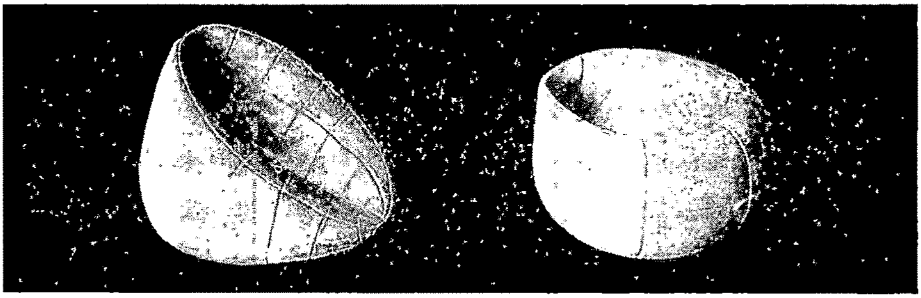
$$(3)\qquad \begin{aligned}x_0 &= t^4 - 1\\ x_1 &= t^4 - 2t^2 + 1\\ x_2 &= \{t^2 - (t_1 - 1/t_1)t - 1\}^2.\end{aligned}$$

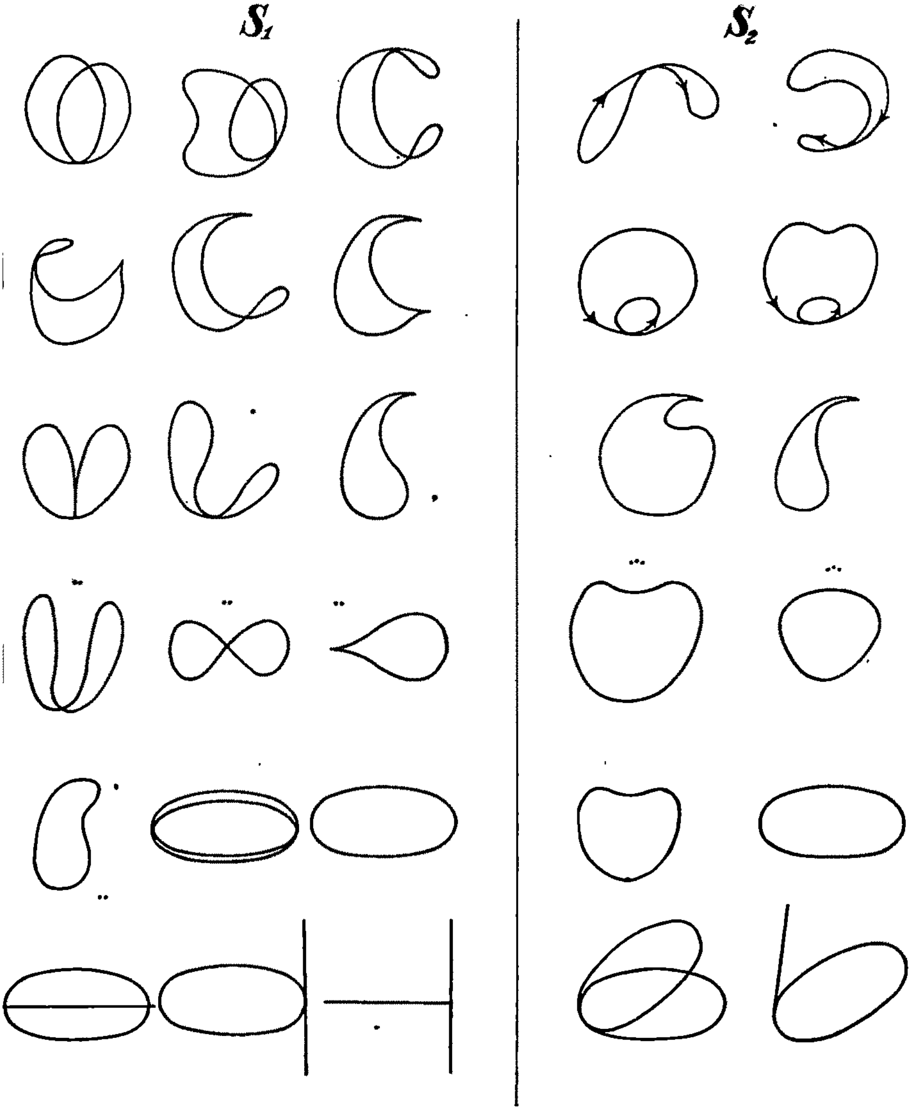
This representation shows that the curve is reflected into itself by $(1, 0, 0)$ and x_0. That this type is self-projective is easily verified by the vanishing of the Morley invariant I_9.

$$(4)\qquad I_9{}^2 = A(AE - I^2) - 8D(I^3 - 9AEI + 54DE)$$

where E is the tacnodal invariant and I is the oscnodal invariant introduced above.

Attention is called, however, to the fact that the invariant form $AI - 8D$ does not vanish for an oscnode. Remembering that it does vanish for the self-projective curve with a tacnode and that the invariant I does not, we have a method for distinguishing between the oscnodal and the self-projective curve with a tacnode.




S_1
S_2

6. *The Tacnodal and Oscnodal Surfaces.* It is well known that all forms of the rational plane quartic with simple singularities are, to within projection, plane sections of the Steiner Surface. This surface has three double lines which give the nodes of the plane sections. Now curves with tacnodes have two coincident nodes, considering this fact and using Salmon's suggestions for modified forms of the equation of the Steiner Surface,* we have constructed a tacnodal surface S_1, which has the equation

$$x_0^4 + x_1^2x_2^2 + x_0^2x_1x_3 = 0.$$

The second surface S_2 we present has one double line and the surface osculates itself along this line and has the equation

$$(x_1^2 - x_0x_2)^2 - x_0^3x_3 = 0.$$

All plane sections of these surfaces are rational quartics with tacnodes and oscnodes and, to within projection, all forms of the curve with these singularities may be cut from these surfaces. These surfaces and types of their plane section are given in the lower part of the figure on page 399.

The curves with isolated tacnodes and oscnodes have these singularities represented by two and three dots respectively and such belong to the curve immediately below or to the left of the dots. The last five drawings under S_1 are degenerate forms which include the conic of the tropes and the squares of the two double lines. The last three drawings under S_2 are also degenerate forms being the trope conic, two osculating conics and the double line tangent to a conic.

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* Salmon-Rogers, *Analytic Geometry of Three Dimensions*, Vol. 2, p. 214.

Pencils of Conics in the Galois Fields of Order 2^n.

BY ALAN D. CAMPBELL.

We define a conic in a Galois field of order 2^n [which field we write more briefly $GF(2^n)$] as the locus of all points whose coordinates x, y, z satisfy an equation of the form

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy = 0, \tag{1}$$

whose coefficients and variables represent numbers in this $GF(2^n)$. The discriminant of (1) is

$$\Delta \equiv fgh + af^2 + bg^2 + ch^2. \tag{2}$$

A pencil of conics then has the equation

$$\lambda C_1 + \mu C_2 = 0, \tag{3}$$

where $C_1 = 0$ and $C_2 = 0$ are two distinct conics. In this paper we derive the classes, or sets of classes, of projectively equivalent pencils and we give a typical pencil for each class, or set of classes. If there is an arbitrary coefficient in the typical pencil we say this pencil represents a set of classes, whenever different values of this coefficient may give nonequivalent pencils and so represent distinct classes. The discriminant Δ of the general conic of a pencil we call the discriminant of the pencil. If we subject (3) to the transformation

$$x = a_1x' + b_1y' + c_1z', \quad y = a_2x' + b_2y' + c_2z', \quad z = a_3x' + b_3y' + c_3z', \tag{4}$$

with determinant $| a_1, b_2, c_3 | \neq 0$, we get for the new pencil

$$\Delta' \equiv | a_1, b_2, c_3 |^2 \Delta.$$

(After such a transformation as this we drop the primes from the variables, without explicit mention of this fact).

We note that in a $GF(2^n)$ every number is a perfect square with just one square root, also $(\alpha x + \beta y + \gamma z)^2 = \alpha^2x^2 + \beta^2y^2 + \gamma^2z^2$. In any $GF(2^n)$ we have the following typical conics

$$x^2 + yz = 0, \tag{5}$$

$$xy = 0, \tag{6}$$

(7) $$x^2 + y^2 + \alpha xy = 0,$$

where $z^2 + \alpha z + 1 = 0$, $z = x/y$, is irreducible in this $GF(2^n)$*,

(8) $$x^2 = 0.$$

We first divide the classes of pencils of conics into the following sets:

Set I. Each pencil of this set contains at least one double line.

Set II. Each pencil has no double line, but has at least one real line pair.

Set III. Each pencil has no double line and no real line pair, but has at least one conjugate imaginary line pair.

Set IV. Each pencil has no degenerate conics of any description.

Set I. We put (3) in a form (A) where $C_1 \equiv x^2$ and $C_2 = 0$ is of the form (1) with $a = 0$. If $f = g = h = 0$, $b \neq 0$ in (A), we put

$$x = x', \quad y = (1/b^{1/2})y' + (c^{1/2}/b^{1/2})z', \quad z = z',$$

and we get the following class with a typical pencil and its discriminant

(9) Class 1, $$\lambda x^2 + \mu y^2 = 0, \quad \Delta \equiv 0.$$

If $h \neq 0$, $f = 0$ in (A),† we put

$$x = x', \quad y = (1/h)y' + (g/h)z', \quad z = z',$$

and we get a pencil (A') with $C_1' \equiv x^2$ and $a' = f' = g' = 0$, $h' = 1$ in C_2'. If $c' \neq 0$ in (A'), we obtain by an obvious transformation

(10) Class 2, $$\lambda x^2 + \mu(z^2 + xy) = 0, \quad \Delta \equiv \mu^3.$$

If $c' = 0$, $b' \neq 0$ in (A'), we have

(11) Class 3, $$\lambda x^2 + \mu(y^2 + xy) = 0, \quad \Delta \equiv 0.$$

The pencil (11) has only one double line and so is not equivalent to (9).

If $b' = c' = 0$ in (A'), we have

(12) Class 4, $$\lambda x^2 + \mu xy = 0, \quad \Delta \equiv 0.$$

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* See L. E. Dickson, "Linear Groups," p. 199.

† If $h = f = 0$, $g \neq 0$, we put $x = x'$, $y = z'$, $z = y'$ and we have again the case $h \neq 0$, $f = 0$.

The pencil (12) has just one double line and so is not equivalent to (9). Any transformation (4) that is to send (11) into (12) must have $b_1 = c_1 = c_2 = 0$, $b_2 c_3 \neq 0$. But then (4) sends C_2 of (11) into $a_2^2 x'^2 + b_2^2 y'^2 + \cdots$ which has a term in y'^2, whereas (12) has no such term.

If $f \neq 0$ in (A), we can reduce this pencil to a form (A''), with $C_1' \equiv x^2$ and C_2' having $a' = h' = g' = 0$, $f' = 1$. If $C_2' = 0$ is reducible to (7) we get

(13) Class 5, $\lambda x^2 + \mu(y^2 + z^2 + \alpha yz) = 0, \quad \Delta \equiv \alpha^2 \lambda \mu^2.$

If $C_2' = 0$ in (A'') is reducible to (6), we get

(14) Class 6, $\lambda x^2 + \mu yz = 0, \quad \Delta \equiv \lambda \mu^2.$

The pencil (14) has a real line pair and so is nonequivalent to (13).

Set II. We put (3) in a form (B) where $C_1 \equiv xy$ and C_2 has $h = 0$. If $cg \neq 0$ in (B)* and $ax^2 + cz^2 + gzx$ is factorable, we can reduce (B) to a form (B') with $C_1' \equiv xy$ and $a' = 0$ $c'g' \neq 0$. If in (B') $b'y^2 + f'yz + c'z^2$ is factorable and $f' \neq 0$, we can reduce (B') to a form (B'') with $C_1'' \equiv xy$ and $c''f''g'' \neq 0$, $a'' = b'' = h'' = 0$, and we obtain

(15) Class 7, $\lambda xy + \mu(z^2 + yz + zx) = 0, \quad \Delta \equiv \lambda\mu(\lambda + \mu).$

If in (B') $f' \neq 0$ and the terms in y and z alone are not factorable, we obtain

(16) Class 8, $\lambda xy + \mu(z^2 + y^2 + \alpha yz + zx) = 0,$

$$\Delta \equiv \mu(\lambda^2 + \mu^2 + \alpha\lambda\mu),$$

where α makes the quadratic factor in Δ irreducible.

If in (B') $f' = 0$, we have

(17) Class 9, $\lambda xy + \mu(z^2 + zx) = 0, \quad \Delta \equiv \lambda^2 \mu.$

If in (B) neither the three terms in x and z alone nor the three terms in y and z alone are factorable, we arrive at

(18) Set 10, $\lambda xy + \mu(x^2 + y^2 + z^2 + \alpha yz + \beta zx) = 0,$

$\alpha\beta \neq 0$, where $x^2 + z^2 + \beta zx$ and $y^2 + z^2 + \alpha yz$ are irreducible, with

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* We must have $g \neq 0$ or $f \neq 0$ or $fg \neq 0$, otherwise $C_2 = 0$ is a double line. If $g = 0$, $cf \neq 0$, we can put $x = y'$, $y = x'$, $z = z'$ and get the case $cg \neq 0$.

$$\Delta \equiv \mu[\lambda^2 + \alpha\beta\lambda\mu + (\alpha^2 + \beta^2)\mu^2].$$

If the quadratic factor of Δ is irreducible, then (18) may possibly be equivalent to (16). However in this case (4) must have $b_1 = c_1 = 0$, $a_1 \neq 0$, $a_2 = c_2 = 0$, $b_2 \neq 0$, $c_3 \neq 0$, $a_1^2 + a_3^2 + \beta a_1 a_3 = 0$ [because (16) has no term in x^2]. But this last equation has only the solution $a_1 = a_3 = 0$. If the quadratic factor of Δ is reducible, then (18) may possibly be equivalent to (15). However (4) must then have

$$a_1 a_2 = 0, \quad b_1 b_2 = 0, \quad a_1^2 + a_2^2 + a_3^2 + \alpha a_2 a_3 + \beta a_1 a_3 = 0,$$
$$b_1^2 + b_2^2 + b_3^2 + \alpha b_2 b_3 + \beta b_1 b_3 = 0,$$

since (15) has no terms in x^2 and y^2. But then we have $a_1 \neq 0$ (say), $a_2 = 0$, which gives us $a_1^2 + a_3^2 + \beta a_1 a_3 = 0$ (an irreducible equation).

If in (B) $c = 0$, $f \neq 0$, we have a pencil reducible to (17).

If in (B) $c = f = 0$, $b \neq 0$, we have

(19) Class 11, $\lambda xy + \mu(y^2 + zx) = 0, \quad \Delta \equiv \mu^3.$

If in (B) $c = f = b = 0$, we obtain

(20) Class 12, $\lambda xy + \mu xz = 0, \quad \Delta \equiv 0.$

If in (B) $f = 0$, $c \neq 0$, and the three terms in x and z alone in C_2 are not factorable, we get

(21) Class 13, $\lambda xy + \mu(x^2 + z^2 + \alpha zx) = 0, \quad \Delta \equiv \lambda^2\mu.$

The line pair $C_2 = 0$ in (21) is conjugate imaginary, so this pencil is non-equivalent to (17).

Set III. We put (3) in a form (C) with $C_1 = 0$ of the form (7) and with $h = 0$ in C_2. We must have $f \neq 0$ or $g \neq 0$ or $fg \neq 0$, otherwise $C_2 = 0$ is a double line. We suppose $g \neq 0$ (the case $g = 0$, $f \neq 0$ is reducible to this). We must also have $c \neq 0$, otherwise (C) contains a real line pair. We now transform (C) to a pencil (C') with $C_1' = 0$ of the form (7) and with $f' = h' = 0$, $c' \neq 0$, $g' = 1$. Also in (C') either $a' \neq 0$ or $b' \neq 0$ or $a'b' \neq 0$, otherwise $C_2' = 0$ is a real line pair.

If $b' \neq 0$ in (C'), we get

(23) Set 14, $\lambda(x^2 + y^2 + \beta xy) + \mu(y^2 + z^2 + \gamma xz) = 0,$

$\beta\gamma \neq 0$, $\Delta \equiv \mu(\beta^2\lambda^2 + \gamma^2\lambda\mu + \gamma^2\mu^2)$. We assume for this set that γ makes the quadratic factor of Δ irreducible.

If the quadratic factor of Δ of (23) is reducible, then we take for $C_1' = 0$ and $C_2' = 0$ in (C') two conjugate imaginary line pairs with vertices at $(0, 0, 1)$ and $(0, 1, 0)$ respectively.* We obtain

$$(24) \quad \text{Set 15,} \qquad \lambda(x^2 + y^2 + \alpha xy) + \mu(x^2 + z^2 + \beta xz) = 0$$

$\alpha\beta \neq 0$, $C_1 = 0$ and $C_2 = 0$ reducible to (7), $\Delta \equiv \lambda\mu(\alpha^2\mu + \beta^2\mu)$. The third degenerate conic of (24), which must be a conjugate imaginary line pair, is

$$(\beta^2/\alpha^2 + 1)x^2 + (\beta^2/\alpha^2)y^2 + z^2 + (\beta^2/\alpha)xy + \beta xz = 0.\dagger$$

This reduces to the form (17) with $\alpha/(\alpha^2 + \beta^2)^{1/2}$ instead of α.

If $b' = 0$ in (C') we get Set 15 again.

We note that there is no pencil with just one (or with just two) conjugate imaginary line pairs for its degenerate conics.

Set IV. We reduce the discriminant of (3) to the form

$$(25) \qquad \Delta \equiv \lambda^3 + \lambda^2\mu + \alpha\mu^3,$$

which is to be an irreducible cubic. Then we put (3) in a form (D) with $C_1 \equiv x^2 + yz$. We must have $g \neq 0$ or $h \neq 0$ or $gh \neq 0$ in C_2 of (D), otherwise (D) has a double line. We suppose $g \neq 0$ (the case $g = 0$, $h \neq 0$ is reducible to this). Now we reduce (D) to a form (D') with $C_1' \equiv x^2 + b''y^2 + yz$ and with $f' = h' = 0$, $g' = 1$ in C_2', by transformations on x, y, z alone. The discriminant $\Delta \equiv \lambda^3 + b'\mu^3 + a'\lambda^2\mu + b''\lambda\mu^2$ must be of the form (25), so we have $b' = \alpha$, $a' = 1$, $b'' = 0$. We get

$$(26) \quad \text{Set 16,} \qquad \lambda(x^2 + yz) + \mu(x^2 + \alpha y^2 + \beta z^2 + xz) = 0,$$

with (25) as its discriminant.

Suppose now that in $GF(2^n)$ we have $2^n = 3k + 1$ and the pencil (3) has a discriminant reducible to $\Delta \equiv \lambda^3 + \alpha\mu^3$, where $\alpha \neq$ cube. We put the discriminant in this form, we reduce $C_1 = 0$ to the form (5); then our pencil has a discriminant $\Delta \equiv \lambda^3 + \Delta_2\mu^3 + c\lambda^2\mu + (fg + h^2)\lambda\mu^2$, where Δ_2 belongs to $C_2 = 0$. So we must have $c = 0$, also $f \neq 0$ or $g \neq 0$ or $fg \neq 0$ (otherwise the pencil has a degenerate conic). We suppose $f \neq 0$ (the case $f = 0$,

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* A pencil containing two conjugate imaginary line pairs with the same vertex contains also a double line.

† This set evidently does not exist in $GF(2)$.

$g \neq 0$ is reducible to this). We perform some evident transformations and get the pencil into the form

$$\lambda(a_1x^2 + z^2 + xy) + \mu(a_2x^2 + b_2y^2 + yz) = 0,$$

with $\Delta \equiv \lambda^3 + a_2\mu^3 + a_1\lambda\mu^2$. So we have $a_1 = 0$, and we get

(27) Set 17, $\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) = 0$,

$\Delta \equiv \lambda^3 + \alpha\mu^3$, $\alpha \neq$ cube.

We now append a list of the classes of pencils of conics, together with their typical pencils and their discriminants.

Class 1, $\lambda x^2 + \mu y^2 = 0$, $\Delta \equiv 0$.
Class 2, $\lambda x^2 + \mu(z^2 + xy) = 0$, $\Delta \equiv \mu^3$.
Class 3, $\lambda x^2 + \mu(y^2 + xy) = 0$, $\Delta \equiv 0$.
Class 4, $\lambda x^2 + \mu xy = 0$, $\Delta \equiv 0$.
Class 5, $\lambda x^2 + \mu(y^2 + z^2 + \alpha yz) = 0$, $\Delta \equiv \alpha^2\lambda\mu^2$.
Class 6, $\lambda x + \mu yz = 0$, $\Delta \equiv \lambda\mu^2$.
Class 7, $\lambda xy + \mu(z^2 + yz + xz) = 0$, $\Delta \equiv \lambda\mu(\lambda + \mu)$.
Class 8, $\lambda xy + \mu(z^2 + y^2 + \alpha yz + xz) = 0$, $\Delta \equiv \mu(\lambda^2 + \mu^2 + \alpha\lambda\mu)$.
Class 9, $\lambda xy + \mu(z^2 + zx) = 0$, $\Delta \equiv \lambda^2\mu$.
Set 10, $\lambda xy + \mu(x^2 + y^2 + z^2 + \alpha yz + \beta xz) = 0$,
$\Delta \equiv \mu[\lambda^2 + \alpha\beta\lambda\mu + (\alpha^2 + \beta^2)\mu^2]$.
Class 11, $\lambda xy + \mu(y^2 + zx) = 0$, $\Delta \equiv \mu^3$.
Class 12, $\lambda xy + \mu xz = 0$, $\Delta \equiv 0$.
Class 13, $\lambda xy + \mu(x^2 + z^2 + \alpha xz) = 0$, $\Delta \equiv \lambda^2\mu$.
Set 14, $\lambda(x^2 + y^2 + \beta xy) + \mu(y^2 + z^2 + \gamma xz) = 0$,
$\Delta \equiv \mu(\beta^2\lambda^2 + \gamma^2\lambda\mu + \gamma^2\mu^2)$.
Set 15, $\lambda(x^2 + y^2 + \alpha xy) + \mu(x^2 + z^2 + \beta xz) = 0$,
$\Delta \equiv \lambda\mu(\alpha^2\lambda + \beta^2\mu)$.
Set 16, $\lambda(x^2 + yz) + \mu(x^2 + \alpha y^2 + \beta z^2 + xz) = 0$,
$\Delta \equiv \lambda^3 + \lambda^2\mu + \alpha\mu^3$.
Set 17, $\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) = 0$,
$\Delta \equiv \lambda^3 + \alpha\mu^3$.

SYRACUSE, NEW YORK.

On Generalized Lacunae.*

By John J. Gergen.

1. *Introduction.* The lacunae of a Taylor's series are those terms with zero coefficients; a series

$$T(x)=\sum_{m=0}^{\infty} c_m x^m$$

having lacunae can be written in form

$$T(x)=\sum_{n=0}^{\infty} c_{\lambda_n} x^{\lambda_n} \tag{1}$$

where $c_m=0$ if $m\neq\lambda_n$, and $c_m\neq 0$ if $m=\lambda_n$. The effects of lacunae in series upon the nature of the singularities of the function defined by the series have been considered by many writers.** Thus, for example, Hadamard † has proved that if in series (1)

$$(\lambda_{n+1}-\lambda_n)/\lambda_n>\lambda>0 \qquad (n=0,1,2,\cdots),$$

then the circle $|x|=R$, where $R^{-1}=\overline{\lim}\,|c_{\lambda_n}|^{1/\lambda_n}$ and is supposed finite but different from zero, is a cut for the function $T(x)$. Fabry ‡ has generalized this theorem by proving that the same conclusion holds if

$$\lim\,(\lambda_{n+1}-\lambda_n)=\infty.$$

Mandelbrojt § has proved that if, for some integer $p\geq 0$

$$\overline{\lim}\,(\lambda_{n+1}-2^p\lambda_n)=\infty,$$

then $T(x)$ cannot be written in the form

$$\phi(x)/[P(x)]^{1/(p+1)},$$

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* Preliminary report presented to the American Mathematical Society, April 16, 1927. A résumé of the theorems appears in the *Comptes Rendus des Séances de l'Académie des Sciences*, t. 184 (1927), pp. 1040-1043.

** For a bibliography and discussion of the work that has been done on the series of Taylor and their analytic prolongation see Hadamard et Mandelbrojt, "La Série de Taylor et Son Prolongement Analytique," *Scientia*, No. 12, Paris (1926).

† Hadamard, *Journal de Mathématiques*, t. 8 (1892), p. 101.

‡ Fabry, *Acta Mathematica*, t. 22 (1898), p. 65.

§ Mandelbrojt, *Annales Scientifiques de l'École Normale Supérieure*, 3e série, t. 40 (1923), p. 413. Dr. Mandelbrojt suggested this problem on generalized lacunae.

where $\phi(x)$ is regular within a circle $|x| = R' > R$, and $P(x)$ is a polynomial. Tsuji [*] and Pólya [†] independently have generalized this theorem by showing that if

$$\overline{\lim}\,(\lambda_{n+1} - \lambda_n) = \infty,$$

then $T(x)$ has on the circle $|x| = R$ at least one singular point which is not an algebraic-logarithmic singular point.[‡]

It is the purpose here to determine facts about the singularities of functions defined by Taylor's series having geenralized lacunae. Mandelbrojt [§] defines the generalized lacunae of the series $\sum_0^\infty a_n x^n$, relative to the entire function $g(z)$, as the ordinary lacunae of the series $\sum_0^\infty g(a_n) x^n$. It should be noted that if one chooses $g(z)$ such that $g(0) = 0$, a series having ordinary lacunae can be made to have generalized lacunae with respect to this function $g(z)$. On the other hand, the fact that a series has generalized lacunae does not imply that it has ordinary lacunae.

2. *Preliminary Definition and Theorem.* Definition. Suppose β an essential singularity of $f(x)$ and that $f(x)$ is regular except at β. We say that β is a singularity of the exponential order q (≥ 0) if the entire [‖] function in x, $f((x\beta + 1)/x)$, is of the apparent order q.[¶] If $f(x)$ is an entire function, the point at infinity is a singularity of the exponential order q if $f(x)$ is of the apparent order q.

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[*] Tsuji, *Japanese Journal of Mathematics*, Vol. 3 (1926), p. 69.

[†] Pólya, *Comptes Rendus des Séances de l'Académie des Sciences*, t. 184 (1927), p. 502.

[‡] The function $f(x)$ is said to have at the point x_0 an algebraic-logarithmic singular point if it can be represented in a circle C with center x_0 by the sum of a finite number of terms of the form

$$(x - x_0)^s\,[\log\,(x - x_0)]^q\,\phi(x),$$

where s is a rational number, q a non-negative integer, and $\phi(x)$ is regular in C.

[§] Mandelbrojt, *Bulletin de la Société Mathématique de France*, t. 53 (1925), p. 235. The above definition is less general than that given by Mandelbrojt but is sufficient for our purposes.

[‖] We do not classify polynomials as entire functions. See Borel, *Leçons sur les Fonctions Entières*, 2nd ed., Paris (1921), p. 1.

[¶] The entire function $f(z)$ is of the apparent order q if corresponding to an arbitrary $\epsilon > 0$ there exists an r_ϵ such that if $|z| = r > r_\epsilon$ then

$$\text{maximum of } |f(z)| \text{ on } |z| = r < e^{r^{q+\epsilon}}$$

and if q is the smallest number having this property. See Borel, *loc. cit.*, p. 75.

Since for x^t, where t is an arbitrary positive integer, we have

$$| x^t | < e^{|x|^\epsilon}$$

for $| x |$ sufficiently large, ϵ being arbitrarily small, we shall say that β is a singular point of $f(x)$ of the exponential order zero if it is a pole of $f(x)$ and $f(x)$ is regular except at β.

Finally, if $F(x) = f(x) + \phi(x)$ and if $\phi(x)$ is regular at β, we shall say that β is a singular point of $F(x)$ of the exponential order q if it is that for $f(x)$.

We then have the preliminary theorem.

THEOREM I. *Let* $f(x) = \sum_0^\infty a_n x^n$ *be a uniform function having as its only possible singularities the pth roots of unity and the point at infinity. Suppose that no singular point of* $f(x)$ *on the circle* $| x | = 1$ *is of the exponential order greater than* q.* *Let* $g(z) = \sum_1^\infty \alpha_m z^m$ *be an entire function such that* †

$$| \alpha_m |^{1/m} = 0 \ (e^{-2^{(q+1)m}})$$

Then the function $G(x) = \sum_0^\infty g(a_n) x^n$ *has as its only possible singularities the pth roots of unity and the point at infinity.*‡

For the function $g(z)$ we might take

$$\sum_1^\infty z^m / m! e^{m 2^{(q+1)m}}$$

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* This assumption implies that if β_k is singular point of $f(x)$, we can write

$$f(x) = \xi_k(x) + \Gamma(x)$$

where $\xi_k(x)$ is regular except at the point β_k and $\Gamma(x)$ is regular at β_k. This is possible if $f(x)$ is uniform.

† This is the notation of Landau. In this instance it means that

$$\lim_{m=\infty} | \alpha_m |^{1/m} e^{2^{(q+1)m}} = 0.$$

‡ For theorems of similar nature see Hadamard et Mandelbrojt, *loc. cit.*, pp. 48-53. See also Soula, *Journal de Mathématiques*, t. 86 (1921), p. 94, and Leau, *Comptes Rendus des Séances de la Société Mathématique de France*, t. 25-26 (1897-98), p. 267, and *Journal de Mathématiques*, t. 5 (1899), p. 365. For the case where one has

$$\lim (x - \beta)^q f(x) = 0$$

for all the singularities β, Leau replaces the above condition on $g(z)$ by one much less restrictive. The above theorem is proved in a manner similar to Leau's.

Proof: Consider the sequence of functions $\{f_m(x)\}$, where

$$f_1(x) = f(x) = \sum_0^\infty a_n x^n,$$

$$f_m(x) = H(f_1, f_{m-1}) = \sum_{n=0}^\infty a_n{}^m x^n \qquad (m = 2, 3, \cdots).$$

By Hadamard's * theorem on the multiplication of singularities $f_m(x)$ has as its only possible singularities the set β consisting of

$$\beta_k = e^{2\pi i(k/p)} \qquad (k = 0, 1, \cdots, p-1),$$

and γ the point at infinity. Moreover, $f_m(x)$ is a uniform function.†

Let $\epsilon_1 > 0$ be arbitrarily small but $< |\beta_1 - \beta_0|/2$ if $p > 2$, or $< 1/2$ if $p \leq 2$. Let R_1 be arbitrarily large but $> 1 + \epsilon_1$. Let D_1 be the open region lying exterior to the circles $|x - \beta_k| = \epsilon_1$ but interior to the circle $|x| = R_1$. We desire to prove that the series

$$(2) \qquad \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) + \cdots$$

converges uniformly in D_1.

Consider the uniform functions $\phi(x) = \sum_0^\infty u_n x^n$, $\psi(x) = \sum_0^\infty v_n x^n$ having as their only possible singularities the set β. Let $\eta > 0$ be arbitrarily small but $\leq \epsilon_1$ and σ be arbitrarily large but $\geq R_1$. Let $\eta' = (1 + \eta)^{\frac{1}{2}} - 1$, $\sigma' = \sigma/(1 - \eta)$. Construct a circle K', $|x| = \sigma'$. Construct the circles b_k', $|x - \beta_k| = \eta'$. Let S' be the open region lying exterior to the circles b_k' but interior to the circle K' and let T' be its boundary. Through the points of intersection of b_k' with the circle $|x| = 1$ construct radii to K'. Let l_k', l_k'' be the segments of these radii from $|x| = 1$ to $|x| = \sigma'$, where

$$\text{angle of inclination of } l_k' = \text{amplitude } \beta_k + \omega,$$
$$\text{angle of inclination of } l_k'' = \text{amplitude } \beta_k - \omega.$$

If we regard the segments l_k' as forming cuts of two sides the curve K_1 consisting of T' and the segments l_k' bounds a simply connected open region S_1. If x is any point in S_1 and z any point on K_1, then x/z is not a point of β, for if it were, we should have either $x = z\beta_k$, whence x would be a point of K_1, or else x would be the point at infinity. The function

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* Hadamard, *Acta Mathematica*, t. 22 (1898), p. 55.

† Borel, *Comptes Rendus des Séances de la Société Mathématique de France*, t. 25-26 (1897-98), p. 238.

$$F_1(x) = \frac{1}{2\pi i} \int_{K_1} \phi(z)\,\psi\left(\frac{x}{z}\right)\frac{dz}{z}$$

is then as in Hadamard's theorem the analytic prolongation of

$$\sum_{0}^{\infty} u_n v_n x^n \tag{3}$$

for x in S_1. Moreover, for these values of x we have

$$F_1(x) = \frac{1}{2\pi i} \int_{T'} \phi(z)\,\psi\left(\frac{x}{z}\right)\frac{dz}{z},$$

since the integral over the segment l_k' is cancelled when the direction of integration is reversed.

Similarly, the curve K_2 consisting of T' and the segments l_k'' bounds a simply connected region S_2, and the function

$$\begin{aligned} F_2(x) &= \frac{1}{2\pi i} \int_{K_2} \phi(z)\,\psi\left(\frac{x}{z}\right)\frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{T'} \phi(z)\,\psi\left(\frac{x}{z}\right)\frac{dz}{z} \end{aligned}$$

is the analytic prolongation of (3) for x in S_2. But $F_1(x) = F_2(x)$ for x in the common parts of S_1 and S_2, since $\phi(x)$ and $\psi(x)$ are uniform functions. Hence $F_1(x)$ can be prolonged analytically across the segments l_k' and its value at each point of S' is that of the integral

$$F_1(x) = \frac{1}{2\pi i} \int_{T'} \phi(z)\psi\left(\frac{x}{z}\right)\frac{z}{dz}.$$

Hence $H(\phi, \psi)$, the analytic prolongation of (3), is given for x in S' by the formula

$$H(\phi, \psi) = \frac{1}{2\pi i} \int_{T'} \phi(z)\,\psi\left(\frac{x}{z}\right)\frac{dz}{z}. \tag{4}$$

The relation (4) is true in particular in the region S lying interior to the circle $|x| = \sigma$ but exterior to the circles $|x - \beta_k| = \eta$. Moreover, for x in S and z on T' we have x/z in S'. For if z is on K', then for all x in S

$$|x/z| < \sigma/\sigma' = 1 - \eta < 1 - \eta',$$

while if z is on some b_k',

$$|x/z| < \sigma/(1 - \eta') < \sigma/(1 - \eta) = \sigma'$$

and
$$|(x/z)-\beta_j| = |(x-z\beta_j)/z| > (\eta-\eta')/(1+\eta') = \eta',$$
since $z\beta_k$ is a point on one of the circles b_k'.

If then
$$|\phi(z)| < N' \text{ for } z \text{ on } T'$$
and
$$|\psi(x)| < N'' \text{ for } x \text{ in } S',$$
it follows that
$$|H(\phi,\psi)| < \frac{\sigma'+\eta' p}{1-\eta'} N'N'' \tag{5}$$
for all x in S.

The inequality (5) permits us to demonstrate that the series (2) converges uniformly in D_1. Let

$$\begin{aligned}\epsilon_r &= (1+\epsilon_{r-1})^{\frac{1}{2}}-1\\ &= (1+\epsilon_1)^{1/2^{r-1}}-1 \qquad (r=2,3,\cdots),\\ R_r &= R_{r-1}/(1-\epsilon_{r-1}) \qquad (r=2,3,\cdots).\end{aligned}$$

Let D_r be the open region interior to the circle $|x| = R_r$ but exterior to the circles $|x-\beta_k| = \epsilon_r$. Let C_r be the boundary of D_r.

Since $f_1(x)$ is holomorphic in D_r and continuous on its boundary there exists an M_r independent of x in D_r and on C_r such that for these values of x

$$|f_1(x)| < M_r \qquad (r=1,2,3,\cdots).$$

It follows that in the region D_1 the series

$$\sum_2^\infty \alpha_m f_m(x) \ll \sum_2^\infty |\alpha_m|\, L_2\cdot L_3\cdots L_m\, M_m{}^m, \tag{6}$$
where
$$L_r = (R_r+\epsilon_r p)/(1-\epsilon_r)$$
for we can conclude that

$$|f_m(x)| < L_2\cdot L_3\cdots L_m\, M_m{}^m$$

for these values of x by applying the results obtained for $\phi(x)$ and $\psi(x)$, replacing $\psi(x)$ in turn by $f_1(x)$, $f_2(x)$, $\cdots$, $f_{m-1}(x)$; $\phi(x)$ in each case by $f_1(x)$; the region S' in turn by D_m, D_{m-1}, $\cdots$, D_2; the region S in turn by D_{m-1}, D_{m-2}, $\cdots$, D_1; N' in each case by M_m; etc.

We note that

$$\epsilon_1 = (1+\epsilon_1)-1 = \epsilon_r\left[(1+\epsilon_1)^{(2^{r-1}-1)/2^{r-1}}+\cdots+1\right],$$

so that

$$\epsilon_1/2^r < \epsilon_r < 1/2^r \qquad (r = 1, 2, \cdots).$$

Moreover,

$$\begin{aligned} R_r &= R_1 \prod_1^{r-1} [1/(1-\epsilon_s)] \\ &< R_1 \prod_1^{\infty} [1/(1-\epsilon_s)] \\ &< R_1 \prod_1^{\infty} [1 + 1/(2^s - 1)] \\ &< R, \end{aligned}$$

where R is finite and independent of r.

Hence

$$L_r < (R+p)/(1-\epsilon_r)$$

and

$$\begin{aligned} L_2 \cdot L_3 \cdot \cdots \cdot L_m &< (R+p)^m \prod_1^{\infty} [1/(1-\epsilon_s)] \\ &< [R(R+p)]^m. \end{aligned}$$

We can then replace (6) by

$$(7) \qquad \sum_2^{\infty} \alpha_m f_m(x) \ll \sum_2^{\infty} | \alpha_m | [R(R+p)M_m]^m.$$

In order to find a bound for M_m we write

$$f_1(x) = \xi_0(x) + \xi_1(x) + \cdots + \xi_{p-1}(x) + \xi_\gamma(x),$$

where $\xi_k(x)$ is regular except possibly at the point β_k and $\xi_\gamma(x)$ is regular except possibly at the point at infinity. The values of $| \xi_k(x) |$ for which $| x - \beta_k | \geq \rho$ are the values of $| \xi_k[(x\beta_k + 1)/x] |$ for which $| x | \leq 1/\rho$. Then, since every singular point of $f(x)$ is of the exponential order $\leq q$, we have for $| x | \leq 1/\rho$,

$$| \xi_k\,(x\beta_k + 1)/x | < N_k e^{(1/\rho)^{q+\epsilon_1}} \qquad (k = 0, 1, \cdots, p-1),$$

where N_k is independent of x. Consequently,

$$| \xi_k(x) | < N_k e^{(1/\rho)^{q+\epsilon_1}}$$

for $| x - \beta_k | \geq \rho$. Hence, for $| x - \beta_k | \geq \epsilon_m$

$$\begin{aligned} | \xi_k(x) | &< N_k e^{(1/\epsilon_m)^{q+\epsilon_1}} \\ &< N_k e^{(2^m/\epsilon_1)^{q+\epsilon_1}}. \end{aligned}$$

Let N_γ be a bound for $| \xi_\gamma(x) |$ for $|x| \leq R$. Let

$$\bar{N} = N_1 + N_2 + \cdots + N_{p-1} + N_\gamma,$$

then

$$M_m < \bar{N}\, e^{(2^m/\epsilon_1)^{q+\epsilon_1}}$$
$$< \bar{M}\, e^{2^{(q+1)m}},$$

where $\bar{M}$ is independent of m.

We may then replace (7) by

$$\sum_{2}^{\infty} \alpha_m f_m(x) \ll \sum_{2}^{\infty} |\, \alpha_m \,|\, M^m\, e^{m2^{(q+1)m}}, \tag{8}$$

where $M = \bar{M}\,(R + p)R$.

By hypothesis

$$|\, \alpha_m \,|^{1/m} = 0\,(e^{-2^{(q+1)m}}).$$

Hence the series

$$\sum_{2}^{\infty} \alpha_m\, e^{m2^{(q+1)m}} x^m$$

converges absolutely for all values of x. It follows that $\sum_{1}^{\infty} \alpha_m f_m(x)$ converges uniformly to a function $G(x)$, holomorphic in D_1. Moreover, by a well-known theorem of Weierstrass

$$\sum_{1}^{\infty} \alpha_m\, f^{(n)}_{(x)} \qquad (n = 2, 3, \cdots),$$

converges uniformly to $G^{(n)}(x)$ in D_1. Hence

$$G(0) = \sum_{m=1}^{\infty} \alpha_m f_m(0) = \sum_{m=1}^{\infty} \alpha_m a_0{}^m = g(a_0)$$

and in general

$$G^{(n)}(0) = \sum_{m=1}^{\infty} \alpha_m f_m{}^{(n)}(0) = n\,!\,.\sum_{m=1}^{\infty} \alpha_m a_n{}^m = n\,!\, g(a_n).$$

We have then as the unique expansion of $G(x)$ at the origin

$$G(x) = \sum_{0}^{\infty} G^{(n)}(0)\, x^n/n\,!$$
$$= \sum_{0}^{\infty} g(a_n) x^n.$$

The function $G(x)$ was proved to be holomorphic in D_1. But since ϵ_1 was arbitrarily small and R_1 arbitrarily large its only possible singularities are the pth roots of unity and the point at infinity.

3. *Notes on the Preliminary Theorem.*

a. An entire function $g(z)$ satisfying the condition of Theorem I is of the apparent order zero since*

$$\lim_{m=\infty} \frac{\log\ (m!)}{\log|\alpha_m|} = 0.$$

It is then of the class zero† and can be written in the form

$$g(z) = A\, z^t \prod_1^\infty [1 - (z/\mu_n)],$$

where A is a constant independent of z, t a positive integer and μ_n the nth zero of $g(z)$, it being supposed that $0 < |\mu_n| \leq \mu_{n+1}$.

If we suppose $g(z)$ a polynomial of degree m,

$$g(z) = \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_m z^m,$$

then

$$g(a_n) = \alpha_1 a_n + \alpha_2 a_n{}^2 + \cdots + \alpha_m a_n{}^m$$

and

$$\sum_2^\infty g(a_n) x^n = \alpha_1 \sum_2^\infty a_n x^n + \alpha_2 \sum_0^\infty a_n{}^2 x^n + \cdots + \alpha_m \sum_0^\infty a_n{}^m x^n.$$

The Theorem I is then true in this case even though we suppose nothing about the order of increasing of $f(x)$ in the neighborhood of its singular points. However, in the theorems that follow we shall consider $g(z)$ an entire function rather than a polynomial.

b. If in Theorem I the only possible singular points of $f(x)$ are the pth roots of unity, the point at infinity being a regular point, then $G(x)$ is regular at the point at infinity. This follows immediately from a theorem of Weierstrass since $f_m(x)$ is continuous on the circle $|x| = \rho > 1$, is regular for $|x| > \rho$, and the series (2) converges uniformly on $|x| = \rho$.

If $f(x)$ is an entire function, then $f_m(x)$ is likewise an entire function and we can replace M_r by a number Q independent of $|x| \leq R$ and of r. We then have instead of (8)

$$\sum_2^\infty \alpha_m f_m(x) \ll \sum_2^\infty |\alpha_m| (QM)^m$$

for $|x| < R_1$, which is sufficient to prove that if $g(z)$ is simply an entire function, then $G(x)$ is regular except possibly at the point at infinity.

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* Hadamard et Mandelbrojt, *loc. cit.*, p. 50.

† Borel, *Fonctions Entières*, *loc. cit.*, p. 76.

c. If in Theorem I the only possible singularities of $f(x)$ are the pth roots of unity then either $G(x)$ reduces to the constant $g(a_0)$ or else it has a singularity at one of the pth roots of unity. It is sufficient for the latter to be true that for a single a_n, $n \neq 0$, $g(a_n) \neq 0$.

4. *Theorems on Generalized Lacunae.* We shall designate by $\{\lambda_n\}$, $\{\lambda'_n\}$ two increasing, infinite sequences of positive integers complimentary with regard to the sequence $\{n\}$. If there exists a series

$$\phi(x) = \sum_{n=0}^{\infty} b_{\lambda'_n} x^{\lambda'_n},$$

with radius of convergence equal to one and having on the circle $|x| = 1$ an algebraic-logarithmic singular point or an isolated singlar point around which $\phi(x)$ is uniform, we shall designate $\{\lambda'_n\}$ by $S_{\lambda'_n}$.

The sequence $\{\lambda'_n\} = \{nk\}$, where k is a positive integer > 1, would be properly designated by $S_{\lambda'_n}$, for the series

$$1/(1-x^k) = \sum_{0}^{\infty} x^{nk}$$

has a pole on the circle $|x| = 1$.

The coefficients of a series

$$\psi(x) = \sum_{0}^{\infty} c_n x^n$$

having the subscripts of a sequence $S_{\lambda'_n}$ we shall designate by $S(c_{\lambda'_n})$.

A theorem proved by Mandelbrojt * and generalized by Pólya † states that if in the series

$$\psi(x) = \sum_{0}^{\infty} c_n x^n$$

with radius of convergence equal to one, each term of a sequence $S(c_{\lambda'_n})$ is zero, then $\psi(x)$ has at least two singular points on the circle $|x| = 1$.

We shall designate by *Eg, q* the set of zeros of an entire function

$$g(z) = \sum_{m=1}^{\infty} \alpha_m z^m$$

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* Hadamard et Mandelbrojt, *loc. cit.*, p. 99.

† Pólya, *loc. cit.*

having the property that for a certain number q

$$| \alpha_m |^{1/m} = 0\,(e^{-2^{(q+1)m}}).$$

We then have

THEOREM II. *Suppose the function* $f(x) = \sum_1^\infty a_n x^n$ *is regular for* $|x| < 1$ *and has a singular point at* $x = 1$. *Suppose that the terms of a sequence* $S(a_{\lambda'_n})$ *take only the values contained in an* $Eg,\, q$ *and that for some* $\lambda_n \neq 0$, $g(a_{\lambda_n}) \neq 0$, *then either* $f(x)$ *has at least two singular points in the extended plane or else the point* 1 *is a singularity of the exponential order greater than* q.

In a word, we suppose that the terms for which $n = \lambda'_n$ are generalized lacunae of $\sum_2^\infty a_n x^n$, relative to the entire function $g(z)$. The proof is immediate. If $f(x)$ is regular except at the point 1, it is a uniform function, and if this point is a singularity of the exponential order $\leq q$, then the function $G(x) = \sum_0^\infty g(a_n) x^n$ is regular except at the point 1, which must be a singular point.

On the other hand, each term of $S(g(a_{\lambda'_n})$ is zero. Hence by the theorem of Mandelbrojt and Pólya stated above the function $G(x)$ has at least two singular points on the circle $|x| = 1$. It follows that either the point 1 is a singular point of the exponential order greater than q or else $f(x)$ has at least two singular points in the extended plane.

Finally, we have

THEOREM III. *Given the function* $f(x) = \sum_0^\infty a_n x^n$, *with integral coefficients* * *and radius of convergence equal to one. Suppose that the terms of a sequence* $S(a_{\lambda'_n})$ *take only the values contained in a set* Eg, q. *Suppose that* $\overline{\lim}\, | g(a_{\lambda_n}) |^{1/\lambda_n} = 1$ *and that* $\lim\, (\lambda_{n+1} - \lambda_n) = \infty$. *Then the circle* $|x| = 1$ *is a cut for the function* $f(x)$.

Here again we are supposing that the terms of $\sum_0^\infty a_n x^n$ for which $n = \lambda'_n$ are generalized lacunae of this series, relative to $g(z)$.

We have first, because of the hypotheses as to integral coefficients and unit radius of convergence, by a theorem of Carlson † and Pólya ‡ that either the circle $|x| = 1$ is a cut or else we can write $f(x)$ in the form

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* a_n is said to be integral if $a_n = b_n + c_n i$, where b_n and c_n are integers.

† Carlson, *Mathematische Zeitschrift*, Bd. 9 (1921), p. 1.

‡ Pólya, *Mathematische Annalen*, Bd. 77 (1916), p. 497.

$$f(x) = \frac{P(x)}{(1-x^p)^t} \quad ,$$

where $P(x)$ is a polynomial and p, t are positive integers. Hence, if we suppose that the circle $|x| = 1$ is not a cut, the only possible singularities o $G(x) = \sum_0^\infty g(a_n)x^n$ are the pth roots of unity and the point at infinity Moreover, $G(x)$ has at least one singular point on the circle $|x| = 1$.

On the other hand, by the theorem of Hadamard and Fabry stated in th Introduction the circle $|x| = 1$ is a cut for the function $G(x)$. It follow that

$$f(x) \neq \frac{P(x)}{(1-x^p)^t}.$$

Hence the statement of the theorem is justified.

THE RICE INSTITUTE.

Properties of Certain Aggregate Functions.*

By Lester S. Hill.

1. *Introductory Notes.* Many mathematical papers have made use of arguments reposing upon the idea that rather general aggregates may conveniently be regarded as functions of other general aggregates. In fact, the notion is merely that of establishing a correspondence between two sets of aggregates, and is thus by no means new; the ordinary functions of elementary analysis, when considered geometrically, involve precisely this notion.

R. L. Moore, in a recent study of the structure of a continuum, was led to propose for consideration what he called "upper semi-continuous collections of continua," and to develop interesting properties of such collections.† But, while the term "upper semi-continuous" is very suggestive of ordinary functional relations, Dr. Moore does not introduce a variable. H. Hahn has, in like manner, closely approximated the formal consideration of upper-semi-continuous functions in which the function values are aggregates of points.‡

W. A. Wilson, in a recent paper,§ has formally introduced a variable with a definite range, and used a specific upper semi-continuous function in the study of continua irreducible between two points.

In the present paper, an attempt is made to develop certain fundamental properties of the aggregate functions thus appearing, implicitly and explicitly, in current investigations of analysis situs. Correspondences in which aggregates of points are functions of the points of an aggregate are considered from the standpoint of the theory of functions of real variables, and a few essential theorems are derived. These theorems constitute generalizations of standard relations in analysis. Thus, for the functions of the general class treated, we may assert that every monotone descending and convergent sequence of upper semi-continuous functions defines an upper semi-continuous function; that every sequence of continuous functions which is uniformly convergent, or which is merely "semi-uniformly" convergent, defines a continuous function; that every function which is upper semi-continuous, or lower semi-continuous, at each point of the set T is at most pointwise discontinuous in T when T is either a bounded region or a bounded perfect set.

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* Communicated by title at the Columbus, Ohio, meeting of the American Mathematical Society, September, 1926.

† R. L. Moore, "Concerning Upper Semi-Continuous Collections of Continua," *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 416-428; also, *Proceedings of the National Academy of Sciences*, Vol. 10 (1924), No. 8.

‡ H. Hahn, "Ueber irreduzible Kontinua," *Sitzungsberichte der Akademie der Wissenschaften*, Wien, Vol. 130 (1921).

§ W. A. Wilson, "On the Structure of a Continuum, Limited and Irreducible between Two Points," *American Journal of Mathematics*, Vol. 48 (1926), No. 3.

But, aside from their intrinsic interest, the theorems presented in the following pages have the merit of providing a convenient basis for a study of very general types of regional frontier. In a paper soon to be offered for publication, it will be shown that certain upper semi-continuous transformations of the circumference of a circle into the frontier F of a bounded plane region can be extended, whenever F is the sum of two rather general continua irreducible between two points, so as to yield homeomorphic transformations of the bounded region into the interior of the circle.

2. *Terms and Notations. Limit Sets.* Let x be any point of the Euclidean space E; let T be a set of points in E; let τ be a point of T. These relations are expressed by $\tau \subseteq T \subseteq E$, or by $E \supseteq T \supseteq \tau$. If ϵ is any positive number ($\epsilon > 0$), we denote by $U_\epsilon(x)$ the set of all points of E whose distances from x are less than ϵ; by $V_\epsilon(\tau)$ the divisor of $U_\epsilon(\tau)$ and T; and by $V_\epsilon^*(\tau)$ the set obtained by omitting the point τ from $V_\epsilon(\tau)$.

The divisor of two sets A and B will be indicated as the logical product $A \cdot B$, and the union as the logical sum $A + B$. By $\bar{A}$ we shall denote the sum of A and its derived set.

If with each point t of a set T lying in the Euclidean space E there be associated in any definite way a set X_t of points lying in the Euclidean space G, we shall say that there is defined the "function" $X_t = f(t)$ of the "variable" t over the "range" T.

Let τ be a point of the range T. Let x be any point of G such that, for every $\epsilon > 0$, and every $\delta > 0$, $U_\epsilon(x)$ contains a point of $X_t = f(t)$ for some t in $V_\delta(\tau)$. If K is the class of points in G satisfying this condition, we say that K is the *upper*[*] *limit* of $X_t = f(t)$ as $t \to \tau$, and we write

$$K = \overline{\operatorname{Lim}}_{t\to\tau} X_t = \overline{\operatorname{Lim}}_{t\to\tau} f(t).$$

Let x be any point of G such that, for every $\epsilon > 0$, there exists a $\delta > 0$, for which $U_\epsilon(x)$ contains a point of $X_t = f(t)$ if t is any point of $V_\delta(\tau)$. If L is the class of points x in G satisfying this condition, we say that L is the *lower limit* as $t \to \tau$ of $X_t = f(t)$, and we write $L = \underline{\mathrm{Lim}}_{t \to \tau} X_t = \underline{\mathrm{Lim}}_{t \to \tau} f(t)$.

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[*] The upper and lower limit sets here defined are natural extensions of the "en-

If $K = L$, we say that $X_t = f(t)$ "converges" to the *limit* $K = L = \mathrm{Lim}_{t \to \tau} X_t = \mathrm{Lim}_{t \to \tau} f(t)$.

If $[t_i]$, $(i = 1, 2, 3, \cdots)$, is a denumerable sequence of points of T, the notations $\overline{\mathrm{Lim}}_{i \to \infty} X_{t_i}$, $\overline{\mathrm{Lim}}_{i \to \infty} X_{t_i}$, $\mathrm{Lim}_{i \to \infty} X_{t_i}$ are almost self-explanatory.*

It is easy to see that the set $\underline{\mathrm{Lim}}_{t \to \tau} X_t$, or the set $\underline{\mathrm{Lim}}_{i \to \infty} X_{t_i}$, may be void (may contain no point). But we note:

THEOREM 1. *If the union of all the X_t sets in G is a bounded set, $\overline{\mathrm{Lim}}_{t \to \tau} X_t$ will contain at least one point:* $\overline{\mathrm{Lim}}_{t \to \tau} X_t \neq 0$.

It is to be observed that some or all of the X_t sets may consist of single points, and that some or all of these sets may be identical.

3. *Continuity and Semi-Continuity of the Function $X_t = f(t)$.* Let τ be a proper limit point, in the ordinary sense, of T. We shall call the function $X_t = f(t)$ upper semi-continuous, supracontinuous, lower semi-continuous, infracontinuous, respectively, at $t = \tau$, according as

$$X_\tau \supseteq \overline{\mathrm{Lim}}_{t \to \tau} X_t, \quad X_\tau = \overline{\mathrm{Lim}}_{t \to \tau} X_t, \quad X_\tau \subseteq \underline{\mathrm{Lim}}_{t \to \tau} X_t, \quad X_\tau = \underline{\mathrm{Lim}}_{t \to \tau} X_t$$

the notations $A \subseteq B$, $B \supseteq A$ implying that every point of A is a point of B. We may agree that $X_t = f(t)$ possesses each of the four types of semi-con-

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semble d'accumulation" and the "ensemble limite" of Z. Janiszewski. Cf. "Sur les continus irreductibles entre deux points," *Journal de l'École Polytechnique*, Ser. II, Vol. 16 (1912).

[*] Let $[X^{(n)}]$ be a sequence of sets lying in the Euclidean space G, and depending upon one index n which ranges over a denumerable sequence of values $[n]$. Then $\overline{\mathrm{Lim}}_{n \to \infty} X^{(n)}$ is the class of all points x of G having the property that $U_\epsilon(x)$, for any $\epsilon > 0$, contains points of $X^{(n)}$ for an infinity of values of n; and $\underline{\mathrm{Lim}}_{n \to \infty} X^{(n)}$ is the class of all points x of G having the property that $U_\epsilon(x)$, for any $\epsilon > 0$, contains points of $X^{(n)}$ for all except (at most) a finite set of values of n. The set $\mathrm{Lim}_{n \to \infty} X^{(n)}$ exists, and is the same as $\overline{\mathrm{Lim}}_{n \to \infty} X^{(n)}$, when the latter set is the same as $\underline{\mathrm{Lim}}_{n \to \infty} X^{(n)}$.

tinuity at an isolated point of T. Evidently, supra-continuity implies upper semi-continuity, and infracontinuity implies lower semi-continuity.

$X_t = f(t)$ is continuous at $t = \tau$ if and only if it possesses there all four types of semi-continuity. It is therefore continuous at an isolated point of T; and it is continuous at the proper limit point τ of T if and only if

$$X_\tau = \operatorname*{Lim}_{t \to \tau} X_t = \overline{\operatorname*{Lim}_{t \to \tau}} X_t = \underline{\operatorname*{Lim}_{t \to \tau}} X_t.$$

A simple example, in two-way space, will be sufficient.

Example: In the ordinary Cartesian plane, with rectangular coordinates, let T be the set of points $-\pi/2 \leqq x \leqq 3$, $y = 0$; and for each point of T, write $t = x$.

We define a function $X_t = f(t)$ over the range T as follows:

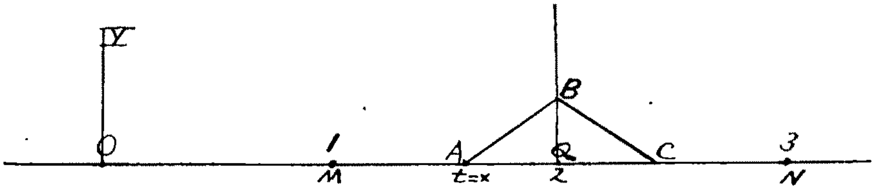
(1) At $t = -\pi/2$, X_t consists of the two points $x = -\pi/2$, $y = 1, -1$.

(2) For $-\pi/2 < t < 0$, X_t is the point $x = t$, $y = \sin t$.

(3) At $t = 0$, X_t is the segment of the Y-axis: $x = 0$, $-1 \leqq y \leqq 1$.

(4) For $0 < t \leqq 1$, X_t consists of the two points: $x = t, y = \sin(1/t)$, $y = \cos(1/t)$.

(5) For $1 < t < 2$, X_t consists of the closed set of all points of the broken line ABC.



with $AB = BC$, and $\tan BAC = (2-t)/(t-1) = AQ/MA$.

(6) At $t = 2$, X_t is the point $x = 2$, $y = 0$.

(7) For $2 < t < 3$, $X_t = X_{t_1}$ with $t_1 = 4 - t$.

(8) At $t = 3$, X_t consists of all points: $x = 3$, $y \geqq 0$.

This function is illustrated in the figure for the interval ($t = x = 1$ to $t = x = 3$) only. If t' and t'' are two points of that interval which are symmetrically located with respect to $t = 2$, then $X_{t'} = X_{t''}$.

There are only four points in T at which the function is not continuous.

At $t = 0$, it is supracontinuous; at $t = -\pi/2$, it is merely upper semi-continuous. At $t = 1$, it is infracontinuous; at $t = 3$, it is merely lower semi-continuous.*

4. *Concerning the Limit Sets of* $X_t = f(t)$. Several theorems expressing useful or significant properties of the limit sets of $X_t = f(t)$ will be recorded without proof.

THEOREM 2. *Let τ be a limit point of the range $T = \{t\}$. For some $\delta > 0$, let the union of the X_t sets, as t ranges over $V_\delta^*(\tau)$, be bounded. Let $K = \overline{\text{Lim}}_{t\to\tau} X_t$, $L = \underline{\text{Lim}}_{t\to\tau} X_t$. Then, for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, such that X_t lies in $U_\epsilon(K)$, and L lies in $U_\epsilon(X_t)$ whenever t lies in $V_{\delta_\epsilon}^*(\tau)$.*

Since $X_t = f(t)$ converges to a limit set, as $t \to \tau$, if and only if $K = L$, Theorem 2 serves to emphasize the essentially bipartite character of convergence. The following theorem throws some further light upon the nature of convergence.

THEOREM 3. *Under the hypotheses of Theorem 2, $X_t = f(t)$ will converge as $t \to \tau$, if and only if there exists a set M having the property that for every denumerable sequence $[t_i]$ of points in T converging to τ, $M = \overline{\text{Lim}}_{i=\infty} X_{t_i}$; and if the set M exists, $M = \text{Lim}_{t\to\tau} X_t$.*

As might be expected, $\overline{\text{Lim}}_{t\to\tau} X_t$ and $\underline{\text{Lim}}_{t\to\tau} X_t$ can be related, respectively, to a divisor set and a union set.

THEOREM 4. *Under the hypotheses of Theorem 2, let B be any closed set in the space G having the property that, for any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that, when t lies in $V_{\delta_\epsilon}^*(\tau)$, the set X_t lies in $U_\epsilon(B)$; and let C be any set in G having the property that, for any $\epsilon > 0$, there exists a*

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* It is to be noted that: $\overline{\text{Lim}}_{t\to1} X_t = \overline{\text{Lim}}_{t\to3} X_t = \underline{\text{Lim}}_{t\to3} X_t$ consists of all the points $x = 1$, $y \geqq 0$ and all the points $x = 3$, $y \geqq 0$; while $\underline{\text{Lim}}_{t\to1} X_t$ consists of the two points $x = 1$, $y = \sin 1$ and $x = 1$, $y = \cos 1$. Notwithstanding that $\overline{\text{Lim}}_{t\to3} X_t = \underline{\text{Lim}}_{t\to3} X_t$, the function $X_t = f(t)$ is not continuous at $t = 3$; the reason being that X_t, for $t = 3$, is a *proper subset* of $\overline{\text{Lim}}_{t\to3} X_t = \underline{\text{Lim}}_{t\to3} X_t = \text{Lim}_{t\to3} X_t$. Continuity at $t = 3$ would imply that $X_3 = \text{Lim}_{t\to3} X_t$.

$\delta_\epsilon > 0$ *such that, when* t *lies in* $V_{\delta_\epsilon}{}^*(\tau)$, *the set* C *lies in* $U_\epsilon(X_t)$. *Let* *be the divisor of all sets* B, *and* U *the union of all sets* C. *Then* $K = \overline{\operatorname*{Lim}}_{t\to\tau} X$ *is the set* D, *and* $L = \underline{\operatorname*{Lim}}_{t\to\tau} X_t$ *is the set* $\bar{U}$.*

With the understanding that a void set is closed, we may assert, eve independently of the hypotheses of Theorem 4, that $\overline{\operatorname*{Lim}}_{t\to\tau} X_t$ and $\underline{\operatorname*{Lim}}_{t\to\tau} X_t$ ar always closed sets.

We now record a kind of uniformity property of $\underline{\operatorname*{Lim}}_{t\to\tau} X_t$.

Theorem 5. *Under the hypotheses of Theorem 2, there exists, corresponding to any* $\epsilon > 0$, *a* $\delta_\epsilon > 0$ *such that, if* t *lies in* $V_{\delta_\epsilon}{}^*(\tau)$, *and if* p *i an arbitrary point of* $\underline{\operatorname*{Lim}}_{t\to\tau} X_t$, *the divisor set* $X_t \cdot U_\epsilon(p)$ *is not void.*

Properties of the limit sets of a function $X_t = f(t)$, and continuity properties of the function, may be stated in terms of the "displacement" of sets Let $d(A, B)$ denote the distance between the sets A and B in a Euclidea space; $d(A, B)$ is the greatest lower bound (minimum) of $d(a, b)$ as th points a, b range independently over A and B. We may define:

$\pi(A, B) =$ the least upper bound (maximum) of $d(A, b)$ as b range over B.

$\pi^\circ(A, B) =$ the greater of the numbers $\pi(A, B)$, $\pi(B, A)$, or their common value.

These displacements were introduced, purely as definitions, by Hausdorff.[†]

It is not difficult to establish the following theorem:

Theorem 6. *In the function* $X_t = f(t)$, *let the* X_t *be closed sets, an let their union in* G *be bounded. Let* τ *be a proper limit point of the range* T *If* $X_t = f(t)$ *is upper semi-continuous at* $t = \tau$, *then* $\lim_{t\to\tau} \pi(X_\tau, X_t) = 0$ *and conversely. If* $X_t = f(t)$ *is lower semi-continuous at* $t = \tau$, *the* $\lim_{t\to\tau} \pi(X_t, X_\tau) = 0$, *and conversely. Hence, for* $f(t)$ *to be continuous a* $t = \tau$, *it is necessary and sufficient that* $\lim_{t\to\tau} \pi^\circ(X_\tau, X_t) = 0$.

It is to be observed that π and π°, as here defined, are real-number functions of familiar type, and that "lim" denotes the ordinary limit of real variable analysis. The equation "lim" $= 0$ implies, of course, that the limit exists.

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* For definition of $\bar{U}$ cf. Section 2.

[†] F. Hausdorff, *Grundzüge der Mengenlehre.*

5. *Concerning Sequences of Functions.* The following theorem is established by a kind of argument which is very familiar in analysis. The proof will be omitted here:

THEOREM 7. *Every monotone descending sequence of closed sets converges to its divisor set. Given the sequence of closed sets* $[X_i]$, *with* $X_i \supseteq X_{i+1}$, *we may write:* $Dv[X_i] = \operatorname*{Lim}_{i\to\infty} X_i$.*

THEOREM 8. *Let* τ *be a proper limit point of* $T = \{t\}$. *Let each function of the denumerable sequence* $X_t^{(n)} = f^{(n)}(t)$, $(n = 1, 2, 3, \cdots)$, *be upper semi-continuous at* $t = \tau$, *and let* $X_t^{(n)} \supseteq X_t^{(n+1)}$. *Finally, let* $X_t = f(t) = \operatorname*{Lim}_{n\to\infty} X_t^{(n)} = \operatorname*{Lim}_{n\to\infty} f^{(n)}(t)$. *Then* $X_t = f(t)$ *is upper semi-continuous at* $t = \tau$.†

Proof: Understanding that a void set is a subset of every set, we note that the theorem is obviously valid for the case $K = \overline{\operatorname*{Lim}}_{t\to\tau} X_t = 0$.

Let $K \neq 0$, and let p be any point of K. For any $\epsilon > 0$, and any $\delta > 0$, there exists in $V_\delta^*(\tau)$ a point t' such that $U_\epsilon(p)$ contains a point of $X_{t'}$. But $X_{t'} = \operatorname*{Lim}_{n\to\infty} X_{t'}^{(n)}$. Hence for n sufficiently great, and therefore, in view of $X_{t'}^{(n)} \supseteq X_{t'}^{(n+1)}$, for any n, there exists in $V_\delta^*(\tau)$ a point t' such that $U_\epsilon(p)$ contains a point of $X_{t'}^{(n)}$. It follows that p is, for every n, a point of $\overline{\operatorname*{Lim}}_{t\to\tau} X_t^{(n)}$; and therewith, in view of the upper semi-continuity of $X_t^{(n)}$, a point of $X_\tau^{(n)}$; and consequently‡ a point of $X_\tau = \operatorname*{Lim}_{n\to\infty} X_\tau^{(n)}$. Thus $X_\tau \supseteq \overline{\operatorname*{Lim}}_{t\to\tau} X_t = K$, and $f(t)$ is upper semi-continuous at $t = \tau$.

Definition: Let $X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions defined over T, and converging, in T, to the function $X_t = \operatorname{Lim} X_t^{(n)}$. We shall call this convergence "semi-uniform" if, corresponding to every $\epsilon > 0$, there exists an infinite set N_ϵ of positive integers, N_ϵ *depending in general upon* ϵ, such that, for any t in T, and for any integer n in N_ϵ, we have:

$$\pi^\circ(X_t, X_t^{(n)}) < \epsilon.$$

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* Cf. footnote, page 421.

† Cf. footnote, page 421.

‡ Since $U_\epsilon(p)$ contains at least one point, p, of $X_\tau^{(n)}$ for every n, we see that p is a point of $\underline{\operatorname*{Lim}}_{n\to\infty} X_\tau^{(n)}$, and therewith a point of $\operatorname*{Lim}_{n\to\infty} X_\tau^{(n)}$. Cf. footnote page 421.

The convergence is "uniform" if, corresponding to every $\epsilon > 0$, th
exists an index n_ϵ such that, for $n > n_\epsilon$, and for any t in T, $\pi^\circ(X_t, X_t^{(n)}) <$

THEOREM 9. *Let τ be a proper limit point of the range T.*
$X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions converging sen
uniformly, in some partial range $Q = V_\delta(\tau)$, to the function $X_t = f($
Then if each function $f^{(n)}(t)$ is continuous at $t = \tau$, so is the functi
$X_t = f(t)$.

Proof: Let $[t_i]$ be any denumerable sequence of points of Q convergi
to τ. Let ϵ be any positive number. There exists an infinite set N_ϵ of i
dices, such that, for every index n in N_ϵ, and for every i, we ha
$\pi^\circ(X_{t_i}, X_{t_i}^{(n)}) < \epsilon$. Thus, for every n in N_ϵ, and for every i, we may writ
(1) $X_{t_i} \subseteq U_\epsilon(X_{t_i}^{(n)})$, and (2) $X_{t_i}^{(n)} \subseteq U_\epsilon(X_{t_i})$.

Denoting by $\bar{U}_\epsilon(A)$, as usual, the union of $U_\epsilon(A)$ and the set of
limit points, we may conclude from (1) that, for every n in N_ϵ, and for eve
i: $\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i} \subseteq \bar{U}_\epsilon(\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}^{(n)})$, and* therefore $\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i} \subseteq \bar{U}_\epsilon(X_\tau^{(n)})$.
establish this conclusion, suppose that $\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}$ contains a point p such th
the distance $d(p, X_\tau^{(n)}) = \epsilon + \zeta$, with $\zeta > 0$. Since $X_\tau^{(n)} = \mathrm{Lim}_{i\to\infty} X_{t_i}$
we have only to make i greater than a definite integer i_0 to insure th
$d(p, X_{t_i}^{(n)}) > \epsilon + \zeta/2$. But, for an infinity of indices i, all greater than
the set X_{t_i} will have points in common with $U_{\zeta/4}(p)$, since p is a point
$\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}$. Let j be one of these indices, and let q be a point of the divis
set $X_{t_j} \cdot U_{\zeta/4}(p)$. Then, clearly, $d(q, X_{t_j}^{(n)}) > \epsilon + \zeta/4$. Thus (1) is co
tradicted for $i = j$.

An analogous argument derives from (2) the formula
$\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}^{(n)} \subseteq \bar{U}_\epsilon(\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i})$, and therewith the formula $X_\tau^{(n)} \subseteq \bar{U}_\epsilon(\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}$
valid for every integer n in N_ϵ.

Recalling that $X_\tau = \mathrm{Lim}_{n\to\infty} X_\tau^{(n)}$, and allowing n to approach infinit
within N_ϵ, we find that, for any $\lambda > \epsilon$, certainly: (3) $\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i} \subseteq \bar{U}_\lambda(X_\tau$
and (4) $X_\tau \subseteq \bar{U}_\lambda(\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i})$. Since ϵ is arbitrarily small, so is λ, and therefore we must have: $X_\tau = \overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}$. By theorem 3, it follows that $X_\tau =$
$\mathrm{Lim}_{t\to\tau} X_t$, and that $X_t = f(t)$ is continuous at $t = \tau$.

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* By hypothesis, $\overline{\mathrm{Lim}}_{i\to\infty} X_{t_i}^{(n)} = \mathrm{Lim}_{i\to\infty} X_{t_i}^{(n)} = X_\tau^{(n)}$.

Corollary: Let τ be a proper limit point of the range T. Let $X_t^{(n)} = f^{(n)}(t)$ be a denumerable sequence of functions converging uniformly, in T, to $X_t = f(t)$. Then, if each function $f^{(n)}(t)$ is continuous at $t = \tau$, so is $f(t)$.

6. *Upper Semi-Continuous Transformation of Bounded Continua.*

THEOREM 10. *Let T be a bounded continuum in the Euclidean space E. In the Euclidean space G, let each X_t be a continuum, and let the union $U\{X_t\}$ of all the X_t sets be bounded. Then, if the function $X_t = f(t)$ is upper semi-continuous in T, the union set $U\{X_t\}$ is a continuum.*

We first show that $U\{X_t\}$ is a connected set. If this is not the case, there exist two sets U_1 and U_2 such that $U = U_1 + U_2$ and $U_1\bar{U}_2 + U_2\bar{U}_1 = 0$. Since U is bounded, we have $d(U_1, U_2) > 0$. If t is any point of T, its image, X_t, will lie wholly in U_1 or wholly in U_2, each set X_t being a bounded continuum. Let T_1, T_2 be those subsets of T whose images lie in U_1, U_2 respectively. We see that $T_1\bar{T}_2 + T_2\bar{T}_1 = 0$. For, plainly, T_1 and T_2 can have no common point, and no point of either can be a limit point of the other. If the point t' of T_1 is a limit point of T_2, there is a denumerable sequence $[t_i]$ in T_2 converging to t'. Since $X_t = f(t)$ is upper semi-continuous, there exists, corresponding to any $\epsilon > 0$, an integer n_ϵ, such that, when $i > n_\epsilon$, we have X_{t_i} lying wholly in $U_\epsilon(X_{t'})$. This is, however, impossible; $X_{t'}$ lies in U_1, while X_{t_i} lies in U_2, and $d(U_1, U_2) > 0$. Since, obviously, $T = T_1 + T_2$, the relation $T_1\bar{T}_2 + T_2\bar{T}_1 = 0$ contradicts the hypothesis that T is a continuum.

To show that $U = U\{X_t\}$ is closed, suppose that it has an improper limit point q in G. Then q is not a point of any X_t.

Let $[q_i]$ be any denumerable sequence of different points in U converging, in the sense of aggregate theory, to the point q. Let $[t_i]$ be any denumerable sequence of points of T such that q_i is a point of X_{t_i} as we may evidently do. It is plain that there are an infinity of *different* points in the sequence $[t_i]$: for otherwise an infinity of points of $[q_i]$ would lie on one set X_{t_i}, so that, X_{t_i} being closed, the limit point q of the convergent sequence $[q_i]$ would lie in X_{t_i}, and therewith in U, contrary to hypothesis. Since T is bounded, the sequence $[t_i]$ will have at least one limit point τ, lying in the closed set T. Consider the closed set X_τ. In view of the fact that q is not a point of X_τ, let $d(q, X_\tau) = h > 0$.

Since $X_t = f(t)$ is upper semi-continuous in T, so that $X_\tau \supseteq \overline{\operatorname{Lim}}_{t \to \tau} X_t$, there exists in $[t_i]$ an infinite[*] subsequence $[s_i]$, $s_i = t_{n_i}$, such that, for $i > i_0$, where i_0 is a determinate integer, we have[†] $X_{s_i} \subseteq U_{h/2}(X_\tau)$, and[‡] consequently $d(q_{n_i}, q) > h/2$. But this is a clear contradiction, q being the limit point of the convergent sequence $[q_i]$.

The set $U\{X_t\}$ has, therefore, no improper limit point, and is closed. Since it is both closed and connected, it is a continuum.

7. *Semi-Continuous Functions on Regional or Perfect Ranges.* We proceed to obtain another generalization of an important theorem of analysis. We shall understand, in what follows, that $X_t = f(t)$ is "semi-continuous" over, or in, the range T if it is upper semi-continuous at each point of T, or else lower semi-continuous at each point of T.

THEOREM 11. *Let the range T be bounded, and let it be either a region or a perfect set. If the function $X_t = f(t)$ is semi-continuous in T, it is at most point-wise discontinuous in T.*

To establish this theorem, we shall make use of two auxiliary real-number functions, and their properties as given in two lemmas. The functions $\delta_\epsilon(\tau)$ and $\delta_\epsilon^*(\tau)$ are defined as follows:

(a) Let τ be a point of T such that, for some point τ' of T, X_τ does not lie wholly in $\bar{U}_\epsilon(X_{\tau'})$, respectively $X_{\tau'}$ does not lie wholly in $\bar{U}_\epsilon(X_\tau)$. Then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is the least upper bound (maximum) of those real numbers δ having the property that, if τ^* is any point of $V_\delta(\tau)$, $X_\tau \subseteq \bar{U}_\epsilon(X_{\tau^*})$, respt $X_{\tau^*} \subseteq \bar{U}_\epsilon(X_\tau)$.

(b) Let τ be a point of T such that, for every point τ' of T, the set X_τ lies wholly in $\bar{U}_\epsilon(X_{\tau'})$, respt. the set $X_{\tau'}$ lies wholly in $\bar{U}_\epsilon(X_\tau)$. Then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, has the value k, where k is a fixed positive number, independent of τ, which is greater than the diameter[§] of T.

If $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is upper semi-continuous in the ordinary sense at $\tau = \tau^*$, we may write:

$$\overline{\lim_{\tau \to \tau^*}}\, \delta_\epsilon(\tau) \leqq \delta_\epsilon(\tau^*), \text{ respt. } \overline{\lim_{\tau \to \tau^*}}\, \delta_\epsilon^*(\tau) \leqq \delta_\epsilon^*(\tau^*)$$

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[*] Any subsequence which converges to τ.

[†] Cf. Theorem 6.

[‡] Note that q_i is a point of X_{t_i}, and q_{n_i} is a point of $X_{s_i} = X_{t_{n_i}}$. The sequence $[q_{n_i}]$ is a subsequence of $[q_i]$.

[§] The diameter of T is the maximum of the numbers $d(t_1, t_2)$ for all pairs of points t_1, t_2 of T.

where $\overline{\lim\limits_{\tau\to\tau^*}}\ \delta_\epsilon(\tau)$, respt. $\overline{\lim\limits_{\tau\to\tau^*}}\ \delta_\epsilon^*(\tau)$, denotes the minimum (greatest lower bound), as $\lambda \to 0$, of the maximum (least upper bound) of $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, in $V_\lambda(\tau^*)$.

LEMMA 1. *If $X_t = f(t)$ is lower semi-continuous, respt. upper semi-continuous, in T, the function $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is everywhere upper semi-continuous in T, in the ordinary sense.*

Proof: Each function is clearly upper semi-continuous wherever it has its maximum value k. We shall prove the Lemma for other points of T in the case of $\delta_\epsilon(\tau)$. A precisely similar argument may be used in the case of $\delta_\epsilon^*(\tau)$.

Assuming that $X_t = f(t)$ is lower semi-continuous in T, let τ^* be a point of T where $\delta_\epsilon(\tau^*) < k$. Suppose, for argument, that $\delta_\epsilon(\tau)$ is not upper semi-continuous at $\tau = \tau^*$. Then, of course:

$$\overline{\lim_{\tau\to\tau^*}}\ \delta_\epsilon(\tau) = \mu + \delta_\epsilon(\tau^*), \text{ with } \mu > 0.$$

Let $\theta = \delta_\epsilon(\tau^*) + \mu/2$, $\phi = \delta_\epsilon(\tau^*) + \mu/4$, $\rho_i = (\epsilon/10)^i$, $\sigma_i = \epsilon + \rho_i$.

There exist: a denumerable sequence of positive numbers, converging steadily to zero: $\zeta_1 > \zeta_2 > \zeta_3 > \cdots$; and a denumerable sequence of points of T, converging to τ^*: $\tau_1, \tau_2, \tau_3, \cdots$; such that the following relations hold:

$$(1)\ \tau_i \text{ in } V_{\zeta_i}(\tau^*), \quad (2)\ \delta_\epsilon(\tau_i) > \theta, \quad (3)\ X_{\tau^*} \subset U_{\rho_i}(X_{\tau_i}),$$

the postulated lower semi-continuity of $X_t = f(t)$ giving (3), in view of Theorem 6.

By definition of $\delta_\epsilon(\tau_i)$, we may assert in view of (2) that, for every i, if τ is any point of $V_\theta(\tau_i)$, we have $X_{\tau_i} \subseteq \bar{U}_\epsilon(X_\tau)$; and therefore, in view of relation (3) above, $X_{\tau^*} \subset U_{\sigma_i}(X_\tau)$.

But τ_i converges to τ^*. Hence, for i sufficiently great, we shall have $V_\phi(\tau^*) \subset V_\theta(\tau_i)$. Thus, if τ is any point of $V_\phi(\tau^*)$, we shall have, for every i, $X_{\tau^*} \subset U_{\sigma_i}(X_\tau)$. This implies that for any point τ of $V_\phi(\tau^*)$, we shall have $X_{\tau^*} \subseteq \bar{U}_\epsilon(X_\tau)$, a result contradicting the definitional property of the function $\delta_\epsilon(\tau)$, in that $\phi > \delta_\epsilon(\tau^*)$.

LEMMA 2. *If $X_t = f(t)$ is lower semi-continuous, respt. upper semi-continuous in T, and if T is a bounded region or a bounded perfect set, then $\delta_\epsilon(\tau)$, respt. $\delta_\epsilon^*(\tau)$, is at most point-wise discontinuous in T.*

This Lemma results immediately from Lemma 1 and a standard theorem of analysis. Compare the formulation of the theorem in question by C. Caratheodory, *Vorlesungen über reelle Funktionen,* p. 144, No. 142: "Jede halbstetige, im Kleinen beschränkte Funktion ist punktiert unstetig, falls ihr Definitionsbereich perfekt ist, oder offen, oder der Durchschnitt einer perfekten und einer offenen Punktmenge."

Proof of Theorem 11: We may argue concurrently the two cases involved by considering at the same time the two point-wise discontinuous functions $\delta_\epsilon(\tau)$ and $\delta_\epsilon^*(\tau)$. First statements made will apply to the case in which $X_t = f(t)$ is lower semi-continuous in T; alternative statements to the case in which $X_t = f(t)$ is upper semi-continuous.

Let $X_t = f(t)$ be everywhere lower semi-continuous, respt. everywhere upper semi-continuous, in T. Let $0 < \epsilon^* < 1/2$, and consider the denumerable sequence of positive numbers $[\epsilon_i]$, with $\epsilon_i = (\epsilon^*/2)^i$. Let τ_1 in T be any point of *continuity* of $\delta_{\epsilon_1}(\tau)$, respt. of $\delta_{\epsilon_1}^*(\tau)$. In view of Lemma 2, if $\bar{\tau}$ is any point of T, and if $\epsilon > 0$, the point τ_1 may be chosen in $V_\epsilon(\bar{\tau})$.

Let $\rho_1 > 0$ be chosen arbitrarily small, and so as to satisfy the requirements:

(a) $\rho_1 < \frac{1}{4}\delta_{\epsilon_1}(\tau_1)$, respt. $\rho_1 < \frac{1}{4}\delta_{\epsilon_1}^*(\tau_1)$

(b) If τ' and τ'' are points of $V_{\rho_1}(\tau_1)$, then

$$| \delta_{\epsilon_1}(\tau') - \delta_{\epsilon_1}(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_1}(\tau_1), \text{ respt. } \quad | \delta_{\epsilon_1}^*(\tau') - \delta_{\epsilon_1}^*(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_1}^*(\tau_1)$$

the absolute value of a real number a being denoted by $| a |$.

In $V_{\rho_1}(\tau_1)$, choose a point τ_2 at which $\delta_{\epsilon_2}(\tau)$, respt. $\delta_{\epsilon_2}^*(\tau)$, is *continuous.* If T is a region, choose τ_2 so that it lies in the same component of T with τ_1. This can be done, by virtue of Lemma 2.

Choose $\rho_2 > 0$ so that:

(a) $\rho_2 < \frac{1}{4}\delta_{\epsilon_2}(\tau_2)$, respt. $\rho_2 < \frac{1}{4}\delta_{\epsilon_2}^*(\tau_2)$

(b) If τ' and τ'' are points of $V_{\rho_2}(\tau_2)$, then

$$| \delta_{\epsilon_2}(\tau') - \delta_{\epsilon_2}(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_2}(\tau_2), \text{ respt. } \quad | \delta_{\epsilon_2}^*(\tau') - \delta_{\epsilon_2}^*(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_2}^*(\tau_2)$$

(c) $\rho_2 < D_1/4$, where D_1 denotes the diameter of the set $V_{\rho_1}(\tau_1)$

(d) $V_{\rho_2}(\tau_2) \subset V_{\rho_1}(\tau_1)$

(e) If T is a region, $V_{\rho_2}(\tau_2)$ lies in the component of T containing τ_2.

This procedure is to be continued indefinitely. In $V_{\rho_{i-1}}(\tau_{i-1})$, choose the point τ_i at which $\delta_{\epsilon_i}(\tau)$, respt. $\delta_{\epsilon_i}{}^*(\tau)$, is continuous; if T is a region, τ_i is to be chosen in the component of T containing τ_{i-1}. Cf. Lemma 2. Choose $\rho_i > 0$ in such manner that:

(a) $\rho_i < \frac{1}{4}\delta_{\epsilon_i}(\tau_i)$, respt. $\rho_i < \frac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$

(b) If τ' and τ'' are points of $V_{\rho_i}(\tau_i)$, then

$$| \delta_{\epsilon_i}(\tau') - \delta_{\epsilon_i}(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } \quad | \delta_{\epsilon_i}{}^*(\tau') - \delta_{\epsilon_i}{}^*(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

(c) $\rho_i < \frac{1}{4}D_{i-1}$, where D_i is the diameter of the set $V_{\rho_i}(\tau_i)$

(d) $V_{\rho_i}(\tau_i) \subset V_{\rho_{i-1}}(\tau_{i-1})$

(e) If T is a region, $V_{\rho_i}(\tau_i)$ lies in the component of T containing τ_i.

By the construction outlined, it is evident that in every case the sequence of points $[\tau_i]$ converges in the usual sense to a definite limit point τ^* in T.

If τ' and τ'' are any two points of $V_{\rho_i}(\tau_i)$, we have simultaneously the two relations:

$$(1) \qquad X_{\tau'} \subset \bar{U}_{\epsilon_i}(X_{\tau''}), \qquad (2) \qquad X_{\tau''} \subset \bar{U}_{\epsilon_i}(X_{\tau'})$$

which are both valid whether $X_t = f(t)$ is lower or upper semi-continuous. To demonstrate these relations, we note that in view of (a) the distance $d(\tau', \tau'')$ is less than $\frac{1}{2}\delta_{\epsilon_i}(\tau_i)$ respt. less than $\frac{1}{2}\delta_{\epsilon_i}{}^*(\tau_i)$; and that in view of (b) we have:

$$| \delta_{\epsilon_i}(\tau_i) - \delta_{\epsilon_i}(\tau') | < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } \quad | \delta_{\epsilon_i}{}^*(\tau_i) - \delta_{\epsilon_i}{}^*(\tau') | < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

as well as

$$| \delta_{\epsilon_i}(\tau_i) - \delta_{\epsilon_i}(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_i}(\tau_i), \text{ respt. } \quad | \delta_{\epsilon_i}{}^*(\tau_i) - \delta_{\epsilon_i}{}^*(\tau'') | < \tfrac{1}{4}\delta_{\epsilon_i}{}^*(\tau_i)$$

It follows that both of the numbers $\delta_{\epsilon_i}(\tau')$, $\delta_{\epsilon_i}(\tau'')$, respt. both of the numbers $\delta_{\epsilon_i}{}^*(\tau')$, $\delta_{\epsilon_i}{}^*(\tau'')$, are greater than $\frac{3}{4}\delta_{\epsilon_i}(\tau_i)$, respt. $\frac{3}{4}\delta_{\epsilon_i}{}^*(\tau_i)$. Consequently, both of the numbers $\delta_{\epsilon_i}(\tau')$, $\delta_{\epsilon_i}(\tau'')$, respt. both of the numbers $\delta_{\epsilon_i}{}^*(\tau')$, $\delta_{\epsilon_i}{}^*(\tau'')$, are greater than $d(\tau', \tau'')$. By definition of the functions $\delta_{\epsilon_i}(\tau)$, $\delta_{\epsilon_i}{}^*(\tau)$, the simultaneous relations (1), (2) immediately follow.

Corresponding to any positive integer i, we can find a positive number q, such that $V_q(\tau^*) \subset V_{\rho_i}(\tau_i)$, τ^* being the limit point of the sequence $[\tau_i]$. Thus we can choose q so that, if τ is any point of $V_q(\tau^*)$, the set X_τ, respt. the set X_{τ^*}, will lie in $\bar{U}_{\epsilon_i}(X_{\tau^*})$, respt. in $\bar{U}_{\epsilon_i}(X_\tau)$.

Given any $\epsilon > 0$, we can find an integer i_0, such that, for $i > i_0$, we have $\epsilon_i < \epsilon$. Hence, for any $\epsilon > 0$, we can find $q > 0$ such that, when τ lies in $V_q(\tau^*)$, we have $X_\tau \subset U_\epsilon(X_{\tau}^*)$, respt. $X_{\tau}^* \subset U_\epsilon(X_\tau)$, the dash over the U being dropped.

For the lower semi-continuous function $X_t = f(t)$, respt. for the upper semi-continuous function, we have found, in the arbitrarily small neighborhood $V_{\rho_1}(\tau_1)$, of the point τ_1, a point τ^* at which $X_{\tau}^* \supseteq \overline{\mathrm{Lim}}_{\tau \to \tau^*} X_\tau$, respt. $X_{\tau}^* \subseteq \underline{\mathrm{Lim}}_{\tau \to \tau^*} X_\tau$; and therewith, in each case, a point τ^* of continuity of the function.*

But τ_1 was chosen arbitrarily from a set of points everywhere dense in T. Hence T contains an everywhere dense set of points at each of which the function $X_t = f(t)$ is continuous, and Theorem 11 is established.

The term "region" has been used to include the case of an open interval, or the union of a set of open intervals, when T lies in one-way space.

The supracontinuous functions employed by W. A. Wilson, in Sections 26-28 of paper cited,† have herewith been shown to be at most point-wise discontinuous. For Theorem 11 applies, *à fortiori*, to functions which are everywhere supracontinuous, or everywhere infracontinuous, in a bounded regional or perfect range T.

It is not difficult to generalize somewhat the hypotheses of Theorem 11. So strong a limitation as that requiring T to be a bounded set is not necessary.

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* From the relation $X_\tau \subseteq U_\epsilon(X_{\tau^*})$, just established, we conclude, on the basis of Theorem 4, that $X_{\tau^*} \supseteq \overline{\mathrm{Lim}}_{t \to \tau^*} X_\tau$. In like manner, Theorem 4 converts the relation $X_{\tau^*} \subseteq U_\epsilon(X_\tau)$ into $X_{\tau^*} \subseteq \underline{\mathrm{Lim}}_{t \to \tau^*} X_\tau$.

† Cf. footnote §, page 419.

The Theory of Group-Reduced Distributions.

By J. Howard Redfield.

In view of the similarity which will be admitted to hold between the subject matters of the Theory of Finite Groups and of Combinatory Analysis, it is somewhat surprising to find that in their literatures the two branches have proceeded on their separate ways without developing their interrelationship, and with scarcely any reference to one another beyond the use by each of certain very elementary results of the other.

In the present paper the connection established between the two branches takes the form of associating every permutation group with a certain symmetric function. By means of these functions and certain differential operations closely allied with those of Hammond and MacMahon, a theory is developed which is in a sense an extension of MacMahon's theory of Chess-Board Diagrams (*Combinatory Analysis,* vol. I, p. 224 ff.). From the point of view here adopted, this theory of MacMahon may be regarded as a special one in which the groups involved are exclusively such as are direct products of symmetric groups on distinct sets of elements, and in which we deal with collections of objects divided into kinds and freely permutable within each kind. In the extended theory the groups may be of any type whatever. We are thus enabled to treat a much wider variety of combinatorial problems; some of them have previously been solved by special devices, but even as to these we gain generality of outlook. One class of such problems, striking because having a convenient geometrical interpretation, though otherwise not more important than some others, has to do with configurations based on the regular polygons and polyhèdra; the groups of rotations involved clearly lie outside the domain of the chess-board diagram type of group.

At the same time nearly every problem which we are able to solve suggests others which do not yield to the methods here developed, but which seemingly should nevertheless be amenable to treatment by methods further perfected along the same lines. Belief is therefore warranted that this borderland region would repay much more extensive investigation. While for the moment Combinatory Analysis will be seen to be the chief beneficiary, some of the results obtained below are not without interest from the point of view of Group Theory, and both branches may be expected to profit by future work in this field.

1. *Preliminary Definitions.* We consider an *assemblage* $A(n, q)$ *of* nq *elements,* divided into q *ranges* $S_1, S_2, \cdots, S_q$ of n elements each. We form therewith correspondences, here termed *range-correspondences,* of type defined by a representation wherein the ranges are written in q lines so as to form a rectangular array of n columns. Range-correspondences will be regarded as identical if their arrays consist of identical columns; regardless of the order of the columns. Thus the number of possible range-correspondences will be $(n!)^{q-1}$.

With each range S_r $(r = 1, 2, \cdots, q)$ we associate a specified permutation group (its *range-group*) G_r, of degree n and order m_r, operating on the elements of S_r. The combined operations of these q groups will determine a group Γ of degree $(n!)^{q-1}$ and order $m_1 m_2 \cdots m_q$, whose elements will be the $(n!)^{q-1}$ range-correspondences, which Γ will interchange among themselves. In general, Γ will not be transitive, that is, the combined groups $G_1, G_2, \cdots, G_q$ will not transform every range-correspondence into every other. Such transformation will however be possible within each of a number θ of closed classes, here termed *group-reduced distributions* (being the transitive constituents of Γ), into which the $(n!)^{q-1}$ range-correspondences are parcelled by the given groups $G_1, G_2, \cdots, G_q$.

By the symbol $N(G_r; p_1^{\pi_1} p_2^{\pi_2} \cdots)$ we denote the number of operations, contained in the group G_r, which exhibit, when written in the usual cycle notation, π_1 cycles of order p_1, π_2 cycles of order p_2, etc.

A partition such as $p_1^{\pi_1} p_2^{\pi_2} \cdots$ which specifies the orders of the different cycles in the symbol of a group operation, will be called the *cycle-partition* of the operation; thus the operation $abc \cdot de \cdot fg \cdot h$ is said to be of cycle-partition $32^2 1$.

THEOREM: *The number* θ *of group-reduced distributions of an assemblage* $A(n, q)$ *determined by the range-groups* $G_1, G_2, \cdots, G_q$ *whose respective orders are* $m_1, m_2, \cdots, m_q$, *has the expression*

$$\theta = \frac{\sum\{(p_1^{\pi_1} p_2^{\pi_2} \cdots \pi_1!\, \pi_2! \cdots)^{q-1} \cdot N(G_1; p_1^{\pi_1} p_2^{\pi_2} \cdots) \cdot N(G_2; p_1^{\pi_1} p_2^{\pi_2} \cdots) \cdots N(G_q; p_1^{\pi_1} p_2^{\pi_2} \cdots)\}}{m_1 m_2 \cdots m_q}$$

in which under Σ *there is a term for every partition* $p_1^{\pi_1} p_2^{\pi_2} \cdots$ *of the degree* n *common to all the range-groups.*

Proof: Since the range-groups operate on distinct sets (ranges) of elements, operations of different range-groups will be commutative. The direct product of the range-groups is a group of degree qn, simply isomorphic with the group Γ defined above; Γ may be called the *group of range-correspondences.*

Let γ be an operation of Γ, and let $J(\gamma)$ denote the number of range-correspondences (i. e. elements of Γ) which are not altered by γ. The sum $\Sigma[J(\gamma)]$, taken for all the $m_1m_2\cdots m_q$ operations of Γ, gives an enumeration in which every range-correspondence β_1 is counted as many times as there are operations in Γ which do not alter β_1. Let $K(\beta_1)$ denote the number of operations of Γ which thus leave β_1 unaltered.

If β_1 is unaltered by γ_1 and γ_2, it is unaltered by γ_1^{-1}, γ_2^{-1}, $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$. Hence those operations of Γ which do not alter β_1 form a sub-group U_1, whose order $K(\beta_1)$ is a divisor of $m_1m_2\cdots m_q$.

Let $1, u_2, u_3, \cdots, u_{K(\beta_1)}$ be the operations of U_1, and let v_2 be any operation of Γ which is not in U_1. Then v_2 will change β_1 into a range-correspondence β_2, different from β_1 but belonging to the same group-reduced distribution. It is clear that β_1 will be changed into β_2 by every operation of the set $v_2, u_2v_2, u_3v_2, \cdots, u_{K(\beta_1)}v_2$, and that β_2 will be unaltered by every operation of the set $1, v_2^{-1}u_2v_2, v_2^{-1}u_3v_2, \cdots, v_2^{-1}u_{K(\beta_1)}v_2^2$, which latter form a sub-group U_2 of Γ, isomorphic with U_1. Using other operations v_3, v_4, etc. we can in like manner reach every range-correspondence of the group-reduced distribution to which β_1 belongs, at the same time accounting for all the operations of Γ. The reasoning is identical with that employed in proving that the order of a group is a multiple of the order of each of its subgroups (e. g Miller, Blichfeldt, and Dickson, *Finite Groups,* p. 22), and establishes that the group-reduced distribution δ of which β_1 is a member contains altogether $m_1m_2\cdots m_q/K(\beta_1)$ range-correspondences. But every range-correspondence in δ is counted $K(\beta_1)$ times in the sum $\Sigma[J(\gamma)]$, and therefore δ itself is counted $K(\beta_1)\times[m_1m_2\cdots m_q/K(\beta_1)]=m_1m_2\cdots m_q$ times. Since this number $m_1m_2\cdots m_q$ is independent of $K(\beta_1)$ and the same for every one of the θ group-reduced distributions, it follows that $\theta=\Sigma[J(\gamma)]/m_1m_2\cdots m_q$.

It remains to evaluate $\Sigma[J(\gamma)]$, and we observe in the first place that $J(\gamma)=0$ for every γ which is determined by a set of range-group operations which are not all of the same cycle-partition. For if to any range-correspondence β_1 we apply an operation (not the identity) of G_1 (say), the range-correspondence will be altered, and, to restore it, we must evidently apply to each of the ranges $S_2, S_3, \cdots, S_q$, an operation of the same cycle-partition with the operation first performed on S_1; while if the first operation is the identity of G_1, all the others must also be the identities of their range-groups.

Supposing, then, that γ is determined by the q operations $g_1, g_2, \cdots, g_q$, all of cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$; let these operations be written in the cycle notation, in q lines, so that they form a rectangular array with cycles of equal order over one another. The elements forming columns will then determine a range-correspondence β_1 which is not altered by γ. Attending now to the first line of the array, the operation g_1 exhibits π_1 cycles of order p_1; each of these can be rewritten so as to begin with any one of its p_1 symbols, and the π_1 cycles can be permuted as wholes in $\pi_1!$ ways. We thus get $p_1^{\pi_1}\pi_1!$ variations, and when the same is done with the π_2 cycles of order p_2, and the rest, we have $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ equivalent notations for the same operation g_1. The matching of the orders of cycles in the q rows of the array is not disturbed, but instead of the single range-correspondence β_1 we now have $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ range-correspondences (including β_1) which are unaltered by γ. Similar variations can be effected on each of the remaining $q-1$ rows of the array. But if *all* the rows are varied, every range-correspondence occurring will be repeated $p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots$ times, since permutations of whole columns of an array do not alter the range-correspondence which it represents. Evidently we shall obtain all the distinct range-correspondences each once only if we keep one row fixed as a reference base and vary independently the remaining $q-1$ rows. We thus find that the total number $J(\gamma)$ of distinct range correspondences which are unaltered by γ, is equal to $(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)^{q-1}$. Hence

$$\theta = \frac{\Sigma[J(\gamma)]}{m_1m_2\cdots m_q} = \frac{\Sigma[(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)^{q-1}]}{m_1m_2\cdots m_q},$$

the summation Σ covering every operation γ of Γ which is determined by a set of operations of $G_1, G_2, \cdots, G_q$ which are all of the same cycle-partition. The theorem as stated in terms of $N(G_r; p_1^{\pi_1}p_2^{\pi_2}\cdots)$ immediately follows.

2. *The Differential Operations* ☊ *and* ☋. In the foregoing developments, for greater generality, no use has been made of the theory of Symmetric Functions, which at this point it becomes convenient to introduce. We shall have occasion to consider only such symmetric functions as are rational and integral, and the term is to be understood throughout with this restriction.

For the details of the theory reference is made to MacMahon's *Combinatory Analysis* (vol. I, sec. I, chap. I), of which the notation will in general be here followed. For convenience however we may here recall that we deal with the symmetric functions of a set of ν indeterminates or symbolic quantities $\alpha_1, \alpha_2, \cdots, \alpha_\nu$. The number ν is not specified, since ordinarily we make no use of expressions which are dependent on its value.

A partition (of a number w) enclosed in round brackets, as $(p_1p_2p_3\cdots p_k)$, the k parts being not necessarily unequal, denotes the sum $\sum[\alpha_{i_1}^{p_1}\alpha_{i_2}^{p_2}\cdots\alpha_{i_k}^{p_k}]$ of all the *different* terms which can be formed by taking for $i_1, i_2, \cdots, i_k$ every possible ordered set of k different suffixes selected from the integers $1, 2, 3, \cdots, \nu$. Specially, s_w is written for (w), a_w for (1^w), and h_w for the sum $\sum[(l_1^{\lambda_1}l_2^{\lambda_2}\cdots)]$ taken for every partition $l_1^{\lambda_1}l_2^{\lambda_2}\cdots$ of w; thus

$$h_4 = (4) + (31) + (2^2) + (21^2) + (1^4).$$

MacMahon (loc. cit., Vol. I, sec. I, chap. I) gives formulae for expressing the s-, a-, and h-functions in terms of one another. These can be derived, by the method of undetermined coefficients or otherwise, from the following generating function identities:

$$\text{(A)}\qquad \begin{cases} (1-\alpha_1x)(1-\alpha_2x)\cdots(1-\alpha_\nu x) \\ = 1 - a_1x + a_2x^2 - a_3x^3 + \cdots \\ = 1/(1 + h_1x + h_2x^2 + h_3x^3 + \cdots) \\ = \exp(-s_1x - \tfrac{1}{2}s_2x^2 - \tfrac{1}{3}s_3x^3 - \cdots). \end{cases}$$

Of the Hammond differential operators, MacMahon makes d_w correspond to s_w, and D_w to a_w. Here however we shall use the equally valid correspondence of δ_w to s_w, and D_w to h_w, since by so doing we can replace certain alternating + and — signs in our expressions by + signs throughout.

It is convenient to take as fundamental the operator δ_w, which MacMahon (loc. cit., Vol. I, p. 36) defines thus:

$$\text{(B)}\qquad \delta_w = \frac{\partial}{\partial h_w} + h_1\frac{\partial}{\partial h_{w+1}} + h_2\frac{\partial}{\partial h_{w+2}} + \cdots$$

We proceed to show that δ_w is equivalent to $w(\partial/\partial s_w)$, a very simple relation of which MacMahon makes no use.

Equating logarithms of reciprocals of the third and fourth expressions of (A), we have

$$\text{(C)}\qquad \log(1 + h_1x + h_2x^2 + \cdots) = s_1x + \tfrac{1}{2}s_2x^2 + \tfrac{1}{3}s_3x^3 + \cdots.$$

Operating with δ_w on the right-hand member of (C), and with the equivalent

$$\frac{\partial}{\partial h_w} + h_1\frac{\partial}{\partial h_{w+1}} + h_2\frac{\partial}{\partial h_{w+2}} + \cdots$$

on the left-hand member, we obtain

$$\text{(D)}\qquad \frac{x^w + h_1x^{w+1} + h_2x^{w+2} + \cdots}{1 + h_1x + h_2x^2 + \cdots}$$
$$= \delta_w s_1 \cdot x + \tfrac{1}{2}\delta_w s_2 \cdot x^2 + \cdots + (1/w)\ \delta_w s_w \cdot x^w + \cdots.$$

The left-hand member of (D) reduces to x^w, and by equating terms containing like powers of x we find that $\delta_w s_v = 0$ if $w \neq v$, while $\delta_w s_w = w$, so that

$$\delta_w = w \ \frac{\partial}{\partial s_w}.$$

It follows then that $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots = 0$ whenever $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$ and $\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots$ are different partitions of the same number, while $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots = \lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots l_1!\, l_2!\cdots$.

We introduce a new connective symbol of operation $\cap$, and write by definition $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cap s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$ for $\delta_{\lambda_1}{}^{l_1}\delta_{\lambda_2}{}^{l_2}\cdots s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$, with the further defining relation $(A + B) \cap C = (A \cap C) + (B \cap C)$, expressing the distributive law with respect to addition. MacMahon's $D_{\lambda_1}{}^{l_1}D_{\lambda_2}{}^{l_2}\cdots h_{\mu_1}{}^{m_1}h_{\mu_2}{}^{m_2}\cdots$ or $D_{\mu_1}{}^{m_1}D_{\mu_2}{}^{m_2}\cdots h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots$ is then readily seen to be equivalent to $h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots \cap h_{\mu_1}{}^{m_1}h_{\mu_2}{}^{m_2}\cdots$.

We extend the definition of $\cap$ so as to write $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cap s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cap \cdots \cdots \cap s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots$ (to q operands) $= (\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots l_1!\, l_2!\cdots)^{q-1}$, while the result is zero if not all the operands exhibit the same partition $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$. Thus extended, the operator $\cap$ is commutative and distributive as to addition, but the result cannot be decomposed into simpler $\cap$-operations, and the notion of associativeness has no relevance.

Another new symbol of operation $\cup$ is defined by $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cup s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots = (s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cap s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots)\, s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots$, with $(A+B) \cup C = (A \cup C) + (B \cup C)$; $s_{\lambda_1}{}^{l_1}s_{\lambda_2}{}^{l_2}\cdots \cup s_{\mu_1}{}^{m_1}s_{\mu_2}{}^{m_2}\cdots$ vanishing as in the case of $\cap$ when $\lambda_1{}^{l_1}\lambda_2{}^{l_2}\cdots$ and $\mu_1{}^{m_1}\mu_2{}^{m_2}\cdots$ are different partitions. Both $\cap$ and $\cup$ connect only symmetric functions of equal weights, but whereas $\cap$ yields a pure number, $\cup$ yields a symmetric function of the same weight with the operands. $\cup$ is evidently commutative, distributive as to addition, and associative, so that we can write unambiguously without parentheses $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_q$. It is also readily verified that $(A \cup B) \cap C = A \cap (B \cup C) = A \cap B \cap C$.

3. *The Group-Reduction Function* of any permutation group G of degree n and order m, is defined to be the symmetric function

$$\operatorname{Grf}(G) = (1/m)\sum[N(G; p_1{}^{\pi_1}p_2{}^{\pi_2}\cdots)\, s_{p_1}{}^{\pi_1}s_{p_2}{}^{\pi_2}\cdots],$$

in which N has the same meaning as in § 1 and Σ covers every partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$ of the weight n. It does not completely characterize its group, since the same G. R. F. may be shared by two or more distinct permutation groups (isomorphic or conformal).

The G. R. F.'s of the symmetric and alternating groups of degree n are, in MacMahon's notation, respectively h_n and $h_n + a_n$. For, the symmetric group of degree n contains all the possible operations of all cycle-partitions which can be formed with n elements. Considering the cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$, we first write in the cycle notation a typical operation, putting blanks — in the places to be occupied by the n elements; thus for the cycle-partition 32^2 of 7 we should write (— — —) (— —) (— —). The blanks can now be filled by the n elements $\alpha_1, \alpha_2, \cdots, \alpha_n$ in any permutation; this gives $n!$ operation symbols, which are however not all distinct. For each cycle of p_r elements can be made to begin with any one of its elements, and if there are π_r such cycles, these can be permuted in $\pi_r!$ ways, and this can be done independently for each value of r; all these varieties are different notations for a single operation. Therefore the number of distinct operations of cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$ in the symmetric group of degree n is

$$n!/(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots), \quad \text{and the G. R. F. is}$$

$$(1/n!)\,\Sigma\left(\frac{n!}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\,s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots\right)$$

$$= \Sigma\left(\frac{s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\right).$$

But this is the expression of h_n in terms of s-functions given by MacMahon (loc. cit., Vol. I, p. 7).—The alternating group of degree n contains the even permutations of the corresponding symmetric group, and its G. R. F. is therefore

$$(2/n!)\,\Sigma\left(\frac{n!}{p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots}\,s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots\right),$$

in which the summation includes only those cycle-partitions which correspond to even permutations, that is, those having an even number of even parts. Now the expression of a_n in terms of s-functions (MacMahon, loc. cit., Vol. I, p. 6) differs from the expression for h_n only in that the terms exhibiting partitions with an odd number of even parts, have the negative sign. In the sum $h_n + a_n$ these terms cancel out, while the terms exhibiting partitions with an even number of even parts are doubled. If the factor 2 thus arising is carried outside the Σ sign, the non-vanishing coefficients under the Σ remain as in the expression of h_n, and we find that the G. R. F. of the alternating group, as given above, is equal to $h_n + a_n$.

The G. R. F.'s of the cyclic groups are of considerable theoretical as well as practical importance. By $\mathrm{Cyc}(p_1^{\pi_1}p_2^{\pi_2}\cdots)$ we shall denote the G. R. F. of a cyclic group generated by an operation whose cycle-partition is $p_1^{\pi_1}p_2^{\pi_2}\cdots$, a partition of the degree w of the group; its order is of course the least common multiple of $p_1, p_2, \cdots$. Such a group is transitive if and only if it is generated by an operation whose cycle-partition is the one-part partition w, and in that case it is easily verified that

$$\mathrm{Cyc}(w) = (1/w)[s_1^w + \phi(\alpha)s_\alpha^{w/\alpha} + \phi(\beta)s_\beta^{w/\beta} + \cdots + \phi(w)s_w],$$

in which $1, \alpha, \beta, \cdots, w$ are all the divisors of w, and $\phi(\zeta)$ denotes the number of positive integers, including 1, less than and prime to ζ. Thus $\mathrm{Cyc}(6) = (1/6)(s_1^6 + s_2^3 + 2s_3^2 + 2s_6)$. In the G. R. F. of an intransitive cyclic group, the coefficients are the same as in the G. R. F. of the transitive cyclic group of the same order, but the literal parts of the terms are s-function products exhibiting the cycle-partitions of the generating operation and its powers. Thus $\mathrm{Cyc}(32^2) = (1/6)(s_1^7 + s_2^2s_1^3 + 2s_3s_1^4 + 2s_3s_2^2)$.

The G. R. F. of a dihedral group derived from a transitive cyclic group of degree w will be denoted by $\mathrm{Dih}(w)$. For odd and even degrees we have respectively

$$\mathrm{Dih}(2k+1) = \tfrac{1}{2}\mathrm{Cyc}\,(2k+1) + \tfrac{1}{2}s_2^k s_1;$$
$$\mathrm{Dih}(2k) = \tfrac{1}{2}\mathrm{Cyc}(2k) + \tfrac{1}{4}s_2^k + \tfrac{1}{4}s_2^{k-1}s_1^2.$$

It is easily seen that the algebraic product of two or more G. R. F.'s is the G. R. F. of the direct product of the groups belonging to the factors, written on distinct sets of symbols.

The enumerating expression for group-reduced distributions can now be rewritten in the form

$$\theta = \mathrm{Grf}(G_1) \Cap \mathrm{Grf}(G_2) \Cap \cdots \Cap \mathrm{Grf}(G_q).$$

This results immediately from the Theorem of § 1 and the definition of $\Cap$ in § 2.

For every group G of degree n, $h_n \Cap \mathrm{Grf}(G) = 1$, and $h_n \Cup Grf(G) = \mathrm{Grf}(G)$. For in $\mathrm{Grf}(G)$, let the coefficient of $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$ *be* $A(p_1^{\pi_1}p_2^{\pi_2}\cdots)$. In h_n the coefficient of $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$ is $1/(p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots)$. (MacMahon, *loc. cit.*, vol. I, p. 7). Then $A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots \Cap s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots/p_1^{\pi_1}p_2^{\pi_2}\cdots\pi_1!\,\pi_2!\cdots = A(p_1^{\pi_1}p_2^{\pi_2}\cdots)$, by § 2. Thus $h_n \Cap \mathrm{Grf}(G) = \Sigma[A(p_1^{\pi_1}p_2^{\pi_2}\cdots)]$ taken for all partitions $p_1^{\pi_1}p_2^{\pi_2}\cdots$ of n. But this is the sum of the coefficients of Grf(G), and we know from the definition of a G. R. F. (§ 3) that this sum is equal to unity.—Also, since

$$A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots \mathbin{\&} s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots / p_1^{\pi_1}p_2^{\pi_2}\cdots \pi_1!\,\pi_2!\cdots$$
$$= A(p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots,$$

it follows immediately that the $\&$-product of h_n and any symmetric function (including any G. R. F.) of degree n, merely reproduces the original symmetric function.

If $h_{n-k}h_1^k \mathbin{\&} \mathrm{Grf}(G) = 1$, then G is at least k-ply transitive, and conversely. For let the two ranges of the assemblage be S_1 and S_2, of which let S_1 be associated with the group G, and S_2 with a group H whose G. R. F. is $h_{n-k}h_1^k$; then H is the direct product of the symmetric group on $n-k$ symbols, by the group of identity on k other symbols. As elements of S_2 we may take $n-k$ similar symbols α, permutable among themselves in all possible ways, together with k distinct symbols $\beta_1, \beta_2, \cdots, \beta_k$, none of which are displaced by any operation of H. Let the elements of S_1 be $\gamma_1, \gamma_2, \cdots, \gamma_n$. If now G is at least k-ply transitive, the k elements of S_1 which are paired with $\beta_1, \beta_2, \cdots, \beta_k$ in any range-correspondence, can by some operation contained in G be replaced by any specified set of k elements of S_1, and this moreover in any one of the $k!$ possible correspondences. Thereafter the $n-k$ elements α can be permuted in any specified way by a suitable operation contained in H. Thus from any range correspondence we can pass to any other by operations of G and H, so that there is only one group-reduced distribution, and $h_{n-k}h_1^k \mathbin{\&} \mathrm{Grf}(G) = 1$. But if G is not k-ply transitive, then not every set of k elements of S_1 can be brought into correspondence with $\beta_1, \beta_2, \cdots, \beta_k$, and there will be more than one group-reduced distribution; $h_{n-k}h_1^k \mathbin{\&} \mathrm{Grf}(G) > 1$. Thus the proposition and its converse are established.

If $h_{n-1}h_1 \mathbin{\&} \mathrm{Grf}(G) = t$, then G has t transitive constituents. For let S_1 as before be associated with the group G, and S_2 with a group H whose G. R. F. is $h_{n-1}h_1$; then we may take, as elements of S_2, $n-1$ similar symbols α, with a distinct symbol β. Starting with any range-correspondence in which an element γ of S_1 corresponds to β, the various operations of G will replace γ by those elements of S_1, and those alone, which belong to the same transitive constituent of G as does γ; while the operations of H, which affect only the α's, will unite into one group-reduced distribution all the range-correspondences in which a particular element of S_1 is paired with β. Thus there will be a group-reduced distribution for each of the t transitive constituents of G, and $h_{n-1}h_1 \mathbin{\&} \mathrm{Grf}(G) = t$.

It is convenient, so far as it can be done significantly and unambiguously, to speak of a G. R. F. as transitive, cyclic, of degree n, etc., when the corresponding group has the property in question.

Examples of Application: (i) Required the number of distinct configurations which can be obtained by placing a solid node • at each of four vertices of a cube, and a hollow node ° at each of the four remaining vertices, configurations differing only in orientation not being regarded as distinct.

The group of rotations of the cube is simply isomorphic with the symmetric group on 4 symbols, and when written on 8 symbols representing the vertices, has the G. R. F.

$$V = (1/24)(s_1^8 + 9s_2^4 + 8s_3^2s_1^2 + 6s_4^2).$$

The group of all node-permutations which do not alter colors of nodes is the direct product of two symmetric groups of degree 4 on distinct sets of symbols, and has as G. R. F.

$$h_4^2 = [(1/24)(s_1^4 + 6s_2s_1^2 + 3s_2^2 + 8s_3s_1 + 6s_4)]^2$$

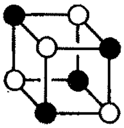
$$= (1/576)(s_1^8 + 12s_2s_1^6 + 42s_2^2s_1^4 + 36s_2^3s_1^2 + 9s_2^4 + 16s_3s_1^5 + 96s_3s_2s_1^3$$
$$+ 48s_3s_2^2s_1 + 64s_3^2s_1^2 + 12s_4s_1^4 + 72s_4s_2s_1^2 + 36s_4s_2^2 + 96s_4s_3s_1$$
$$36s_4^2).$$

The computation of $h_4^2 \mathbin{☊} V$ is as follows:

$$\begin{array}{rcr}
1 \times 1 \times 1^8 \times 8! & = & 40320 \\
9 \times 9 \times 2^4 \times 4! & = & 31104 \\
64 \times 8 \times 3^2 1^2 \times (2!)^2 & = & 18432 \\
36 \times 6 \times 4^2 \times 2! & = & \underline{6912} \\
\text{Product of orders:}\quad 576 \times 24 = 13824 &) & \overline{96768} \\
\text{Number of configurations:}\quad h_4^2 \mathbin{☊} V = & & 7
\end{array}$$

Be it noted that only those cycle-partitions need be considered which are represented in *all* the G. R. F.'s entering as operands. This "orthogonality" of the s-function products with respect to ☊ and ☋ greatly simplifies the computations. In the present case, however, and usually when one of the operands is a monomial product of h-functions, it would be simpler to use the D-operators and to compute D_4^2V by the rules which MacMahon gives (*loc. cit.*, p. 43).

The actual configurations, shown below, cannot be determined by the methods of the present theory, but must be found, as in all other cases, by detailed consideration of the groups involved, and this may of course be very laborious, except in simple cases, or where special devices are available.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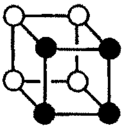
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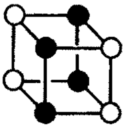
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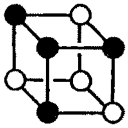
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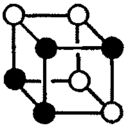
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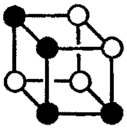
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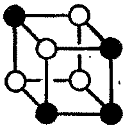

(7)

In connection with the present example we may note without proof certain other simple results obtainable.

Thus if in V we substitute $x^r + y^r$ for every s_r, we obtain the polynomial

$$x^8 + x^7y + 3x^6y^2 + 3x^5y^3 + 7x^4y^4 + 3x^3y^5 + 3x^2y^6 + xy^7 + y^8,$$

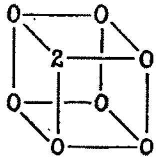
in which the coefficient of x^ty^{8-t} enumerates the distinct configurations possible with t nodes • and $8 - t$ nodes °.

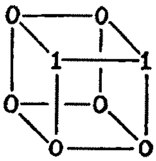
The sum of the coefficients in the above expression is 23, which is the total number of configurations when the numbers of nodes of the two colors are not specified. This enumeration is also effected by substituting 2 for every s_r in V. Similarly if k colors are available we substitute k for every s_r; thus with 3 colors there are $(1/24)(3^8 + 9.3^4 + 8.3^4 + 6.3^2) = 333$ possible configurations.

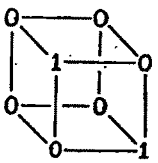
If in V we put $1/(1 - x^r)$ for every s_r, we obtain the infinite series

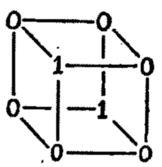
$$1 + x + 4x^2 + 7x^3 + 21x^4 + 37x^5 + \cdots,$$

in which the coefficient of x^t enumerates the distinct configurations obtained by placing a zero or a positive integer at every vertex of the cube, subject to the condition that the sum of the 8 numbers is always t. For $t = 2$, the 4 configurations are









If in V we put 2 for every s_{2k} and 0 for every s_{2k+1}, we enumerate the configurations in which it is possible to change the color of every node into the opposite color by a suitable rotation of the cube. The number is found to be $(1/24)(0+9.2^4+0+6.2^2)=7$, and it will be seen that in fact all the 7 configurations shown above have this property. In other analogous cases the like is not always true; thus for the 12 vertices of the icosahedron there are 24 distinct configurations with 6 nodes of each color, only 16 of which interchange colors by rotation.

(ii) Let n points be placed at the vertices of a regular n-gon. Starting from any one of these points, a closed route is traced, consisting of straight segments running from each point to the next in any order. Required the number of distinct configurations obtained, when no account is taken of differences of orientation or of any particular starting point of a route.

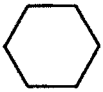
If we distinguish (a) between routes traversed in opposite senses and (b) between a configuration and its reflection, we shall have as range-groups two cyclic groups of degree and order n, and the number of configurations will be $\mathrm{Cyc}(n) \mathbin{\&} \mathrm{Cyc}(n)$. Thus if $n=6$ we shall have

$$(1/6)(s_1^6+s_2^3+2s_3^2+2s_6) \mathbin{\&} (1/6)(s_1^6+s_2^3+2s_3^2+2s_6)$$
$$=(1/36)(6\,!+2^3.3\,!+2^2.3^2.2\,!+2^2.6)=24.$$

If we disregard one only of the distinctions (a) and (b) above, the number of configurations will be $\mathrm{Cyc}(n) \mathbin{\&} \mathrm{Dih}(n)$. For $n=6$ we shall have

$$(1/6)(s_1^6+s_2^3+2s_3^2+2s_6) \mathbin{\&} (1/12)(s_1^6+3s_2^2s_1^2+4s_2^3+2s_3^2+2s_6)$$
$$=(1/72)(6\,!+4.2^3.3\,!+2^2.3^2.2\,!+2^2.6)=14.$$

In the diagrams shown below, distinction (a) is disregarded.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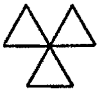
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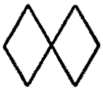
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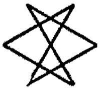
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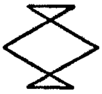
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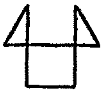
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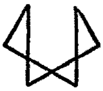
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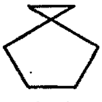
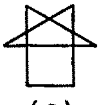
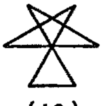
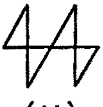
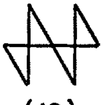
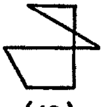

(8)

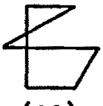

(9)


(10)


(11)


(12)


(13)


(14)

If we specify the direction of travel, as by arrow heads, we obtain two diagrams from each of the above; but these are not distinct in the cases of (3), (4), (11), and (12), since a reversal of direction is there equivalent to a rotation through 180° about an axis through the center of the figure and perpendicular to its plane. The remaining 10 pairs are distinct and increase the total to $14 + 10 = 24$, as computed.

If we abolish *both* distinctions (a) and (b), the enumeration is given by $\mathrm{Dih}(n) \cap \mathrm{Dih}(n)$; for $n = 6$ this is

$$1/12)(s_1^6 + 3s_2^2s_1^2 + 4s_2^3 + 2s_3^2 + 2s_6) \cap (1/12)(s_1^6 + 3s_2^2s_1^2 + 4s_2^3 + 2s_3^2 + 2s_6)$$
$$= (1/144)(6! + 3^2.2^2.1^2.2!^2 + 4^2.2^3.3! + 2^2.3^2.2! + 2^2.6) = 12.$$

And in fact diagram (12) is the reflection of (11), and (14) of (13); thus the number of distinct configurations is now reduced to $14 - 2 = 12$.

§ 4. *Decomposition of $\cup$-Products.*

THEOREM: *If* $\mathrm{Grf}(G_1) \cap \mathrm{Grf}(G_2) \cap \cdots \cap \mathrm{Grf}(G_q) = \theta$, *then* $\mathrm{Grf}(G_1) \cup \mathrm{Grf}(G_2) \cup \cdots \cup \mathrm{Grf}(G_q)$ *can be expressed as the sum* $\Phi_1 + \Phi_2 + \cdots + \Phi_\theta$ *of* θ *G. R. F.'s, which can be associated in one-one correspondence with the* θ *group-reduced distributions* $\delta_1, \delta_2, \cdots, \delta_\theta$ *determined by* $G_1, G_2, \cdots, G_q$, *in such wise that if* Φ_σ *is the correspondent of* δ_σ, *and* β *is any range-correspondence of* δ_σ, *then those operations of any range-group* G_r *which leave* β *unaltered when they are performed in combination with suitable similar operations of the remaining G's, form a sub-group whose G. R. F. is* Φ_σ. For different β's belonging to the same δ these sub-groups are similar, but not necessarily identical.

Proof: In the Proof of the Theorem of § 1, we saw that

$$\theta = \sum[J(\gamma)]/m_1m_2 \cdots m_q,$$

and in § 3 we saw that this expression could be written in the form $\mathrm{Grf}(G_1) \cap \mathrm{Grf}(G_2) \cap \cdots \cap \mathrm{Grf}(G_q)$. We also saw (§ 1, Proof) that whenever $J(\gamma)$ does not vanish, γ is determined by a set of operations of $G_1, G_2, \cdots, G_q$ which are all of the same cycle-partition, say $p_1^{\pi_1}p_2^{\pi_2}\cdots$. If now we attach to each $J(\gamma)$ a literal multiplier $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$ embodying the cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$ characterizing γ, it is evident that the modified expression $\sum[J(\gamma)\cdot s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots]/m_1m_2\cdots m_q$ will be equal to $\mathrm{Grf}(G_1) \cup \mathrm{Grf}(G_2) \cup \cdots \cup \mathrm{Grf}(G_q)$.

But $\sum[J(\gamma)]$ contains a unit, and therefore $\sum[J(\gamma)\cdot s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots]$ contains a summand $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$, for every instance in which a range-correspondence β_1 is associated with an operation γ, of cycle-partition $p_1^{\pi_1}p_2^{\pi_2}\cdots$, which leaves β_1 unaltered. If we attend to the summands $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$ associated with a single range-correspondence β_1, we see, since the corresponding operations form a group U_1 of order $K(\beta_1)$, that all these $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots$ taken together give an expression equal to $K(\beta_1)$ times the G. R. F. of U_1. The groups U_2, U_3, $\cdots$, etc., belonging to the range-correspondences $\beta_2, \beta_3, \cdots$ which are in the same group-reduced distribution with β_1, are all similar to U_1 and so have the same G. R. F. If we extend our summation to include this whole group-reduced distribution, which contains $m_1 m_2 \cdots m_q / K(\beta_1)$ range-correspondences, the result is

$$\frac{m_1 m_2 \cdots m_q}{K(\beta_1)} \cdot K(\beta_1)\,\Phi_\sigma = m_1 m_2 \cdots m_q\, \Phi_\sigma,$$

in which Φ_σ is the G. R. F. belonging indifferently to U_1, U_2, U_3, $\cdots$, etc. Finally, if we include all the θ group-reduced distributions, we obtain

$$m_1 m_2 \cdots m_q (\Phi_1 + \Phi_2 + \cdots + \Phi_\theta),$$

in which $\Phi_1, \Phi_2, \cdots, \Phi_\theta$ is the complete set of G. R. F.'s of which Φ_σ is a typical one. But since we have now used all the summands $s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \cdots$, the above expression must be equal to $\sum[J(\gamma) \cdot s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \cdots]$, whence:

$$\begin{aligned} \Phi_1 + \Phi_2 + \cdots + \Phi_\theta &= \sum[J(\gamma) \cdot s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \cdots]/m_1 m_2 \cdots m_q \\ &= \mathrm{Grf}(G_1) \between \mathrm{Grf}(G_2) \between \cdots \between \mathrm{Grf}(G_q). \end{aligned}$$

The decomposition is thus proved to be theoretically possible in all cases. But for actually effecting it for the $\between$-product of given G. R. F.'s, no general method has yet been found. Moreover, if the $\between$-product involves operands which can belong to two or more distinct groups, the decomposition may not be unique.

Thus, the G. R. F. $(1/4)(s_1^6 + 3s_2^2 s_1^2)$ is shared by the two isomorphic but distinct groups

$$\begin{aligned} Q_1 &= [1;\ ab{\cdot}cd{\cdot}e{\cdot}f;\ ab{\cdot}c{\cdot}d{\cdot}ef;\ a{\cdot}b{\cdot}cd{\cdot}ef]; \\ Q_2 &= [1;\ ab{\cdot}cd{\cdot}e{\cdot}f;\ ac{\cdot}bd{\cdot}e{\cdot}f;\ ad{\cdot}bc{\cdot}e{\cdot}f]. \end{aligned}$$

Both are intransitive, with 3 transitive constituents, as shown by

$$h_5 h_1 \between (1/4)(s_1^6 + 3s_2^2 s_1^2) = 3;$$

but the transitive constituents are of degrees 2 2 2 and 4 1 1 respectively. $h_5 h_1 \between (1/4)(s_1^6 + 3s_2^2 s_1^2) = (1/4)(6s_1^6 + 6s_2^2 s_1^2)$, which has the two decompositions $3[(1/2)(s_1^6 + s_2^2 s_1^2)]$ and $2[(1/4)(s_1^6 + 3s_2^2 s_1^2)] + s_1^6$, the first corresponding to Q_1 and the second to Q_2.

Reverting to the example on cube configurations (§ 3, i), we have

$$\begin{aligned} h_4^2 \between (1/24)&(s_1^8 + 9s_2^4 + 8s_3^2 s_1^2 + 6s_4^2) \\ &= (1/24)(70s_1^8 + 54s_2^4 + 32s_3^2 s_1^2 + 12s_4^2), \end{aligned}$$

which decomposes into

(1)	$(1/12)(s_1^8 + 3s_2^4 + 8s_3^2s_1^2)$
(2)	$+ (1/4)(s_1^8 + s_2^4 + 2s_4^2)$
(3)	$+ (1/4)(s_1^8 + 3s_2^4)$
(4)	$+ (1/3)(s_1^8 + 2s_3^2s_1^2)$
(5) and (6)	$+ 2[(1/2)(s_1^8 + s_2^4)]$
(7)	$+ s_1^8.$

where the numbers to the left indicate the correspondence of the component G. R. F.'s with the seven diagrams shown in § 3, i. The reader will easily verify that each of them is the G. R. F. of the group of rotations which does not alter colors of nodes in the corresponding cube configuration. The decomposition

$$(1/4)(s_1^8 + s_2^4 + 2s_4^2) + 2[(1/3)(s_1^8 + 2s_3^2s_1^2)] + 4[(1/2)(s_1^8 + s_2^4)]$$

is algebraically possible and involves none but admissible common sub-groups, yet it is not valid; this will give some notion of the difficulty of the general problem of decomposition.

5. *Special Cases of Decomposition.* The decomposition of $\mathbin{\cup\!\!\!\!\cap}$-products into G. R. F.'s can be accomplished uniquely in two important special classes of cases:

CASE 1: THEOREM: *When the operands $F_1, F_2, \cdots, F_q$ are* all *of them G. R. F.'s of the form $h_{\lambda_1}^{l_1}h_{\lambda_2}^{l_2}\cdots$, then $\Phi = F_1 \mathbin{\cup\!\!\!\!\cap} F_2 \mathbin{\cup\!\!\!\!\cap} \cdots \mathbin{\cup\!\!\!\!\cap} F_q$ is a sum of G. R. F.'s each of which is likewise a monomial product of h-functions, and the decomposition is found by transforming Φ, when it presents itself in terms of s-functions, into an expression in h-functions, using the formulae given by MacMahon* (loc. cit., Vol. I, p. 6).

Proof: For let the q ranges be $S_1, S_2, \cdots, S_q$, associated with groups $G_1, G_2, \cdots, G_q$, each G_r having a G. R. F. such as $h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}\cdots$. We take, as elements of each S_r, p_1 symbols of each of π_1 kinds, p_2 symbols of each of π_2 kinds, etc., so that G_r admits all possible permutations among symbols of the same kind, but admits no interchange of symbols of different kinds. Let the ranges be written in a rectangular array of q rows and n columns, the elements of each range being in any arbitrary order; thus for 3 ranges of 7 elements each, the G. R. F.'s being $h_3h_2^2$, $h_3^2h_1$, $h_4h_2h_1$, we might have

$$\begin{array}{ccccccc} A & A & B & A & C & B & C \\ a & b & b & a & c & b & a \\ \alpha & \gamma & \alpha & \alpha & \beta & \alpha & \beta \end{array}$$

Such an array symbolizes a range-correspondence, which is not altered by permutations of entire columns. But such permutations of entire columns are admitted by $G_1, G_2, \cdots, G_q$ if and only if the elements interchanged in each row are alike, which requires that the entire columns interchanged be alike; and any set of like columns admits under $G_1, G_2, \cdots, G_q$ all its possible permutations. Thus the group of admissible column-permutations for any range-correspondence, is a direct product of symmetric groups, having a G. R. F. of the form $h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots$. In the above example, the columns $Aa\alpha$ and $Bb\alpha$ occur twice each, the others once each; so that the range-correspondence represented admits a group whose G. R. F. is $h_2{}^2h_1{}^3$. The same being true of every range-correspondence and of every group-reduced distribution involved in the problem, it follows that the $\mho$-product of all the G. R. F.'s can be decomposed into a sum of G. R. F.'s each of which is a product of h-functions.

Since the $\mho$-product is a symmetric function, and every symmetric function has a unique expression as a linear function of h-function products, it follows that this expression, when found by the formulae mentioned above, will give the required decomposition.

Chess Board Diagrams. MacMahon's theory of Chess Board Diagrams comes under this case (loc. cit., Vol. I, sec. V, chap. II). The coefficient of $h_{\lambda_1}{}^{l_1}h_{\lambda_2}{}^{l_2}\cdots$ in the decomposition of a $\mho$-product of monomial h-function products enumerates the diagrams in which the assemblage of numbers appearing in the various diagram compartments forms the partition $\lambda_1{}^{l_1}\lambda^{l_2}\cdots$. Using three or more operands we can enumerate analogous diagrams in three or more dimensions; the theory of these, though not explicitly given as such, is contained in Vol. I, sec. II, chap. V of *Combinatory Analysis.* The example which MacMahon gives at pp. 85-86 can be looked upon as an enumeration of three-dimensional block diagrams, having 2 1 1, 2 1 1, and 2 2 units in respective rows, columns, and layers. The enumerating expression for all the diagrams is $h_2h_1{}^2 \,\Omega\, h_2h_1{}^2 \,\Omega\, h_2{}^2 = 38$. The corresponding $\mho$-product is $h_2h_1{}^2 \,\mho\, h_2h_1{}^2 \,\mho\, h_2{}^2$

$$= (1/2)(s_1{}^4 + s_2s_1{}^2) \,\mho\, (1/2)(s_1{}^4 + s_2s_1{}^2) \,\mho\, (1/4)(s_1{}^4 + 2s_2s_1{}^2 + s_2{}^2)$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\{[1 \times 1 \times 1 \times (1^4\cdot 4!)^2]s_1{}^4 + [1 \times 1 \times 2 \times (2.1^2.2!)^2]s_2s_1{}^2\}$$
$$= 36s_1{}^4 + 2s_2s_1{}^2 = 34s_1{}^4 + 4[(1/2)(s_1{}^2 + s_2)]s_1{}^2 = 34h_1{}^4 + 4h_2h_1{}^2;$$

so that there are 34 "unitary" diagrams with one unit in each of 4 compartments, and 4 diagrams with 2 units in one compartment and one unit in each of 2 others. If in MacMahon's expressions involving the magnitudes b_μ (loc. cit.) we put h_μ for every b_μ, we obtain the expression of a ☋-product in terms of h-functions..

CASE 2: THEOREM: *If of the operands $F_1, F_2, \cdots, F_q$ at least one is the G. R. F. of a cyclic group, say* Cyc$(\lambda_1^{l_1}\lambda_2^{l_2}\cdots)$, *then* $\Phi = F_1$ ☋ F_2 ☋ $\cdots$ ☋ F_q *can be expressed as a sum of G. R. F.'s of cyclic groups, each of which is either* Cyc $(\lambda_1^{l_1}\lambda_2^{l_2}\cdots)$ *itself, or the G. R. F. of a cyclic group generated by some power of an operation of cycle-partition $\lambda_1^{l_1}\lambda_2^{l_2}\cdots$. The decomposition is made as follows: From Φ obliterate the term $Cs_{\mu_1}^{m_1}s_{\mu_2}^{m_2}\cdots$ which corresponds to the operations of highest order involved, by subtracting a suitable multiple k* Cyc$(\mu_1^{m_1}\mu_2^{m_2}\cdots)$ *of the G. R. F. of the cyclic group generated by a similar operation; treat the remainder $\Phi - k$* Cyc $(\mu_1^{m_1}\mu_2^{m_2}\cdots)$ *in the same way; and continue the process until all the terms of Φ are obliterated; the G. R. F.'s thus subtracted will constitute the decomposition required.*

Proof: For every summand composing Φ is (by § 4, Theorem) the G. R. F. of a sub-group of each of the groups belonging to the F's, of which groups one, being by hypothesis cyclic, can have as sub-groups only itself and the cyclic groups generated by the powers of its own generator.

If Φ contains no terms which represent operations of higher order than does $Cs_{\mu_1}^{m_1}s_{\mu_2}^{m_2}\cdots$, then this term must be contributed by a component Cyc$(\mu_1^{m_1}\mu_2^{m_2}\cdots)$ of Φ, repeated a number k of times sufficient to make up the coefficient C; and the same reasoning applies to the successive remainders left after subtraction of k Cyc $(\mu_1^{m_1}\mu_2^{m_2}\cdots)$ and its analogues.

Thus h_3h_2 ☋ Cyc(32)

$$= (1/12)\,(s_1^5 + 4s_2s_1^3 + 3s_2^2s_1 + 2s_3s_1^2 + 2s_3s_2)$$
$$\text{☋}\ (1/6)\,(s_1^5 + s_2s_1^3 + 2s_3s_1^2 + 2s_3s_2)$$
$$= (1/72)\,(120s_1^5 + 48s_2s_1^3 + 24s_3s_1^2 + 24s_3s_2)$$
$$= (1/6)\,(10s_1^5 + 4s_2s_1^3 + 2s_3s_1^2 + 2s_3s_2).$$

Subtraction of Cyc(32) or $(1/6)\,(s_1^5 + s_2s_1^3 + 2s_3s_1^2 + 2s_3s_2)$ leaves

$$(1/6)\,(9s_1^5 + 3s_2s_1^3);$$

from this subtracting Cyc (21^3) or $(1/2)\,(s_1^5 + s_2s_1^3)$ leaves s_1^5; from this subtracting Cyc(1^5) or s_1^5 leaves 0. Therefore

h_3h_2 ☋ Cyc(32) = Cyc(32) + Cyc(21^3) + Cyc(1^5).

6. *Derivation of G. R. F.'s of Isomorphs.* If $\phi(\alpha_1, \alpha_2, \cdots, \alpha_n)$, a function of n arguments, preserves the same value when these arguments are permuted in its symbol by any operation of a specified group H of degree n and order μ, but changes its value when its arguments are permuted by any operation not contained in H, then $\phi(\alpha_1, \alpha_2, \cdots, \alpha_n)$ assumes $n!/\mu$ different values in all when the same set of n arguments is inserted in its symbol in all possible permutations. This is a well-known result (comp. e. g. C. Jordan, *Traité des Substitutions*, pp. 25, 26), but we may here point out that it is readily established by a simple application of the theory of Group-Reduced Distributions, by taking as one range the n arguments $\alpha_1, \alpha_2, \cdots, \alpha_n$, with G. R. F. Grf$(H)$, and as the second range the n fixed positions in the symbol of ϕ, with the group of identity on n elements as range-group, G. R. F. $s_1{}^n$. Now Grf(H) is always of the form $(1/\mu)(s_1{}^n + \cdots)$, the omitted terms of which are irrelevant in the Ω-product $s_1{}^n \mathbin{\Omega} (1/\mu)(s_1{}^n + \cdots) = n!/\mu$, which is the number of distinct values which ϕ can assume. It is unnecessary to think of ϕ as a particular algebraic or other function; it is merely a symbol which we agree to regard as equivalent to any similar symbol obtainable from it by any operation of H performed on its arguments.

Any permutation of the n α's determines a corresponding permutation of the $n!/\mu$ ϕ's, and since all the α's are on the same footing as to the positions which they may occupy in the symbol $\phi(—, —, \cdots, —)$, all permutations on the α's which are of a given cycle-partition P (a partition of n) will be in correspondence with permutations of the ϕ's which are of one and the same cycle-partition P' (a partition of $n!/\mu$).

The G. R. F. of any group G of degree n can be transformed, according to the correspondence so established, into the G. R. F. of a group G' of degree $n!/\mu$, isomorphic with G. In this transformation all the coefficients of Grf(G) are unaltered, and only the literal symbols $s_{\lambda_1}{}^{l_1} s_{\lambda_2}{}^{l_2} \cdots$ are changed in accordance with the correspondence of cycle-partitions. The isomorphism is not necessarily simple, and the G. R. F.'s determined by various groups H are not always distinct.

We adopt for the G. R. F. of the derived group G' the notation Grf$(H) \mathbin{\mathfrak{z}}$ Grf(G). We also introduce the concept of the "powers" of a cycle-partition as follows: If P denotes any partition, P^k will denote that partition which is the cycle-partition of the k-th power u^k of any permutation-operation u which is itself of cycle-partition P. Thus if $P = 43$, we shall have

$$P = P^5 = P^7 = P^{11} = 43; \quad P^2 = P^{10} = 32^2; \quad P^3 = P^9 = 41^3;$$
$$P^4 = P^8 = 31^4; \quad P^6 = 2^2 1^3; \quad P^{12} = 1^7.$$

THEOREM: *To every operation of order* ξ *and cycle-partition* P, *in a group* G *of degree* n, *there corresponds, in the derived group* G' *whose G. R. F. is* $\text{Grf}(H) \mathbin{♂} \text{Grf}(G)$, *an operation of cycle-partition* $1^{A_1}\alpha^{A_\alpha}\beta^{A_\beta}\cdots\xi^{A_\xi}$, *where* $1, \alpha, \beta, \cdots, \xi$ *are the divisors of* ξ, *and the* A*'s are the coefficients in the expression* $A_1\text{Cyc}(P) + A_\alpha\text{Cyc}(P^\alpha) + A_\beta\text{Cyc}(P^\beta) + \cdots + A_\xi s_1{}^n$, *obtained by the decomposition of* $\text{Grf}(H) \mathbin{☋} \text{Cyc}(P)$ *by Case 2 of § 5 above.*

Proof: We introduce for ϕ a new notation

$$\phi\begin{pmatrix}\alpha_f & \alpha_g & \cdots & \alpha_l \\ \beta_p & \beta_q & \cdots & \beta_z\end{pmatrix},$$

in which $f, g, \cdots, l$ and $p, q, \cdots, z$ are permutations of the numbers $1, 2, \cdots, n$, and a column such as $\begin{matrix}\alpha_g \\ \beta_q\end{matrix}$ indicates that α_g occupies the qth position in the old notation $\phi(—, —, \cdots, —)$. The meaning of the new symbol is evidently not changed by permutations of whole columns; this purely notational invariance is to be distinguished from the special invariance determined by the group H operating on the upper row alone.

The α's and the β's may be regarded as two ranges S_α, S_β, with H as the group associated with S_α, while with S_β we associate a cyclic group C, generated by an operation u of order ξ and cycle-partition P, so that $\text{Grf}(C) = \text{Cyc}(P)$. Let η be the smallest integer such that the operation u^η, performed on the β's of a given ϕ, yields a symbol ϕ' equivalent to ϕ in virtue of the invariance under H. Then η must be either ξ or a divisor of ξ, and the operations $u, u^2, \cdots, u^\eta$ will yield successively η symbols $\phi_1, \phi_2, \cdots, \phi_{\eta-1}, \phi$ all distinct under the invariance determined by H. In the cyclic group C', into which H transforms C, the operation u will thus be represented by an operation u' of degree $n!/\mu$ and cycle-partition P', P' being a partition of $n!/\mu$ containing a part η, since one of the cycles of u' will be $(\phi\phi_1\phi_2\cdots\phi_{\eta-1})$. On the other hand the two-rowed symbol of ϕ can be read as a range-correspondence which is unaltered only by such operations as combine u^η or one of its powers with a suitable similar operation of H. But u^η generates a cyclic group whose G. R. F. is $\text{Cyc}(P^\eta)$, and so $\text{Grf}(H) \mathbin{☋} \text{Grf}(C) = \text{Grf}(H) \mathbin{☋} \text{Cyc}(P)$ will have (by the Theorems of § 4 and of § 5 Case 2) a component $\text{Cyc}(P^\eta)$, in correspondence with a part η in P'. Extension to the complete ☋-product then establishes the rule stated for finding P'.

Example: Take $\text{Grf}(H) = h_3h_2$; this being of order $3!2! = 12$ will transform any G. R. F. of degree 5 into an isomorphic G. R. F. of degree

$5!/12 = 10$. We found above (§ 5 Case 2) $h_3h_2 \mathbin{\text{☋}} \mathrm{Cyc}(32) = \mathrm{Cyc}(32) + \mathrm{Cyc}(21^3) + \mathrm{Cyc}(1^5)$. Cyc(32) belongs to a group whose order 6 has the divisors 1, 2, 3, 6. We have $P = 32$, $P^2 = 31^2$, $P^3 = 21^3$, $P^6 = 1^5$; also $A_1 = A_3 = A_6 = 1$, $A_2 = 0$; whence $1^{A_1}2^{A_2}3^{A_3}6^{A_6} = 631$. Thus we obtain the cycle-partition correspondence 32 : 631. Moreover, since a cycle-partition correspondence between two group operations implies that the same correspondence holds between like powers of those operations, we have at once the correspondences $31^2 : 3^31$, $21^3 : 2^31^4$, and $1^5 : 1^{10}$. Further correspondences are given by $h_3h_2 \mathbin{\text{☋}} \mathrm{Cyc}(41) = \mathrm{Cyc}(2^21) + 2\mathrm{Cyc}(1^5)$, namely $41 : 4^22$, and $2^21 : 2^41^2$, and by $h_3h_2 \mathbin{\text{☋}} \mathrm{Cyc}(5) = 2\mathrm{Cyc}(1^5)$, namely $5 : 5^2$. Then we have for the complete set of 7 partitions of 5:

Partitions of 5:	5	41	32	31^2	2^21	21^3	1^5
Partitions of 10:	5^2	4^22	631	3^31	2^41^2	2^31^4	1^{10}.

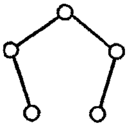
This set of correspondences serves to derive an isomorph of degree 10 from any group of degree 5. Thus from $(1/5)(s_1^5 + 4s_5) = \mathrm{Cyc}(5)$ comes $(1/5)(s_1^{10} + 4s_5^2) = \mathrm{Cyc}(5^2)$. Or taking for G the symmetric group of degree 5, we have from

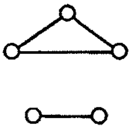
$$h_5 = (1/120)(s_1^5 + 10s_2s_1^3 + 15s_2^2s_1 + 20s_3s_1^2 + 20s_3s_2 + 30s_4s_1 + 24s_5)$$

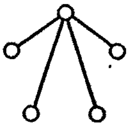
the isomorphic G. R. F.

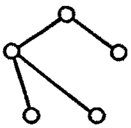
$$h_3h_2 \mathbin{\text{♂}} h_5 = (1/120)(s_1^{10} + 10s_2^3s_1^4 + 15s_2^4s_1^2 + 20s_3^3s_1 + 30s_4^2s_2 + 24s_5^2 + 20s_6s_3s_1).$$

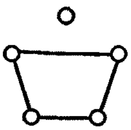
The function last written is typical of a class of G. R. F.'s $h_{n-2}h_2 \mathbin{\text{♂}} h_n$, of degree $n(n-1)/2$ and order $n!$, connected with the enumeration of the symmetrical aliorelative dyadic "relation-numbers" (Whitehead and Russell, *Principia Mathematica,* Vol. II, p. 301) on a field of n elements. If we represent field members by nodes o, and the holding of a typical relation by connecting lines, then there are 10 possible connecting lines for 5 nodes. If exactly 4 of these 10 are drawn, we get the configurations shown below, which are enumerated by $h_6h_4 \mathbin{\text{☊}} (h_3h_2 \mathbin{\text{♂}} h_5) = 6$.

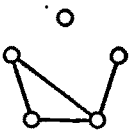

(1)


(2)


(3)


(4)


(5)


(6)

Configurations (5) and (6), which contain isolated nodes, do not, according to the accepted definitions, represent relation-numbers on a field of *five* elements. What we have enumerated are in fact the relation-numbers of a certain type on fields of *five or fewer* elements, and to remove the redundancy we must subtract the corresponding expression for fields of four or fewer elements, thus: $h_6 h_4 \mathbin{\text{℘}} (h_3 h_2 \mathbin{\text{♂}} h_5) - h_2 h_4 \mathbin{\text{℘}} (h_2^2 \mathbin{\text{♂}} h_4) = 6 - 2 = 4$.

By putting every $s_r = 2$ (see § 3 ex. i) in the expressions $h_{n-2} h_2 \mathbin{\text{♂}} h_n$ for various values of n, we find that the total number of symmetrical aliorelative dyadic relation-numbers on fields of n elements is as follows:

Elements in Field:	1	2	3	4	5	6	7	. . .
Relation-Numbers:	0	1	2	7	23	122	888	. . .

Other species of relation-numbers can be enumerated in a similar manner. Thus for asymmetrical aliorelative dyadic relation-numbers we use the functions $h_{n-2} h_1^2 \mathbin{\text{♂}} h_n$: for symmetrical aliorelative triadic relation numbers the functions $h_{n-3} h_3 \mathbin{\text{♂}} h_n$; for aliorelative triadic relation-numbers symmetrical in two only of their arguments, the functions $h_{n-3} h_2 h_1 \mathbin{\text{♂}} h_n$.

Sometimes the isomorphism is not simple but j-fold; in such case every derived operation will be repeated j times, and this will give a leading term $j s_1^n$ inside the parentheses of the G. R. F. When this factor j is carried outside the parentheses, the G. R. F. will appear as of the correct order, so that no exception arises to the rule for finding the expression for $\mathrm{Grf}(H) \mathbin{\text{♂}} \mathrm{Grf}(G)$.

Other isomorphs can be obtained by using as transforming function, instead of a G. R. F. $\mathrm{Grf}(H)$, any symmetric function which is the sum of two or more G. R. F.'s of the proper degree. Thus for enumerating the symmetrical dyadic relation numbers which are not restricted to be aliorelative, we use functions $(h_{n-2} h_2 + h_{n-1} h_1) \mathbin{\text{♂}} h_n$. Such isomorphs are necessarily intransitive. On the other hand, when a single G. R. F. is used as transforming function, the derived isomorph *may* be intransitive, and if it is, still other isomorphs can be got by omitting some of the transitive constituents. The general problem of actually separating an intransitive G. R. F. into transitive constituents has however not been solved, and seems to be closely connected in nature and difficulty with the decomposition of ℘-products into sums of G. R. F.'s.

Some important general relations are (for G and H of degree n, H of order μ): $h_n \mathbin{\text{♂}} \mathrm{Grf}(G) = s_1$ for every G; $(h_n + a_n) \mathbin{\text{♂}} \mathrm{Grf}(G) = (1/2)(s_1^2 + s_2)$ or s_1^2 according as G has or has not odd permutations: $h_{n-1} h_1 \mathbin{\text{♂}} \mathrm{Grf}(G) = \mathrm{Grf}(G)$;

$$h_{(n!/\mu)-1} h_1 \mathbin{\text{℘}} [\mathrm{Grf}(H) \mathbin{\text{♂}} \mathrm{Grf}(G)] = \mathrm{Grf}(H) \mathbin{\text{℘}} \mathrm{Grf}(G).$$

7. *Determination of the G. R. F. from Given Generators.* If two symbols α and β occur in the same cycle of one of the given generators of a group G whose G. R. F. is to be determined, they belong to the same transitive constituent of G. By examination of the given generators from this point of view, the number θ of transitive constituents can be found. Writing $\mathrm{Grf}(G)$ in the general form $(1/m)\sum[N(G; p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots]$, the relation $h_{n-1}h_1 \cap (1/m)\sum[N(G; p_1^{\pi_1}p_2^{\pi_2}\cdots)s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\cdots] = \theta$ gives an equation with m and the N's as unknowns. Again, taking any known group H of degree n and order μ, we can form a function ϕ, of the original n symbols, which is invariant under H, and then determine the forms of the new generators, each containing now $n!/\mu$ symbols, and corresponding to the original generators, which generate the isomorph whose G. R. F. is $\mathrm{Grf}(H) \cup \mathrm{Grf}(G)$; and another equation is given by

$$h_{(n!/\mu)-1}\, h_1 \cap [\mathrm{Grf}(H) \cup \mathrm{Grf}(G)] = \mathrm{Grf}(H) \cap \mathrm{Grf}(G) = \theta',$$

θ' being found by inspection of the new generators just as θ was found previously. Continuing in the same way, using various other groups for H, a sufficient number of equations can in theory be found to determine m and all the N's. In practice some of the isomorphs will be of inconveniently high degree, a difficulty which can be mitigated to some extent by choosing the groups H of as high order as possible.

It is not necessary, however, to set up as many equations as there are unknowns. For without actually evaluating $\mathrm{Grf}(H) \cap \mathrm{Grf}(G)$, we know that it is a positive integer. Also $s_{\lambda_1}^{l_1}s_{\lambda_2}^{l_2}\cdots \cap \mathrm{Grf}(G)$ is a positive integer or 0 for every partition $\lambda_1^{l_1}\lambda_2^{l_2}\cdots$ of n; while if H' is a sub-group of H, it is easily shown that $\mathrm{Grf}(H') \cap \mathrm{Grf}(G) \geqq \mathrm{Grf}(H) \cap \mathrm{Grf}(G)$. These and similar relations furnish a variety of congruences and diophantine inequalities which, in connection with a partial set of definite equations, will frequently suffice for the determination of $\mathrm{Grf}(G)$.

Even when so simplified, it may be doubted whether the method offers any substantial advantage over the alternative, frequently impracticable of course, of actually deriving the whole set of operations of the group from the given generators and then forming the G. R. F. by counting the operations of various cycle-partitions. It is here given rather as illustrating a principle on which it may be possible to base an effective method. For the present we must in practice rely on special devices when dealing with groups of high orders.

8. *Possible Algebraic Forms of G. R. F.'s.* The congruences and diophantine inequalities referred to in § 7 constitute a set of *necessary* conditions that a given symmetric function be the G. R. F. of an existent group. By means of them we can determine the possible forms of G. R. F.'s having specified properties, as for instance those belonging to groups of given degree and order. We can often thus demonstrate the non-existence of certain classes of groups, or narrow the possibilities to one or a few forms. We cannot however so demonstrate the existence of any proposed group, and a completely *sufficient* set of purely algebraic conditions has not been discovered, though nothing indicates that this would be impossible. Such methods might be useful in checking for omissions the lists of groups compiled by Cayley and others (*Quarterly Journal of Mathematics,* Vol. XXV, p. 71, etc.).

WAYNE, PA.
SEPTEMBER, 1926.

Closure of the Tangential Process on the Rational Plane Cubic.

By F. E. Allen.

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1. In a series of recent papers, Winger * has developed various features of the satellite theory of the plane cubic by use of the parameter form of representation and involutions on rational curves. Classic theorems and extensions of satellite properties are proved with great ease and beauty.

A prominent feature of these papers is the discussion of systems of contact curves. Of special interest is Winger's conic N, the envelope of lines joining pairs of contact of tangents from points of the rational cubic, and tangent to the cubic at the sextactic points. This conic is one of a pencil of conics which touch the nodal tangents where they meet the line of inflexions. Since both the cubic and each conic of this pencil are invariant under a dihedral G_6, the pencil and the conic N play an important part in the construction of special and general sets of the group.

The problem of closure of the tangential process on an elliptic cubic or higher curve has been extensively discussed from the number-theory standpoint,† leading to the determination of sets of real, rational points on a curve. With this has been developed the configurations of polygons, both inscribed and circumscribed to a curve of third order or higher.

The present paper takes up the problem of determining those points on a rational cubic which coincide with their nth tangentials from the more general projective standpoint. It is found that the pencil of conics above described is of fundamental importance.

The problem exists for nodal cubics only since the nth tangential of a point on a cuspidal cubic approaches the cusp as a limit.‡

2. To determine those points on a rational cubic which coincide with their third tangentials.

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* *Bulletin of the American Mathematical Society*, October, 1918, November, 1919; *American Journal of Mathematics*, Vol. 42 (1920).

† Sylvester, *American Journal of Mathematics*, Vol. 2, 3; Story, *Ibid.*, Vol. 3; Picquet, *Journal de l'École Polytechnique*, Vol. 54; Porter, *Transactions of the American Mathematical Society*, Vol. 2.

‡ Salmon, *Higher Plane Curves* (2nd ed.), p. 177.

The equation used is that of the crunodal cubic in the form

$$x^3 + y^3 - 3xyz = 0,$$

where the nodal tangents are the x and y axes respectively, and the line of inflexions is the z-axis. The interchange of real and imaginary elements for the acnodal case may be made at any time.

The parameter form is

$$x : y : z = 3t^2 : 3t : t^3 + 1.$$

The first tangential of the point t is $-1/t^2$,[†] the second tangential is $-t^4$, etc., the nth tangential being $-t^{(-2)^n}$.

(a) If $t = -1/t^2, \quad t^3 + 1 = 0.$

The roots are the parameters of the inflexions.

(b) If $t = -t^4, \quad t(t^3 + 1) = 0.$

The roots are the parameters of the node and of the inflexions.

(c) If $t = -1/t^8, \quad t^9 + 1 = 0.$

$$(t^3 + 1)(t^6 - t^3 + 1) = 0.$$

After removing the factor which corresponds to the inflexions, the six remaining points are seen to lie on a conic since they belong to the involution $s_6 = 1$. These are the six "coincidence points" of the rational cubic.[‡] This is easily identified with the conic

$$xy - 3z^2 = 0,$$

which belongs to the pencil already described. It is a special case for the nodal cubic of the conics described by Porter [†] in his discussion of the coincidence points of the general cubic.

The following construction may be shown for this coincidence-conic: It passes through the points in which a line joining the node to an inflexion meets the tangents at the sextactic points of the other two inflexions.

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[*] Winger, *loc. cit.*

[†] Halphen, *Mathematische Annalen,* Vol. 15.

[‡] *Loc. cit.*

3. To determine the satellite conic of a conic of the system

$$xy - \lambda z^2 = 0.$$

If the satellite belongs to the same system, its equation is of the form

$$xy - \lambda' z^2 = 0.$$

If the primary point is $(3t^2, 3t, t^3 + 1)$ the tangential is $(3t^2, -3t^4, t^6 - 1)$. The t-equations giving the parameters of the points of intersection of these conics with the cubic are

$$(1) \qquad t^6 + [2 - (9/\lambda)]t^3 + 1 = 0.$$

$$(2) \qquad t^{12} - [2 - (9/\lambda')]t^6 + 1 = 0.$$

Equation (2) must contain (1) as a factor, the other factor being the function corresponding to its other primary, found by replacing t by $-t$. Eliminating t, there results

$$\lambda' = \lambda^2/(4\lambda - 9),$$

which transforms the parameter of a conic of the system into that of its satellite.

$$\text{If } \lambda' = \lambda, \ \lambda = 0 \text{ or } 3.$$

$\lambda = 0$ gives the degenerate case of the nodal tangents. Hence, the only proper conic of the pencil which is its own satellite is the coincidence-conic which cuts out of the cubic two cyclic groups of points which coincide with their third tangentials:

$$t = e^{\pi i/9}, \quad e^{7\pi i/9}, \quad e^{13\pi i/9},$$
$$t = e^{5\pi i/9}, \quad e^{17\pi i/9}, \quad e^{11\pi i/9}.$$

To $\lambda = 9/4$, Winger's conic N, corresponds $\lambda' = \infty$, the line of inflexions doubled.

4. To determine the points which coincide with their fourth tangentials.

$$t = -t^{16} \quad t(t^3 + 1)(t^{12} + \cdots + 1) = 0.$$

The solution lies in the two conics of the third factor. This will factor into

$$(1) \qquad t^6 + [2 - (9/\lambda)]t^3 + 1 = 0$$

with roots, t_1, $1/t_1$, ωt_1, ω/t_1, $\omega^2 t_1$, ω^2/t_1, and

(2) $$t^6 + [2 - (9/\lambda')]t^3 + 1 = 0$$

with roots t_2, $1/t_2$, ωt_2, ω/t_2, $\omega^2 t_2$, ω^2/t^1, and

Let $t = e^{(2k+1)\pi i/15}$ $2k + 1 \not\equiv 0 \pmod 5$

Substituting in (1) and equating to zero the real (or imaginary) part:

$$\cos 2(2k+1)(\pi/5) + [2 - (9/\lambda)] \cos(2k+1)(\pi/5) + 1 = 0,$$
$$2\cos^2(2k+1)(\pi/5) + [2 - (9/\lambda)] \cos(2k+1)(\pi/5) = 0.$$

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/10)}.$$

$$\lambda_1 = \frac{9}{4\cos^2 18^\circ} \qquad \lambda_2 = \frac{9}{4\cos^2 54^\circ}.$$

By the formula

$$\lambda' = \frac{\lambda^2}{4\lambda - 9},$$

it may be shown that λ_1 is the satellite of λ_2, and conversely. The cyclic groups of points are shown by the following table:

$$t = e^{(2k+1)\pi i/15}$$

λ	k	t	k	t	k	t
λ_1	1	$e^{3\pi i/15}$	3	$e^{7\pi i/15}$	8	$e^{17\pi i/15}$
λ_2	4	$e^{9\pi i/15}$	0	$e^{\pi i/15}$	5	$e^{11\pi i/15}$
λ_1	13	$e^{27\pi i/15}$	6	$e^{13\pi i/15}$	11	$e^{23\pi i/15}$
λ_2	10	$e^{21\pi i/15}$	9	$e^{19\pi i/15}$	14	$e^{29\pi i/15}$

5. Points which coincide with their fifth tangentials.

$$t = -1/t^{32} \qquad t^{33} + 1 = 0 \qquad (t^3 + 1)(t^{30} + \cdots + 1) = 0.$$

The last factor yields five conics by the process of the preceding section.

$$t = e^{(2k+1)\pi i/33} \qquad (2k+1) \not\equiv 0 \pmod{11}$$

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/22)}$$

$$\lambda_1 = \frac{9}{4\cos^2(\pi/22)}, \qquad \lambda_2 = \frac{9}{4\cos^2(9\pi/22)}, \qquad \lambda_3 = \frac{9}{4\cos^2(7\pi/22)}$$

$$\lambda_4 = \frac{9}{4\cos^2(3\pi/22)}, \qquad \lambda_5 = \frac{9}{4\cos^2(5\pi/22)}.$$

λ	k	k	k	k	k	k
λ_1	0	11	22	10	21	32
λ_2	15	26	4	28	6	17
λ_3	18	29	7	25	3	14
λ_4	12	23	1	31	9	20
λ_5	24	2	13	19	30	8

6. Points which coincide with their sixth tangentials.

$$t = -t^{64} \qquad t(t^{63}+1) = 0 \qquad t(t^3+1)(t^{60}+\cdots+1) = 0$$

$$t = e^{(2k+1)\pi i/63} \qquad 2k+1 \not\equiv 0 \pmod{21}$$

If $2k+1 \equiv 0 \pmod 7$ the solutions are those of a previous case, $t^9+1=0$. Hence there are 9 conics for the remaining 54 values of k.

$$\lambda = \frac{9}{4\cos^2(2k+1)(\pi/42)}\,.$$

There are 6 conics which form a cyclic satellite-group, and 3 conics such that each coincides with its third satellite conic, as follows:

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$k=$	0	30	33	27	39	15
$k=$	20	53	50	56	44	5
$k=$	21	51	54	48	60	36
$k=$	41	11	8	14	2	26
$k=$	42	9	12	6	18	57
$k=$	62	32	29	35	23	47

	λ_7	λ_8	λ_9	λ_7	λ_8	λ_9
$k=$	4	22	49	58	40	13
$k=$	25	43	7	16	61	34
$k=$	46	1	28	37	19	55

The following values of k determine points on the coincidence conic, $\lambda=3$:

3, 24, 45; 17, 59, 38.

7. General Case. Points which coincide with their nth tangentials.

$$t = -t^{(-2)^n}$$

Case 1. n even: $t = -t^{2^n}$

$$t(t^{2^n-1} + 1) = 0$$
$$2^n - 1 \equiv 0 \quad (\mathrm{mod}\ 3)$$
$$t(t^3 + 1)(t^{2^n-4} + \cdots + 1) = 0$$
$$2^n - 4 \equiv 0 \quad (\mathrm{mod}\ 12).$$

The number of conics is

$$(2^n - 4)/6 = (2/3)(2^{n-2} - 1)$$

including as special cases all solutions for the prime factors of n.

Case 2. n odd. $t = -(1/t^{2^n})$ $t^{2^n+1} + 1 = 0$

$$2^n + 1 \equiv 0 \quad (\mathrm{mod}\ 3)$$
$$(t^3 + 1)(t^{2^n-2} + \cdots + 1) = 0$$
$$2^n - 2 \equiv 0 \quad (\mathrm{mod}\ 6).$$

The number of conics is

$$(2^{n-1} - 1)/3$$

which includes the solutions for the prime factors of n.

If n is an odd prime, this formula gives the exact number of proper solutions. By Fermat's theorem

$$2^{n-1} - 1 \equiv 0 \quad (\mathrm{mod}\ n)$$

and hence the number of cyclic groups of n conics is

$$(2^{n-1} - 1)/3n \qquad n \neq 3.$$

Oskar Bolza

New Results in Elimination.[*]

By F. Morley and A. B. Coble.

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Book No. ....

Elimination is a fundamental process in algebra. One of its most general problems may be stated in homogeneous form as follows:

Given j forms,

$$\alpha_{1y}^{m_1},\ \alpha_{2y}^{m_2},\ \cdots,\ \alpha_{jy}^{m_j} \qquad (m_1 \leqq m_2 \leqq m_3 \cdots \leqq m_j)$$

homogeneous and of the orders m indicated in the variables $y_1, y_2, \cdots, y_j$, to find that polynomial in the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_j$ whose vanishing expresses the necessary and sufficient condition that the forms vanish simultaneously for a value system $x_1, x_2, \cdots, x_j$. This polynomial R is the so-called resultant or eliminant i. e. the result of eliminating $y_1 : y_2 : \cdots : y_j$ from the j equations, $\alpha_{1y}^{m_1} = 0$, $\alpha_{2y}^{m_2} = 0, \cdots, \alpha_{jy}^{m_j} = 0$. As thus defined a numerical factor in R is unessential though in certain cases (e. g. if R is included among the members of a complete system of invariants of the given forms) some convention as to such a factor may be desirable. In geometric language $R = 0$ is the necessary and sufficient condition that the j spreads, $\alpha_{1y}^{m_1} = 0, \cdots, \alpha_{jy}^{m_j} = 0$, of dimension $j - 2$ in S_{j-1} have a common point x. The degree of R in the coefficients of a particular form $\alpha_{ky}^{m_k}$ is known to be $m_1 . m_2 . \cdots . m_j / m_k$. A subsidiary problem is that of exhibiting in rational form the common solution x which exists when $R = 0$.

Other problems such as those of "restricted" elimination appear e. g. to express the condition that three conics with two known common points (say conics without terms in y_1^2, y_2^2) shall have a further common point. This memoir is confined to the general problem formulated above.

The binary case, $j = 2$, of elimination of one non-homogeneous variable, or two homogeneous variables, from two equations has been satisfactorily solved. The dialytic determinant of Sylvester provides the resultant in the convenient form now commonly presented. A much earlier method of Bézout[†] modified later by Cayley [‡]) for two forms of the same order, extended by Rosenhain [§] to different orders, furnishes the resultant as a more contracted determinant. Conditions for further common solutions have been found.[¶]

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[*] An investigation pursued under the auspices of the Carnegie Institution of Washington, D. C.

[†] Paris Memoirs (1764).

[‡] *Journal für Mathematik*, Vol. 53 (1857.)

[§] *Journal für Mathematik*, Vol. 28 (1844).

[¶] Further references are found in the *Encyklopädie*, Vol. 1, pp. 245-250; 260-274; and in Pascal's *Repertorium*, Leipzig (Teubner) (1900), Vol. 1, pp. 86-89; 274-278; 322-323.

Until recently however little progress had been made, in the elimination of more variables, toward the formation of a pure resultant, i. e. one free of extraneous factors which contain the coefficients of one or more of the given forms. It is clear that, as the variables are successively eliminated from selected pairs of equations by the dialytic or other method, a result R' is obtained which is the resultant R multiplied by a factor which depends upon the order in which the variables are eliminated and upon the order in which the equations are selected. As this order changes a variety of results R' appear of which R is the G. C. D. Thus elimination of any number of variables is properly regarded as a rational operation but the explicit form of the end result which we seek here has not been exhibited in the general case.[*]

Little effective progress has been made by the methods of invariant theory such as that devised by Gordan[†] for three ternary forms.

The first notable extension of the Cayley-Bézout determinant to higher cases was presented by Professor Morley to the Toronto Congress (1924).[‡] This furnished the resultant R for three ternary forms of the same order.

It is the purpose of this paper to extend this result to a wide variety of ternary cases and a more limited number of cases in more than three variables. The orders m for which R is explicitly obtained as a determinant are, in ascending magnitude, the following:

$j=3$: $m_1,\ m_2,\ m_2+r \quad (r=0,\ 1,\ \cdots,\ m_1+1)$;
$j=4$: $m_1,\ m_1,\ m_1,\ m_1+r \quad (r=0,\ 1,\ 2)$,
$m_1,\ m_1,\ m_1+1,\ m_1+1$,
$m_1,\ m_1+r,\ m_1+r,\ m_1+r \quad (r=1,2)$,
$m_1,\ m_1+1,\ m_1+1,\ m_1+2$;
$j=5$: $3,\ 3,\ 3,\ 3,\ 3$,
$2,\ 2,\ 2,\ 2,\ 2+r \quad (r=0,1)$;
$j=6$: $2,\ 2,\ 2,\ 2,\ 2,\ 2$.

Further cases for which some of the orders m are unity are given in § 2.

This is accomplished by the use of a covariant J_k introduced by Morley for the particular case mentioned. J_k may be defined as the sum of jacobians of polars of x as to the given forms whose order in y is k; or more specifically

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[*] For other rational methods see Salmon's *Higher Algebra*, (1876), pp. 79-83.

[†] *Mathematische Annalen*, Vol. 50 (1897).

[‡] Published in this *Journal*, Vol. 47 (1925), pp. 91-97, under the title: "The Eliminant of a Net of Curves."

$$(1)\qquad J_k = \sum_{k_1,\,\ldots,\,k_j} (\alpha_1\alpha_2\cdots\alpha_j)\,\alpha_{1x}^{m_1-k_1-1}\alpha_{1y}^{k_1}\cdots\alpha_{jx}^{m_j-k_j-1}\alpha_{jy}^{k_j}$$
$$(k_1+\cdots+k_j=k).$$

The importance of this particular sum of jacobians is doubtless due to the recurrence relations (4) which it satisfies.

In addition to J_k we use the *modulus* F defined by the given forms, $\alpha_{1y}^{m_1}, \cdots, \alpha_{jy}^{m_j}$, i. e. the totality of algebraic forms comprised under the formula,

$$(2)\qquad F = \beta_{1y}^{k-m_1}\cdot\alpha_{1y}^{m_1} + \beta_{2y}^{k-m_2}\cdot\alpha_{2y}^{m_2} + \cdots + \beta_{jy}^{k-m_j}\cdot\alpha_{jy}^{m_j}$$

where k is any order and $\beta_{1y}^{k-m_1}, \cdots, \beta_{jy}^{k-m_j}$ are *arbitrary* forms of the orders indicated. If in particular the coefficients β all vanish we note that a form which vanishes identically is contained in the modulus.

The method is based on the following fundamental theorem:

(I) *If the equations,* $\alpha_{1y}^{m_1}=0, \cdots, \alpha_{jy}^{m_j}=0$, *have a common solution* x, *then all of the covariants* J_k *for* $k=0, 1, \cdots, m_1+\cdots+m_j-j$ *are contained in the modulus* F.

For $k=0, 1, \cdots, m_1-1$ the order of J_k in y is too low to permit of an effective expression (2) whence, for these values of k, $J_k\equiv 0$. Thus in Morley's case, $m_1=m_2=m_3$, we have the syzygies, $J_{m_1-1}\equiv 0$, $J_{m_1-2}\equiv 0$, which he used to form the resultant.

The identity in y given by the fundamental theorem furnishes a number of equations which contain, in J_k on the left, products of powers of the unknown common solution x, and on the right the unknown coefficients β of a form of the modulus. Additional equations may be obtained by multiplying $\alpha_{1x}^{m_1}=0, \cdots, \alpha_{jx}^{m_j}=0$ by such products of powers of $x_1, \cdots, x_j$ as will produce the order, $\sum m_i - j - k$, of J_k in x. From such equations, linear in the products x and the coefficients β, these quantities may be eliminated in the cases mentioned and the resultant R appears as a determinant.

We observe that the coefficients β in the expression (2) of F are no longer unique if $k \geqq m_1+m_2$. For

$$\beta_{1y}^{k-m_1}\cdot\alpha_{1y}^{m_1} + \beta_{2y}^{k-m_2}\cdot\alpha_{2y}^{m_2} \equiv$$
$$[\beta_{1y}^{k-m_1} + \gamma_y^{k-m_1-m_2}\cdot\alpha_{2y}^{m_2}]\cdot\alpha_{1y}^{m_1} + [\beta_{2y}^{k-m_2} - \gamma_y^{k-m_1-m_2}\cdot\alpha_{1y}^{m_1}]\cdot\alpha_{1y}^{m_2}.$$

If then we hold k at the upper limit m_1+m_2-1 which avoids this ambiguity in the coefficients β, the order in x, $\sum m_i - j - k$, also should not exceed m_1+m_2-1. Otherwise the additional equations will not be independent since the one obtained by multiplying $\alpha_{1x}^{m_1}=0$ by $\alpha_{2x}^{m_2}$ coincides with the one obtained by multiplying $\alpha_{2x}^{m_2}=0$ by $\alpha_{1x}^{m_1}$. Thus our equations are independent only if $\sum m_i - j - k < m_1 + m_2$ $(k = m_1 + m_2 - 1)$ i. e. if

$$m_3 + m_4 + \cdot\cdot\cdot + m_j < m_1 + m_2 + j - 1,$$

an inequality satisfied only by the cases noted above.

The discriminant of a spread, $\alpha_y^m = 0$, in S_{j-1} is the resultant of the j first derivatives of order $m-1$. Hence we provide a determinant expression for the discriminant of any plane curve (Morley), of any surface, of the cubic and quartic spread in S_4, and of the cubic spread in S_5.

The details of the argument are given in the sections which follow. A proof of the fundamental theorem (I) appears in § 1. In § 2 we show that the equations obtained from $J_{\sum m_i - j - (m_1+m_2-1)}$ and the additional equations obtained from the given forms permit of eliminating the coefficients β and products x to obtain a determinant R. In § 3 it is proved that R has the same degree in the coefficients of the given forms as the resultant and therefore either R is the resultant or $R \equiv 0$. But $R \not\equiv 0$ since it is verified that $R = \pm 1$ for the particular forms, $y_1^{m_1}$, $y_2^{m_2}$, $\cdot\cdot\cdot$, $y_j^{m_j}$. There is given in § 4 an interesting extension of the fundamental theorem which applies to $j+1$ general forms. In § 5 we prove that the resultant R, when properly bordered, furnishes in general the unique apolar form of the j given forms and therefore, when $R=0$, a power of ξ_x, the common solution. The existence of other forms of the resultant, also determinants, here becomes clear. These alternative forms are frequently determinants of lower order than those given in § 2.

The extension of the fundamental theorem which appears in § 4 is in effect a generalized form of the syzygy which Morley used as a starting point. In § 6 a new proof along his earlier line is given and the application to the formation of resultants appears as a corollary.

1. *Proofs of the fundamental theorem.* The first proof of theorem (I) is based only on the determinant identities and on the recursion formulae (4) below. These persist for any number of variables so that it will be sufficient to carry the proof through for four variables and then to state the general result.

We taken then four quaternary forms or surfaces,

$$(1) \qquad \alpha_y^m,\ \beta_y^n,\ \gamma_y^p,\ \delta_y^q \qquad (m \gtrless n \gtrless p \gtrless q),$$

all on a point x. The form J_k, the sum of jacobians of polars of x whose degree in y is k, we write as

$$(2) \qquad J_k \equiv (\alpha\beta\gamma\delta) \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-1}{k}, \qquad (k > 0 \text{ else } J_k = 0).$$

In this $(\alpha\beta\gamma\delta)$ is the usual symbolic determinant and the second parenthesis represents a sum of terms, namely

$$(3) \qquad \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-1}{k}$$
$$\equiv \sum_{k_1, \ldots, k_4} \alpha_x{}^{m-k_1-1}\alpha_y{}^{k_1}\beta_x{}^{n-k_2-1}\beta_y{}^{k_2}\,\gamma_x{}^{p-k_3-1}\gamma_y{}^{k_3}\delta_x{}^{q-k_4-1}\delta_y{}^{k_4}$$
$$(k_1 + k_2 + k_3 + k_4 = k).$$

We observe that every term of this sum contains a factor α_y or, if not α_y, a factor $\alpha_x{}^{m-1}$; also that every term contains a factor α_x, or, if not α_x, a factor $\alpha_y{}^{m-1}$. Thus

$$(4.1\alpha) \quad \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-1,}{k} = \alpha_y \genfrac{(}{)}{0pt}{}{m-2,\, n-1,\, p-1,\, q-1}{k-1} + \alpha_x{}^{m-1} \genfrac{(}{)}{0pt}{}{0,\, n-1,\, p-1,\, q-1}{k};$$

$$(4.2\alpha) \quad \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-1}{k} = \alpha_x \genfrac{(}{)}{0pt}{}{m-2,\, n-1,\, p-1,\, q-1}{k} + \alpha_y{}^{m-1} \genfrac{(}{)}{0pt}{}{0,\, n-1,\, p-1,\, q-1}{k-m+1};$$

and similar formulae are valid for β, γ, and δ. If any of the integers in these parentheses are negative the corresponding terms do not appear.

Let ξ_x be an arbitrary linear form which represents a plane ξ not on the point x common to the four surfaces. We multiply J_k by ξ_x and replace $(\alpha\beta\gamma\delta)\xi_x$ from

$$(5) \qquad (\alpha\beta\gamma\delta)\xi_x \equiv (\alpha\beta\gamma\xi)\delta_x - (\alpha\beta\delta\xi)\gamma_x + (\alpha\gamma\delta\xi)\beta_x - (\beta\gamma\delta\xi)\alpha_x.$$

If now to the last term in α_x we apply (4.1α) the second term on the right of (4.1α) contains the factor $\alpha_x{}^m$ which by hypothesis vanishes. Thus

$$(6) \quad \xi_x \cdot J_k = (\alpha\beta\gamma\xi) \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-2}{k-1} \delta_x\delta_y - (\alpha\beta\delta\xi) \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-2,\, q-1}{k-1} \gamma_x\gamma_y$$
$$+ (\alpha\gamma\delta\xi) \genfrac{(}{)}{0pt}{}{m-1,\, n-2,\, p-1,\, q-1}{k-1} \beta_x\beta_y - (\beta\gamma\delta\xi) \genfrac{(}{)}{0pt}{}{m-2,\, n-1,\, p-1,\, q-1}{k-1} \alpha_x\alpha_y.$$

In the first term on the right of (6) substitute for $(\alpha\beta\gamma\xi)\delta_y$ from

$$(\alpha\beta\gamma\xi)\delta_y \equiv (\alpha\beta\gamma\delta)\xi_y + (\alpha\beta\delta\xi)\gamma_y - (\alpha\gamma\delta\xi)\beta_y + (\beta\gamma\delta\xi)\alpha_y.$$

We have then on the right one term in $(\alpha\beta\gamma\delta)$ and two terms in each of $(\alpha\beta\delta\xi)$, $(\alpha\gamma\delta\xi)$, $(\beta\gamma\delta\xi)$. The pair in $(\alpha\beta\delta\xi)$ is

$$(\alpha\beta\delta\xi)\gamma_y \left[\delta_x \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-1,\, q-2}{k-1} - \gamma_x \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-2,\, q-1}{k-1}\right].$$

To reduce this pair we apply (4.2γ) to the first parenthesis and (4.2δ) to the second and obtain

$$(\alpha\beta\delta\xi) \left[\delta_x \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, 0,\, q-2}{k-p} \gamma_y{}^p - \gamma_y\gamma_x\delta_y{}^{q-1} \genfrac{(}{)}{0pt}{}{m-1,\, n-1,\, p-2,\, 0}{k-q}\right].$$

With similar modifications in the other two pairs of terms we have

$$
\begin{aligned}
(7)\quad \xi_x \cdot J_k = \xi_y \cdot (\alpha\beta\gamma\delta)\ &\left(m-1,\, \underset{k-1}{n-1},\, p-1,\, q-2\right)\delta_x \\
&+ (\alpha\beta\delta\xi)\left[\delta_x\left(m-1,\, \underset{k-p}{n-1},\, 0,\, q-2\right)\cdot\gamma_y{}^p - \gamma_y\gamma_x\delta_y{}^{q-1}\left(m-1,\, \underset{k-q}{n-1},\, p-2,\, 0\right)\right] \\
&- (\alpha\gamma\delta\xi)\left[\delta_x\left(m-1,\, \underset{k-n}{0},\, n-1,\, q-2\right)\cdot\beta_y{}^n - \beta_y\beta_x\delta_y{}^{q-1}\left(m-1,\, \underset{k-q}{n-2},\, p-1,\, 0\right)\right] \\
&+ (\beta\gamma\delta\xi)\left[\delta_x\left(0,\, \underset{k-m}{n-1},\, p-1,\, q-2\right)\cdot\alpha_y{}^m - \alpha_y\alpha_x\delta_y{}^{q-1}\left(m-2,\, \underset{k-q}{n-1},\, p-1,\, 0\right)\right].
\end{aligned}
$$

We observe at this point that if $k < q$ the three terms in $\delta_y{}^{q-1}$ will not appear because of the negative $(k-q)$ in the parentheses. Then in the first term on the right of (7) we can replace, by virtue of (4.2δ),

$$\left(m-1,\, \underset{k-1}{n-1},\, p-1,\, q-2\right)\delta_x \text{ by } \left(m-1,\, \underset{k-1}{n-1},\, p-1,\, q-1\right)$$

and we have a preliminary expression for J_k in terms of the modulus $(J_{k-1}, \alpha, \beta, \gamma, \delta)$ which reads:

$$
\begin{aligned}
(8)\quad \xi_x \cdot J_k = \xi_y \cdot J_{k-1} &+ (\alpha\beta\delta\xi)\left(m-1,\, \underset{k-p}{n-1},\, 0,\, q-2\right)\delta_x\cdot\gamma_y{}^p \\
&-(\alpha\gamma\delta\xi)\left(m-1,\, \underset{k-n}{0},\, p-1,\, q-2\right)\delta_x\cdot\beta_y{}^n + (\beta\gamma\delta\xi)\left(0,\, \underset{k-m}{n-1},\, p-1,\, q-2\right)\delta_x\cdot\alpha_y{}^m, \\
&\qquad (k < q).
\end{aligned}
$$

In this if $k < p$ the term in $\gamma_y{}^p$ also will not occur; if $k < n$, only the term in $\alpha_y{}^m$ appears; and if further $k < m$ we shall have merely that

$$(9)\qquad (k < m)\qquad \xi_x \cdot J_k = \xi_y \cdot J_{k-1}.$$

Since J_0, the ordinary jacobian, vanishes at the common point x the recursion formula (9) yields, by repeated multiplication by ξ_x, the result

$$(10)\qquad (k < m)\qquad J_k \equiv 0.$$

Thus for three ternary forms and equal orders $m = n = p$ we have Morley's syzygies:

$$J_{m-1} \equiv 0, \quad J_{m-2} \equiv 0.$$

If $k = m < n$, formula (8) becomes

$$(11)\qquad \xi_x \cdot J_m \equiv (\beta\gamma\delta\xi)\left(0,\, n-1,\, \underset{0}{p-1},\, q-1\right)\cdot\alpha_y{}^m \qquad (m < n \leqq p \leqq q),$$

a rather remarkable expression for a general surface $\alpha_y{}^m$ in terms of a point x on it and any three other surfaces of higher order which also are on x.

We return to the general formula (7) and modify the first term on the right by substituting for $\left(m-1,\, \underset{k-1}{n-1},\, p-1,\, q-2\right)\delta_x$ from (4.2δ) which brings in a new term in $\delta_y{}^{q-1}$ i. e.

$$\delta_x \binom{m-1,\, n-1,\, p-1,\, q-2}{k-1} = \binom{m-1,\, n-1,\, p-1,\, q-1}{k-1} - \delta_y^{q-1} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q}.$$

We modify also the three terms in (7) which already contain δ_y^{q-1} by using

$$\gamma_x \binom{m-1,\, n-1,\, p-2,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \gamma_y^{p-1} \binom{m-1,\, n-1,\, 0,\, 0}{k-p-q+1},$$

$$\beta_x \binom{m-1,\, n-2,\, p-1,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \beta_y^{n-1} \binom{m-1,\, 0,\, p-1,\, 0}{k-n-q+1},$$

$$\alpha_x \binom{m-2,\, n-1,\, p-1,\, 0}{k-q} = \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} - \alpha_y^{m-1} \binom{0,\, n-1,\, p-1,\, 0}{k-m-q+1}.$$

Then the right member of (7) will contain first a term $\xi_y \cdot J_{k-1}$; second, four terms in $\delta_y^{q-1} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q}$ with coefficients,

$$-(\alpha\beta\gamma\delta)\xi_y - (\alpha\beta\delta\xi)\gamma_y + (\alpha\gamma\delta\xi)\beta_y - (\beta\gamma\delta\xi)\alpha_y = -(\alpha\beta\gamma\xi)\delta_y,$$

which therefore contribute the single term

$$-(\alpha\beta\gamma\xi) \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} \cdot \delta_y^q;$$

third, two terms in $(\alpha\beta\delta\xi) \cdot \gamma_y^p$ with coefficients

$$\delta_x \binom{m-1,\, n-1,\, 0,\, q-2}{k-p} + \delta_y^{q-1} \binom{m-1,\, n-1,\, 0,\, 0}{k-p-q+1} = \binom{m-1,\, n-1,\, 0,\, q-1}{k-p}$$

which therefore contribute the single term $(\alpha\beta\delta\xi) \binom{m-1,\, n-1,\, 0,\, q-1}{k-p} \cdot \gamma_y^p$; and finally two similar terms in each of $(\alpha\gamma\delta\xi) \cdot \beta_y^n$ and $(\beta\gamma\delta\xi) \cdot \alpha_y^m$. Thus (7) takes the recursion form:

$$(12) \quad \begin{aligned} \xi_x \cdot J_k \equiv\ & \xi_y \cdot J_{k-1} \\ & - (\alpha\beta\gamma\xi) \binom{m-1,\, n-1,\, p-1,\, 0}{k-q} \cdot \delta_y^q + (\alpha\beta\delta\xi) \binom{m-1,\, n-1,\, 0,\, q-1}{k-p} \cdot \gamma_y^p \\ & - (\alpha\gamma\delta\xi) \binom{m-1,\, 0,\, p-1,\, q-1}{k-n} \cdot \beta_y^n + (\beta\gamma\delta\xi) \binom{0,\, n-1,\, p-1,\, q-1}{k-m} \cdot \alpha_y^m. \end{aligned}$$

The formula (12) furnishes an expression for J_k in terms of the modulus $(J_{k-1}, \alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$. If we multiply (12) by ξ_x and replace on the right $\xi_x \cdot J_{k-1}$ by using (12) for the value $k-1$ we have an expression for J_k in terms of the modulus $(J_{k-2}, \alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$. On continuing this process and recalling from (10) that $J_k \equiv 0$ if $k < m$ we find the following expression for J_k in terms of the smaller modulus $(\alpha_y^m, \beta_y^n, \gamma_y^p, \delta_y^q)$:

$$(13) \quad \begin{aligned} \xi_x^{k-m+1} \cdot J_k \equiv\ & \\ & - (\alpha\beta\gamma\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, n-1,\, p-1,\, 0}{k-q-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \delta_y^q \\ & + (\alpha\beta\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, n-1,\, 0,\, q-1}{k-p-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \gamma_y^p \\ & - (\alpha\gamma\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{m-1,\, 0,\, p-1,\, q-1}{k-n-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \beta_y^n \\ & + (\beta\gamma\delta\xi) \left\{ \sum_{r=0}^{k-m} \binom{0,\, n-1,\, p-1,\, q-1}{k-m-r} \xi_x^{k-m-r} \xi_y^r \right\} \cdot \alpha_y^m. \end{aligned}$$

If now we adopt for J_k the more specific notation

$$J_k \equiv J_k\,(\alpha^m,\ \beta^n\ \gamma^p,\ \delta^q) \tag{14}$$

then it is clear from the definition of J_k that the coefficients of the modulus in (13) are also of the J-type and we write the final form of the identity:

$$\begin{aligned} \xi_x^{k-m+1}\cdot J_k(\alpha^m,\ \beta^n,\ \gamma^p,\ \delta^q) \equiv & -J_{k-q}(^m,\ \beta^n,\ \gamma^p,\ \xi^{k-m+1})\cdot\delta_y^q \\ & + J_{k-p}(\alpha^m,\ \beta^n,\ \delta^q,\ \xi^{k-m+1})\cdot\gamma_y^p - J_{k-n}(\alpha^m,\ \gamma^p,\ \delta^q,\ \xi^{k-m+1})\cdot\beta_y^n \\ & + J_{k-m}(\beta^n,\ \gamma^p,\ \delta^q,\ \xi^{k-m+1})\cdot\alpha_y^m. \end{aligned} \tag{15}$$

We extend this at once to the general case and give the following more explicit statement of the theorem (I):

If the spreads $\alpha_{1y}^{m_1},\ \alpha_{2y}^{m_2},\ \cdots,\ \alpha_{jy}^{m_j}$ in S_{j-1} have a common point x, the form $J_k(\alpha_1^{m_1},\ \cdots,\ \alpha_j^{m_j})$ of order k in y belongs to the modulus defined by the j given forms. If ξ_y is a fixed linear form such that $\xi_x \neq 0$, the explicit expression for J_k in terms of the modulus is

$$\begin{aligned} & \xi_x^{k-m_1+1}\cdot J_k(\alpha_1^{m_1},\cdots,\alpha_j^{m_j}) \equiv \\ & \sum_{s=1}^{j}(-1)^{j-s+1}\,J_{k-m_s}\,(\alpha_1^{m_1},\ \cdots,\ \alpha_{s-1}^{m_{s-1}},\ \alpha_{s+1}^{m_{s+1}},\ \cdots,\ \alpha_j^{m_j},\ \xi^{k-m_1+1})\cdot\alpha_{sy}^{m_s}. \end{aligned} \tag{16}$$

2. *Formation of the Resultant.* If the j equations, $\alpha_{iy}^{m_i}=0$ $(i=1,\ \cdots,\ j)$ are given subject merely to the condition that they have an unknown common solution x, then, in the syzygy [i. e. the identical relation of order k in y (16)] where ξ is fixed, we have the constant $\xi_x^{k-m_1+1}$ and the forms J_{k-m_s} of order $k-m_s$ in y whose coefficients also contain the unknown x. Thus after division by $\xi_x^{k-m_1+1}$ we have an identity of order k in y of the form

$$J_k(\alpha_1^{m_1},\ \cdots,\ \alpha_j^{m_j}) + \beta_{1y}^{k-m_1}\cdot\alpha_{1y}^{m_1} + \cdots + \beta_{jy}^{k-m_j}\cdot\alpha_{jy}^{m_j} \equiv 0 \tag{17}$$

in which there occur the unknown coefficients of the forms β and (in J_k) the combinations, also unknown, of the coordinates x, which are of degree $\sum m_i - j - k$. We seek to eliminate these t unknowns from a set of t equations linear in them. For this it is essential that the values of the unknowns be unique and that the additional equations which may be obtained from $\alpha_{1x}^{m_1}=0,\ \alpha_{2x}^{m_2}=0,\ \cdots$ by multiplication with such combinations of x as raise them to the degree $\sum m_i - j - k$ shall themselves be linearly independent. As explained in the introduction this leads to the inequalities

$$\sum m_i - j - k \leqq m_1 + m_2 - 1, \quad k \leqq m_1 + m_2; \quad \text{or to}$$
$$m_3 + \cdots + m_j < j + k, \quad k < m_1 + m_2.$$

Thus the most favorable range for the orders $m_3, \cdots, m_j$ is obtained by taking the maximum value for k, i. e. $m_1 + m_2 - 1$; and the original inequalities and the one just deduced are

$$(18) \qquad m_1 \leqq m_2 \leqq \cdots \leqq m_j,$$
$$m_3 + \cdots + m_j < m_1 + m_2 + (j - 1).$$

The case $m_i = 1$ may be discarded since if any of the given forms are linear the elimination may at once be reduced to a lower dimension. To this point however we shall recur. Since $m_3 + m_4 \geqq m_1 + m_2$ the largest range for $m_5, \cdots, m_j$ occurs when $m_1 = m_2 = m_3 = m_4$ when

$$m_5 + \cdots + m_j < j - 1.$$

Since $m_i \geqq 2$ the largest range for j appears when $m_i = 2$ and therefore when $2(j - 4) < j - 1$ or $j < 7$. For $j = 6$ and the most favorable cases indicated we have $m_5 + m_6 < 5$ or $m_5 = m_6 = 2$ and necessarily $m_1 = \cdots = m_4 = 2$. For $j = 5$ and $m_3 + m_4 + m_5 < m_1 + m_2 + 4$ the largest value of m_5 is obtained when $m_3 + m_4 = m_1 + m_2$ i. e. when $m_1 = \cdots = m_4$ and $m_5 < 4$. If $m_5 = 3$, $m_1, \cdots, m_4$ must all be 3 or 2; if $m_5 = 2$, $m_1, \cdots, m_4$ must all be 2. For $j = 4$ and $m_3 + m_4 < m_1 + m_2 + 3$ it is convenient to set $m_2 = m_1 + r$, $m_3 = m_1 + r + s$, $m_4 = m_1 + r + s + t$ $(r, s, t \geqq 0)$. Then the inequality becomes $r + 2s + t < 3$ whose seven solutions, $(r, s, t) = (0, 0, 0), (1, 0, 0), (0, 0, 1), (2, 0, 0), (0, 0, 2), (1, 0, 1), (0, 1, 0)$ furnish the cases mentioned in the introduction. For $j = 3$ and $m_3 < m_1 + m_2 + 2$ we set $m_3 = m_2 + r$ and find that $r < m_1 + 2$. We have thus accounted for the range of cases listed in the introduction.

If one of the forms and therefore necessarily the first is linear the elimination may in a sense be reduced to elimination in a lower dimension. The procedure for space for forms α_y, $\beta_y{}^n$, $\gamma_y{}^p$, $\delta_y{}^q$ would be as follows. If z, z', z'' are the vertices of a fixed triangle on the given plane α, any point y of the plane is given by $y = \lambda z + \lambda' z' + \lambda'' z''$. On substituting this value of y in the other three forms we have three ternary forms of orders n, p, q in $\lambda, \lambda', \lambda''$ whose eliminant for λ is the required resultant of the four original forms. This resultant however is of degree npq in each of z, z', z'' and in order to express it in terms of the original coefficients α we should have to modify it in such a way as to exhibit the z, z', z'' only in the combinations $\alpha_i = | \, z \, z' z'' \, |_i$.

Such modification would not in general be practicable and thus it is desirable to see how many cases in which at least one form is linear come under our general method. If then we set $m_1 = 1$, $m_2 = 1 + r_1$, $m_3 = 1 + r_1 + r_2, \cdots$ $(r_1, \cdots, r_{j-1} \geqq 0)$ the original inequality becomes

$$(j-3)r_1 + (j-2)r_2 + (j-3)r_3 + j-4)r_4 \\ + \cdots + 3r_{j-3} + 2r_{j-2} + r_{j-1} < 3.$$

A discussion of this inequality shows that for $j \geqq 6$ there are always four cases for which $(m_{j-1}, m_j) = (1, 1), (1, 2), (1, 3), (2, 2)$ and the other m's all are unity. If in the first three of these the last form is $\alpha_{jy}{}^r$ $(r = 1, 2, 3)$, the resultant obviously is $(\alpha_1\alpha_2 \cdots \alpha_{j-1}\alpha_j)^r$. In the last of the four cases we form the resultant of two binary quadratics in symbolic form and then immediately write it in the symbolic form for the higher case by using the Clebsch translation principle. For $j = 3, 4$ we find precisely the cases already listed in the introduction for which $m_1 = 1$. For $j = 5$ there is in addition to the four cases which always occur when $j \geqq 6$ just one case 1, 2, 2, 2, 2. Our method will in this case furnish a resultant explicit in the coefficients α_1 of the plane α_{1y}, or if these coefficients be replaced by variable coordinates ξ it will furnish the equation of the 16 common points of four quadrics in space S_4.

We consider now the cases listed in the introduction and begin with the three ternary forms of orders

$$j = 3: \qquad m_1,\ m_2,\ m_2 + r \qquad (r = 0, 1, \cdots, m_1 + 1).$$

The fundamental syzygy now reads:

$$J_k + \beta_{1y}{}^{k-m_1} \cdot \alpha_{1y}{}^{m_1} + \beta_{2y}{}^{k-m_2} \cdot \alpha_{2y}{}^{m_2} + \beta_{3y}{}^{k-m_2-r} \cdot \alpha_{3y}{}^{m_2+r} \equiv 0.$$

It is convenient to obtain as many independent equations as possible from $\alpha_{1x}{}^{m_1} = 0$, $\alpha_{2x}{}^{m_2} = 0$, $\alpha_{3x}{}^{m_2+r} = 0$ by multiplication with products of powers of x since such equations contain the coefficients $\alpha_1, \alpha_2, \alpha_3$ in the simplest way and therefore we allow x to enter to the maximum order $m_1 + m_2 - 1$. Since the jacobian of the three forms has the order $m_1 + 2m_2 + r - 3$ the order of J_k in y is $k = m_2 + r - 2$. Thus the syzygy (an identity in y) yields $\binom{k+2}{2}$ equations linear in the $\binom{m_1+m_2+1}{2}$ products π_x of powers of x of order $m_1 + m_2 - 1$ and linear in the coefficients $\beta_1, \beta_2, \beta_3$. But for particular values of m_2 and r some or all of the forms β will vanish identically so that the linear equations will differ formally in certain cases. Thus since $k = m_2 + r - 2$ then $k - m_2 - r < 0$ and $\beta_{3y}{}^{k-m_2-r}$ is always identically zero.

If $r < 2$, $k - m_2 < 0$ and $\beta_{2y}{}^{k-m_2} \equiv 0$. If $r < 2$ and $k = m_2 + r - 2 < m_1$ where also $m_1 \lessgtr m_2$ then $\beta_{1y}{}^{k-m_1} \equiv 0$. For $r = 0$ this yields two cases m_1, m_1, m_1 and $m_1, m_1 + 1, m_1 + 1$ the first of which is treated by Morley and will not be considered further here. For $r = 1$ we have one case $m_1, m_1, m_1 + 1$. The last two cases we combine under

Case III_a: $\quad m_{,1}\, m_1 + r, m_1 + 1 \quad (r = 0, 1) \ : \ J_{m_1-1} \equiv 0.$

If $r < 2$ and $k = m_2 + r - 2 \geqq m_1$ then always $m_2 \gtreqless m_1$. Thus m_2 may have any order and r may be 0, 1 which yields

Case III_b: $\quad m_1, m_2, m_2 + r \quad (r = 0, 1) \ : \ J_{m_2-2+r} \equiv \beta_{1y}{}^{m_2-m_1-2+r} \cdot \alpha_{1y}{}^{m_1}$

$$(m_2 \geqq m_1 + 2 - r),$$

where the last inequality bars Case III_a.

For $2 < r < m_1 + 2$ we have the more general

Case III_c: $\quad m_1, m_2, m_2 + r \quad (r = 2, \cdots, m_1 + 1) \ : \ J_{m_2-2+r}$

$$\equiv \beta_{1y}{}^{m_2-m_1-2+r} \cdot \alpha_{1y}{}^{m_1} + \beta_{2y}{}^{r-2} \cdot \alpha_{2y}{}^{m_2}.$$

The linear equations obtained from the syzygy, and the additional equations linear in π_x alone obtained by raising the orders of $\alpha_{1x}{}^{m_1} = 0$, $\alpha_{2x}{}^{m_2} = 0$, $\alpha_{3x}{}^{m_3} = 0$ to $m_1 + m_2 - 1$ have the matrices indicated below.

Case III_a: $(r = 0, 1)$ $\qquad$ Case III_b: $(r = 0, 1)$

$$(19) \quad \begin{array}{c|c|} & \binom{2m_1+r+1}{2} \\ \binom{m_1+1}{2} & |\alpha_1\alpha_2\alpha_3| \\ \binom{m_1+r+1}{2} & |\alpha_1| \\ \binom{m_1+1}{2} & |\alpha_2| \\ \binom{m_1+r}{2} & |\alpha_3| \end{array} \ ; \quad \begin{array}{c|cc|} & \binom{m_1+m_2+1}{2} & \binom{m_2-m_1+r}{2} \\ \binom{m_2+r}{2} & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| \\ \binom{m_2+1}{2} & |\alpha_1| & |0| \\ \binom{m_1+1}{2} & |\alpha_2| & |0| \\ \binom{m_1+1-r}{2} & |\alpha_3| & |0| \end{array} \ ;$$

Case III_c:

$$\begin{array}{c|ccc|} & \binom{m_1+m_2+1}{2} & \binom{m_2+r-m_1}{2} & \binom{r}{2} \\ \binom{m_2+r}{2} & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| \\ \binom{m_2+1}{2} & |\alpha_1| & |0| & |0| \\ \binom{m_1+1}{2} & |\alpha_1| & |0| & |0| \\ \binom{m_1+1-r}{2} & |\alpha_3| & |0| & |0| \end{array}$$

The numbers at the top of each matrix give the number of products π_x, of coefficients β_1, and of coefficients β_2 eliminated i. e. the number of columns of the interior matrices. The numbers at the left indicate the number of equations obtained from the syzygy and from each of the given equations in order. The interior matrices have elements which are linear in the coefficients indicated except for the matrices $|\,0\,|$ whose elements are zero.

We have to show first that these three matrices are square and thus are determinants which we identify later with the resultant. It is easy to verify that

$$
\begin{aligned}
&\binom{2m_1+r+1}{2}+\binom{r}{2}=2\binom{m_1+1}{2}+\binom{m_1+r}{2}+\binom{m_1+r+1}{2}\,,\\
(20)\qquad &\binom{m_1+m_2+1}{2}+\binom{m_2-m_1+r}{2}+\binom{r}{2}=\binom{m_2+r}{2}+\binom{m_2+1}{2}\\
&\qquad\qquad+\binom{m_1+1}{2}+\binom{m_1+1-r}{2}\,.
\end{aligned}
$$

For $\binom{r}{2}=0$ i. e. $r=0,\,1$ these prove at once that the matrices III_a and III_b are square, and the second one proves the same for III_c for any value of r.

To the four cases for $j=4$ listed in the introduction we attach the syzygies:

$$\mathrm{IV}_a:\quad J_{2m_1+r-3}+\beta_{1y}{}^{m_1+r-3}\cdot\alpha_{1y}{}^{m_1}+\cdots+\beta_{3y}{}^{m_1+r-3}\cdot\alpha_{3y}{}^{m_1}+\beta_{4y}{}^{m_1-3}\cdot\alpha_{4y}{}^{m_1+r}\equiv 0;$$

$$\mathrm{IV}_b:\quad J_{2m_1-1}+\beta_{1y}{}^{m_1-1}\cdot\alpha_{1y}{}^{m_1}+\beta_{2y}{}^{m_1-1}\cdot\alpha_{2y}{}^{m_1}+\beta_{3y}{}^{m_1-2}\cdot\alpha_{3y}{}^{m_1+1}$$
$$+\beta_{4y}{}^{m_1-2}\cdot\alpha_{4y}{}^{m_1+1}\equiv$$

$$\mathrm{IV}_c:\quad J_{2m_1+2r-3}+\beta_{1y}{}^{m_1+2r-3}\cdot\alpha_{1y}{}^{m_1}+\beta_{2y}{}^{m_1+r-3}\cdot\alpha_{2y}{}^{m_1+r}+\cdots\beta_{4y}{}^{m_1+r-3}\cdot\alpha_{4y}{}^{m_1+r}\equiv$$

$$\mathrm{IV}_d:\quad J_{2m_1}+\beta_{1y}{}^{m_1}\cdot\alpha_{1y}{}^{m_1}+\beta_{2y}{}^{m_1-1}\cdot\alpha_{2y}{}^{m_1+1}+\beta_{3y}{}^{m_1-1}\cdot\alpha_{3y}{}^{m_1+1}+\beta_{4y}{}^{m_1-2}\cdot\alpha_{4y}{}^{m_1+2}\equiv$$

The matrices of the corresponding systems of linear equations are:

$\mathrm{IV}_a\quad (r=0,\,1,\,2)$:

$$
(21)\qquad
\begin{array}{c|ccccc|}
 & \binom{2m_1+2}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1}{3}\\
\binom{2m_1+r}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4|\\
\binom{m_1+2}{3} & |\alpha_1| & |0| & |0| & |0| & |0|\\
\binom{m_1+2}{3} & |\alpha_2| & |0| & |0| & |0| & |0|\\
\binom{m_1+2}{3} & |\alpha_3| & |0| & |0| & |0| & |0|\\
\binom{m_1-r+2}{3} & |\alpha_4| & |0| & |0| & |0| & |0|
\end{array}\;;
$$

IV_b:

$$
\begin{array}{c|c|cccc|}
 & \binom{2m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+1}{3} & \binom{m_1+1}{3}\\
\binom{2m_1+2}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4|\\
\binom{m_1+2}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+1}{3} & |\alpha_2| & & & & \\
\binom{m_1+1}{3} & |\alpha_4| & & & &
\end{array}\;;
$$

IV_c $(r=1,2)$:

$$\begin{array}{c|c|cccc|}
 & \binom{2m_1+r+2}{3} & \binom{m_1+2r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} & \binom{m_1+r}{3} \\
\binom{2m_1+2r}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+r+2}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+2}{3} & |\alpha_3| & & & & \\
\binom{m_1+2}{3} & |\alpha_4| & & & &
\end{array};$$

IV_d:

$$\begin{array}{c|c|cccc|}
 & \binom{2m_1+3}{3} & \binom{m_1+3}{3} & \binom{m_1+2}{3} & \binom{m_1+2}{3} & \binom{m_1+1}{3} \\
\binom{2m_1+3}{3} & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| \\
\binom{m_1+3}{3} & |\alpha_1| & & & & \\
\binom{m_1+2}{3} & |\alpha_2| & & 0 & & \\
\binom{m_1+2}{3} & |\alpha_3| & & & & \\
\binom{m_1+1}{3} & |\alpha_4| & & & &
\end{array}.$$

That here the number of columns is equal to the number of rows, obviously true in the cases IV_b, IV_d, may easily be verified in the cases IV_a, IV_c if in these two cases $\binom{r}{} = 0$, i. e. if $r = 0, 1, 2$.

The determinant IV_a takes a different form when $m_1 = 2$, a form which varies according as $r = 0, 1, 2$. Thus we have three special sub-cases:

IV_{a1} $(m_1 = 2, r = 0)$

$$(22)\qquad \begin{array}{c|c|}
 & 20 \\
4 & |\alpha_1\alpha_2\alpha_3\alpha_4| \\
4 & |\alpha_1| \\
4 & |\alpha_2| \\
4 & |\alpha_3| \\
4 & |\alpha_4|
\end{array};$$

IV_{a2} $(m_1 = 2, r = 1)$

$$\begin{array}{c|c|ccc|}
 & 20 & 1 & 1 & 1 \\
10 & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\
4 & |\alpha_1| & & & \\
4 & |\alpha_2| & & 0 & \\
4 & |\alpha_3| & & & \\
1 & |\alpha_4| & & &
\end{array};$$

IV_{a3} $(m_1 = 2, r = 2)_3$

$$\begin{array}{c|c|ccc|}
 & 20 & 4 & 4 & 4 \\
20 & |\alpha_1\alpha_2\alpha_3\alpha_4| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\
4 & |\alpha_1| & & & \\
4 & |\alpha_2| & & 0 & \\
4 & |\alpha_3| & & &
\end{array}.$$

The further cases listed have numerical orders. The two for $j = 5$, V_a and V_b, have syzygies,

$$J_5 + \beta_{1y}^2 \cdot \alpha_{1y}^3 + \cdots + \beta_{5y}^2 \cdot \alpha_{5y}^3 \equiv 0,$$
$$J_{2+r} + \beta_{1y}^r \cdot \alpha_{1y}^2 + \cdots + \beta_{4y}^r \cdot \alpha_{4y}^2 + \beta_5 \cdot \alpha_{5y}^{2+r} \equiv 0 \quad (r = 0, 1),$$

and corresponding determinants

V_a:

$$(23) \qquad \begin{array}{c|c|ccccc|} & 126 & 15 & 15 & 15 & 15 & 15 \\ \hline 126 & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| & |\alpha_5| \\ 15 & |\alpha_1| & & & & & \\ 15 & |\alpha_2| & & & & & \\ 15 & |\alpha_3| & & & 0 & & \\ 15 & |\alpha_4| & & & & & \\ 15 & |\alpha_5| & & & & & \end{array} ;$$

$V_b \ (r = 0, 1)$:

$$\begin{array}{c|c|ccccc|} & \binom{7}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{r+4}{4} & \binom{4}{4} \\ \hline \binom{r+6}{4} & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & |\alpha_3| & |\alpha_4| & |\alpha_5| \\ \binom{5}{4} & |\alpha_1| & & & & & \\ \binom{5}{4} & |\alpha_2| & & & & & \\ \binom{5}{4} & |\alpha_3| & & & 0 & & \\ \binom{5}{4} & |\alpha_4| & & & & & \\ \binom{5-r}{4} & |\alpha_5| & & & & & \end{array} .$$

In V_b for both cases $r = 0, 1$ the matrix is square.

The isolated case 1, 2, 2, 2, 2 for $j = 5$ yields a case V_c for which the syzygy is

$$J_2 + \beta_{1y} \cdot \alpha_{1y} + \beta_2 \cdot \alpha_{2y}^2 + \cdots + \beta_5 \alpha_{5y}^2 \equiv 0,$$

and the determinant is

$$(24) \qquad \begin{array}{c|c|cccc|} & 15 & 5 & 1 & \cdots & 1 \\ \hline 15 & |\alpha_1 \cdots \alpha_5| & |\alpha_1| & |\alpha_2| & & |\alpha_5| \\ 5 & |\alpha_1| & & & & \\ 1 & |\alpha_2| & & 0 & & \\ \vdots & & & & & \\ 1 & |\alpha_5| & & & & \end{array} .$$

Finally for $j = 6$ we have the one case VI for which the syzygy is

$$J_3 + \beta_{1y} \cdot \alpha_{1y}^2 + \cdots + \beta_{6y} \cdot \alpha_{6y}^2 \equiv 0,$$

and the determinant is

VI:

$$(25)\qquad \begin{array}{c|c|cccc||}
 & 56 & 6 & 6 & \cdots & 6 \\
56 & |\alpha_1\cdots\alpha_6| & |\alpha_1| & |\alpha_2| & \cdots & |\alpha_6| \\
6 & |\alpha_1| & & & & \\
6 & |\alpha_2| & & 0 & & \\
\vdots & \vdots & & & & \\
6 & |\alpha_6| & & & &
\end{array} .$$

In the next section we prove that the determinants (19), (21), (22), (23), (24), and (25) are the resultants of their respective forms.

3. *Identification of the Resultant.* If the j given equations have a common solution the determinants of the preceding section, which obviously have an invariantive character, must vanish since the linear equations from which they were derived are then consistent. If the determinants are not identically zero, they are invariants of the given forms which vanish when the resultant R vanishes and which therefore contain R as a factor. If the degrees of the determinants in the coefficients of the forms coincide with the degrees of R then the determinants coincide with R. We prove first that the determinants have the same degree as R and second that the determinants have, in a particular case, the value ± 1 and therefore do not vanish identically. The argument on each of these two points is given in just a few illustrative examples and the verifications made in the remaining cases are not reproduced here.

All of the determinants given have the typical form

$$(26)\qquad \begin{array}{c|c|cccc||}
 & p_0 & p_1 & p_2 & \cdots & p_j \\
q_0 & |\alpha_1\cdots\alpha_j| & |\alpha_1| & |\alpha_2| & \cdots & |\alpha_j| \\
q_1 & |\alpha_1| & & & & \\
q_2 & |\alpha_2| & & 0 & & \\
\vdots & \vdots & & & & \\
q_j & |\alpha_j| & & & &
\end{array}$$

where the p's and q's represent respectively the number of columns and the number of rows in the interior matrices under or opposite them. Then $\sum_{i=0}^{j} p_i = \sum_{i=0}^{j} q_j = N$. The upper left matrix has elements linear in the coefficients of each of the forms, the other non-zero matrices have for elements precisely the coefficients of the one form indicated. For particular determinants some of these latter matrices do not appear and the corresponding numbers p or q are zero.

In the terms of the expansion of such a determinant there are

$$q_0 - \sum_{i=1}^{j} p_i = p_0 - \sum_{i=1}^{j} q_j = p_0 + q_0 - N$$

elements from the upper left matrix, p_i elements from the matrix $|\alpha_i|$ to the right, and q_i elements from the matrix $|\alpha_i|$ underneath. Hence the degree of the determinant in the coefficients of $\alpha_{i y}{}^{m_i}$ is

$$p_0 + q_0 + p_i + q_i - N =$$

$$p_0 - q_1 - q_2 \cdots - q_{i-1} + p_i - q_{i+1} \cdots - q_j$$
$$= q_0 - p_1 - p_2 \cdots - p_{i-1} + q_i - p_{i+1} \cdots - p_j.$$

We use the last expression for the degree in α_i since some of the p's are frequently zero.

Thus the determinant of Case III$_c$ in (19) has the following degrees in α_1, α_2, α_3 respectively:

$$\begin{aligned}
\binom{m_2+r}{2} + \binom{m_2+1}{2} - \binom{r}{2} &= m_2 \cdot (m_2 + r), \\
\binom{m_2+r}{2} + \binom{m_1+1}{2} - \binom{m_2+r-m_1}{2} &= m_1 \cdot (m_1 + r), \\
\binom{m_2+r}{2} + \binom{m_1+1-r}{2} - \binom{m_2+r-m_1}{2} - \binom{r}{2} &= m_1 \cdot m_2,
\end{aligned}$$

which are the degrees of the resultant.

Similarly in Case IV$_c$ of (21) we find for the degree in α_1 and the equal degrees in α_2, α_3, α_4 that

$$\begin{aligned}
\binom{2m_1+2r}{3} + \binom{m_1+r+2}{3} - 3\binom{m_1+r}{3} &= (m_1 + r) \cdot (m_1 + r) \cdot (m_1 + r), \\
\binom{2m_1+2r}{3} + \binom{m_1+2}{3} - \binom{m_1+2r}{3} - 2\binom{m_1+r}{3} &= m_1 \cdot (m_1 + r) \cdot (m_1 + r) - 12\binom{r}{3},
\end{aligned}$$

where $\binom{r}{3}$ vanishes for the values of r (1, 2) in this case.

The similar verification which has been made for the other cases is not reproduced here.

We prove that the determinants do not vanish identically by showing that they take the value ± 1 for the particular case when the forms are perfect powers such as

$$y_1^{m_1},\ y_2^{m_2},\ y_3^{m_3},\ \cdots,\ y_j^{m_j}.$$

We use the obvious fact that a determinant whose elements are all zero except for a single element 1 in each row and column has the value ± 1.

Again the verification which has been carried out for all cases, is given only for the cases, III$_c$ in (19) and IV$_c$ in (21), examined above.

For Case III$_c$ the syzygy now reads

$$J_{m_2-2+r} + \beta_{1y}{}^{m_2-m_1-2+r} \cdot y_1{}^{m_1} + \beta_{2y}{}^{r-2} \cdot y_2{}^{m_2} \equiv 0 \quad (r = 2, \cdots, m_1 + 1).$$

In this the forms β are arbitrary forms of their order and the form J_{m_2-2+r} has the simple expression

$$\sum_{k_1,k_2,k_3} y_1{}^{k_1} y_2{}^{k_2} y_3{}^{k_3} x_1{}^{m_1-k_1-1}\, x_2{}^{m_2-k_2-1} x_3{}^{m_2+r-k_3-1}$$

with the following restrictions on the exponents:

$$k_1 + k_2 + k_3 = m_2 - 2 + r,$$

$$\text{(a)} \qquad 0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_2, \quad 0 \leqq k_3 < m_2 + r.$$

The product terms of the syzygy contribute terms in $y_1{}^{k_1}y_2{}^{k_2}y_3{}^{k_3}$ for which in the first product $k_1 \geqq m_1$, and in the second $k_2 \geqq m_2$. When we equate to zero the terms of the syzygy in $y_1{}^{k_1}y_2{}^{k_2}y_3{}^{k_3}$, then if $k_1 \geqq m_1$ the term according to (a) does not appear in J. It must appear in the product $\beta_{1y}{}^{m_2-m_1-2+r} \cdot y_1{}^{m_1}$ with a coefficient β_1 which is eliminated leaving an element 1 for the determinant. It also cannot occur in the second product since if $k_1 \geqq m_1$ and $k_2 \geqq m_2$ then $k_1 + k_2 \geqq m_1 + m_2$ which contradicts $k_1 + k_2 + k_3 = m_2 - 2 + r$ $(r = 2, \cdots, m_1 + 1)$. Similarly a term for which $k_2 \geqq m_2$ furnishes an equation which contributes a single element $+1$ to the determinant. If finally $k_1 < m_1$ and $k_2 < m_2$ the term can appear only in J with a single coefficient π_x which is eliminated leaving a single element 1 for the determinant. Thus every row of the determinant which arises from the syzygy has a single element 1. Obviously the rows which arise from equations obtained by multiplying $x_1{}^{m_1}$, $x_2{}^{m_2}$, $\cdots$ by individual products π_x have only one element 1 since the x's are eliminated.

It is similarly obvious that the columns of the determinant which arise from the elimination of β_1 and β_2 have a single element 1. The column which arises from the elimination of

$$x_1{}^{k_1}x_2{}^{k_2}x_3{}^{k_3} \qquad (k_1 + k_2 + k_3 = m_1 + m_2 - 1)$$

will if simultaneously

$$0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_2, \quad 0 \leqq k_3 < m_2 + r - 1$$

have one element 1 from an equation of the syzygy. If however any one of the three relations,

$$k_1 \geqq m_1, \quad k_2 \geqq m_2, \quad k_3 \geqq m_2 + r,$$

is valid, the column will have one and only one element 1 from one of the three further sets of equations since no two of the above inequalities can hold simultaneously. Thus the determinant has but one non-zero element in each row and column and is not zero.

In Case IV$_c$ for forms $y_1^{m_1}$, $y_2^{m_1+r}$, $y_3^{m_1+r}$, $y_4^{m_1+r}$ $(r=1,\,2)$ the syzygy reads:

$$J_{2m_1+2r-3} + \beta_{1y}^{m_1+2r-3}\cdot y_1^{m_1} + \beta_{2y}^{m_1+r-3}\cdot y_2^{m_2} + \cdots + \beta_{y4}^{m_1+r-3}\cdot y_4^{m_1+r} \equiv 0$$

where

$$J_{2m_1+2r-3} = \sum_{k_1,\ldots,k_4} y_1^{k_1} y_2^{k_2} y_3^{k_3} y_4^{k_4} x_1^{m_1-k_1-1}\, x_2^{m_1+r-k_2-1} \cdots x_4^{m_1+r-k_4-1}$$

and

$$k_1+k_2+k_3+k_4 = 2m_1+2r-3$$

$$0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_1+r, \quad \cdots, \quad 0 \leqq k_4 < m_1+r$$

As in the preceding case these restrictions are inconsistent with

$$k_1 \geqq m_1, \quad k_2 \geqq m_1+r, \quad k_3 \geqq m_1+r, \quad k_4 \geqq m_1+r$$

and any two of the latter are inconsistent with each other. Hence each equation which arises from the syzygy contributes to a row of the determinant a single element 1 and the same is true of the additional equations.

The columns which arise from the elimination of

$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4} \qquad (k_1+k_2+k_3+k_4 = m_1+m_2-1)$$

will if simultaneously

$$0 \leqq k_1 < m_1, \quad 0 \leqq k_2 < m_1+r, \quad 0 \leqq k_3 < m_1+r, \quad 0 \leqq k_4 < m_1 +$$

have just one element 1, an element which arises from J. If however any one of the four mutually exclusive relations,

$$k_1 \geqq m_1, \quad k_2 \geqq m_1+r, \quad k_3 \geqq m_1+r, \quad k_4 \geqq m_1+r,$$

is valid, the column again has just one element 1, an element which arises from one of the additional equations. Again the determinant is ± 1 and is not identically zero. This completes the identification of the determinants given in § 2 with the resultants of the corresponding set of forms.

4. *An Identity Connecting $j+1$ General Forms in j Variables.* In scanning the proof of the formula (12) of § 2 the only point at which the assumption, that α_y^m, β_y^n, γ_y^p, δ_y^q had a common solution x, was used, was in the derivation of (6) where terms in $\alpha_x^m, \cdots$ were dropped. If we retain these on the hypothesis that the four forms are general and proceed with the remaining terms as in the derivation of (12) we obtain the formula

$$
\begin{aligned}
(27)\qquad \xi_x \cdot J_k \equiv {} & \xi_y \cdot J_{k-1} \\
& + (\alpha\beta\gamma\xi)\left[\binom{m-1,\ n-1,\ p-1,\ 0}{k} \cdot \delta_x^q - \binom{m-1,\ n-1,\ p-1,\ 0}{k-q} \cdot \delta_y^q\right] \\
& - (\alpha\beta\delta\xi)\left[\binom{m-1,\ n-1,\ 0,\ q-1}{k} \cdot \gamma_x^p - \binom{m-1,\ n-1,\ 0,\ q-1}{k-p} \cdot \gamma_y^q\right] \\
& + (\alpha\gamma\delta\xi)\left[\binom{m-1,\ 0,\ p-1,\ q-1}{k} \cdot \beta_x^n - \binom{m-1,\ 0,\ p-1,\ q-1}{k-n} \cdot \beta_y^n\right] \\
& - (\beta\gamma\delta\xi)\left[\binom{0,\ n-1,\ p-1,\ q-1}{k} \cdot \alpha_x^m - \binom{0,\ n-1,\ p-1,\ q-1}{k-m} \cdot \alpha_y^m\right].
\end{aligned}
$$

If we multiply this by ξ_x and replace on the right $\xi_x \cdot J_{k-1}$ in terms of $\xi_y \cdot J_{k-2}$ by using (27) for $k-1$, and if we continue this process until we have on the left $\xi_x^r \cdot J_k$ then in the notation of (14) we shall have

$$
\begin{aligned}
(28)\qquad \xi_x^r \cdot J_k(\alpha^m, \beta^n, \gamma^p, \delta^q) \equiv {} & \xi_y^r \cdot J_{k-r}(\alpha^m, \beta^n, \gamma^p, \delta^q) \\
& + J_k(\alpha^m, \beta^n, \gamma^p, \xi^r) \cdot \delta_x^q - J_{k-q}(\alpha^m, \beta^n, \gamma^p, \xi^r) \cdot \delta_y^q \\
& - J_k(\alpha^m, \beta^n, \delta^q, \xi^r) \cdot \gamma_x^p + J_{k-p}(\alpha^m, \beta^n, \delta^q, \xi^r) \cdot \gamma_y^p \\
& + J_k(\alpha^m, \gamma^p, \delta^q, \xi^r) \cdot \beta_x^n - J_{k-n}(\alpha^m, \gamma^p, \delta^q, \xi^r) \cdot \beta_y^n \\
& - J_k(\beta^n, \gamma^p, \delta^q, \xi^r) \cdot \alpha_x^m + J_{k-m}(\beta^n, \gamma^p, \delta^q, \xi^r) \cdot \alpha_y^m.
\end{aligned}
$$

If in this we replace ξ_x^r by an arbitrary form ϵ_x^r we have the following noteworthy formula, an identity in both x and y which is satisfied by any five quaternary forms:

$$
\begin{aligned}
(29)\qquad & J_k(\alpha^m, \beta^n, \gamma^p, \delta^q) \cdot \epsilon_x^r - J_k(\alpha^m, \beta^n, \gamma^p, \epsilon^r) \cdot \delta_x^q \\
& + J_k(\alpha^m, \beta^n, \delta^q, \epsilon^r) \cdot \gamma_x^p - J_k(\alpha^m, \gamma^p, \delta^q, \epsilon^r) \cdot \beta_x^n + J_k(\beta^n, \gamma^p, \delta^q, \epsilon^r) \cdot \alpha_x^m \equiv \\
& J_{k-r}(\alpha^m, \beta^n, \gamma^p, \delta^q) \cdot \epsilon_y^r - J_{k-q}(\alpha^m, \beta^n, \gamma^p, \epsilon^r) \cdot \delta_y^q \\
& + J_{k-p}(\alpha^m, \beta^n, \delta^q, \epsilon^r) \cdot \gamma_y^p - J_{k-n}(\alpha^m, \gamma^p, \delta^q, \epsilon^r) \cdot \beta_y^n + J_{k-m}(\beta^n \gamma^p, \delta^q, \epsilon^r) \cdot \alpha_y^m.
\end{aligned}
$$

For $k = 0$, $m = n = p = q = r = 1$ this reduces to the usual determinant identity. For $k = 0$ alone, it is a long known dual relation which connects five forms and their jacobians.

For $j + 1$ forms in j variables,

$$\alpha_{1y}^{m_1},\ \alpha_{2y}^{m_2},\ \cdots,\ \alpha_{j+1,y}^{m_{j+1}},$$

the identity is

$$
\begin{aligned}
(30)\qquad \sum_{s=1}^{j+1} (-1)^{s+1} \big[& J_k(\alpha_1^{m_1}, \cdots, \alpha_{s-1}^{m_{s-1}}, \alpha_{s+1}^{m_{s+1}}, \cdots, \alpha_{j+1}^{m_{j+1}}) \cdot \alpha_{sx}^{m_s} \\
& - J_{k-m_s}(\alpha_1^{m_1}, \cdots, \alpha_{s-1}^{m_{s-1}}, \alpha_{s+1}^{m_{s+1}}, \cdots, \alpha_{j+1}^{m_{j+1}}) \cdot \alpha_{sy}^{m_s}\big] \equiv 0.
\end{aligned}
$$

5. *Alternative Determinantal Expressions for R. The Common Solution when $R = 0$; the unique Apolar Form when $R \neq 0$.* That the resultant R may be expressed as a determinant in more than one way, at least in certain

of the above cases, was pointed out by Morley who gave two determinants for the case $m_1 = m_2 = m_3$. We examine this situation more closely. The resultants are obtained from a syzygy containing J_k of order k in y. To bring out the order l in x of J_k we set

$$J_k \equiv J_{k,l}, \qquad k + l = \sum m_i - j. \tag{31}$$

These orders k, l in x, y respectively were subject to the limitations

$$k < m_1 + m_2, \qquad l < m_1 + m_2 \tag{32}$$

in order to secure, in the first case, the uniqueness of the coefficients β of the syzygy and, in the second case, the linear independence of the additional equations. The elimination of k and l from (31), (32) leads to the inequality (18) of § 2 which was used to separate the cases III, IV, V, VI. Throughout the discussion in § 2 we had used uniformly the maximum value $m_1 + m_2 - 1$ of l.

We denote by

$$d = | l - k | \tag{33}$$

the *disparity of the syzygy* which contains $J_{k,l}$. As l decreases, k increases so that the maximum value D of d for any one of the cases occurs when k or l is $m_1 + m_2 - 1$. Thus D, *the disparity of the case,* has the value

$$D = m_1 + m_2 - m_3 - m_4 - \cdots - m_j + j - 2. \tag{34}$$

For the case III with equal orders, and for the cases III_a, III_b, III_c, IV_a, IV_b, IV_c, IV_d, V_a, V_b, VI the values of D are respectively

$$\begin{aligned} &m_1 + 1, \quad m_1 + r \ (r = 0, 1), \quad m_1 - r + 1 \ (r = 0, 1), \\ &m_1 - r + 1 \ (r = 2, \cdots, m_1 + 1), \quad 2 - r \ (r = 0, 1, 2), \quad 0, \\ &2 - r \ (r = 1, 2), \quad 0, \quad 0, \quad 1 - r \ (r = 0, 1), \quad 0. \end{aligned} \tag{35}$$

We have used in each case of § 2 that syzygy $J_{k,l}$ for which d had the maximum value D for the case. There is however in the early cases at least a considerable range for d from D to 0 or 1 according as D is even or odd. Thus for each value of d $\left({}^0_1 \leqq d \leqq D\right)$ there are two syzygies $J_{k,l}$ and $J_{l,k}$ each of which leads to a resultant.

If we examine the two syzygies with leading terms $J_{k,l}$ and $J_{l,k}$ we find that they give rise to resultants whose determinants are the same except for the interchange of rows and columns. Indeed we pass from $J_{k,l}$ to $J_{l,k}$ by interchanging x and y so that the matrix $| \alpha_1, \cdots, \alpha_j |$ of the determinant is transposed. And if we eliminate the coefficients of $\beta_i y^{k-m_i}$ for the one syzygy in the same order as we introduce additional equations to the other syzygy by multiplying $\alpha_{ix}{}^{m_i}=0$ by products $(x)^{k-m_i}$ the rest of the determinants are likewise transposed. We may state then the theorem:

(36) *In each case of § 2 of disparity D there are* $\left\{\begin{array}{l}(D/2+1 \ (D \text{ even})\\ (D+1)/2 \ (D \text{ odd})\end{array}\right.$ *pairs of syzygies, $J_{k,l}$ and $J_{l,k}$, of disparity d $(0 \lesssim d \lesssim D)$. Each syzygy provides a determinant form of the resultant but the determinants arising from a pair are merely transposed. The orders of these* $\left\{\begin{array}{l}(D+1)/2\\ (D+1)/2\end{array}\right.$ *distinct determinants decrease with d, and for $d=0$ when D is even the determinant is symmetric.*

From the values of D given in (35) we note that these symmetric cases occur frequently.

Two illustrative examples may be worth while. For three ternary quartics, $m_1=m_2=m_3=4$, the jacobian is of order 9 and the syzygies available for resultants are $J_{2,7}$, $J_{3,6}$, $J_{4,5}$, $J_{5,4}$, $J_{6,3}$, and $J_{7,2}$. The first three furnish the following forms of R:

$$\begin{array}{r|c|} & 36 \\ 6 & |\alpha_1\alpha_2\alpha_3| \\ 10 & |\alpha_1| \\ 10 & |\alpha_2| \\ 10 & |\alpha_3| \end{array} \; ; \quad \begin{array}{r|c|} & 28 \\ 10 & |\alpha_1\alpha_2\alpha_3| \\ 6 & |\alpha_1| \\ 6 & |\alpha_2| \\ 6 & |\alpha_3| \end{array} \; ; \quad \begin{array}{r|c|ccc|} & 21 & 1 & 1 & 1 \\ 15 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 3 & |\alpha_1| & & & \\ 3 & |\alpha_2| & & 0 & \\ 3 & |\alpha_3| & & & \end{array} \; ;$$

determinants of orders 36, 28, 24 respectively whose transposed determinants arise from the last three syzygies in reverse order.

For three ternary quintics, $m_1=m_2=m_3=5$, and jacobian of order 12, the available syzygies are $J_{3,9}$, $J_{4,8}$, $J_{5,7}$, $J_{6,6}$, $J_{7,5}$, $J_{8,4}$, $J_{9,3}$. Four distinct forms of the resultant appear as determinants of order 55, 45, 39, and 37 of which the last is symmetric. They are

$$\begin{array}{r|c|} & 55 \\ 10 & |\alpha_1\alpha_2\alpha_3| \\ 15 & |\alpha_1| \\ 15 & |\alpha_2| \\ 15 & |\alpha_3| \end{array} \; ; \quad \begin{array}{r|c|} & 45 \\ 15 & |\alpha_1\alpha_2\alpha_3| \\ 10 & |\alpha_1| \\ 10 & |\alpha_2| \\ 10 & |\alpha_3| \end{array} \; ; \quad \begin{array}{r|c|ccc|} & 36 & 1 & 1 & 1 \\ 21 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 6 & |\alpha_1| & & & \\ 6 & |\alpha_2| & & 0 & \\ 6 & |\alpha_3| & & & \end{array} \; ;$$

$$\begin{array}{r|c|ccc|} & 28 & 3 & 3 & 3 \\ 28 & |\alpha_1\alpha_2\alpha_3| & |\alpha_1| & |\alpha_2| & |\alpha_3| \\ 3 & |\alpha_1| & & & \\ 3 & |\alpha_2| & & 0 & \\ 3 & |\alpha_3| & & & \end{array} \; .$$

Let $R_{k,l}$ be that determinant form of R which arises from the syzygy

$$J_{k,l} + \beta_{1y}{}^{k-m_1} \cdot \alpha_{1y}{}^{m_1} + \cdot \cdot \cdot + \beta_{jy}{}^{k-m_j} \cdot \alpha_{jy}{}^{m_j} \equiv 0$$

which exists when $R = 0$. If to this syzygy we add the additional term $c \cdot \eta_y{}^k$ (η_y a linear form), then when $R = 0$ the new syzygy still is valid for the value $c = 0$ and arbitrary η. If to the equations obtained from this new syzygy and the original set of additional equations we add the equation, $\xi_x{}^l = 0$, then the new system of linear equations has a determinant $\Delta(\eta^k, \xi^l)$ which is formed from the determinant R by bordering it with ξ^l under, and η^k opposite, the matrix $| \alpha_1, \cdot \cdot \cdot, \alpha_j |$ arising from $J_{k,l}$. When $R = 0$ $\Delta(\eta^k, \xi^l)$ is a form of orders k, l in η, ξ which, for any η, vanishes when and only when $\xi_x = 0$. Hence $\Delta(\eta^k, \xi^l) = \xi_x{}^l \cdot \eta_s{}^k$ where $\eta_s{}^k$ is some form to be determined.

If we start afresh with the syzygy containing $R_{l,k}$ and proceed in the same way by adding a term $c \cdot \xi_y{}^l$ and an additional equation, $\eta_x{}^k = 0$, the eliminant of the system of linear equations is $\Delta'(\xi^l, \eta^k)$ which factors as before into $\eta_x{}^k \cdot \xi_s{}^l$. But the bordered determinant Δ' is merely the bordered determinant with rows D columns interchanged. Hence $\Delta = \Delta'$ and $\xi_x{}^l \cdot \eta_s{}^k = \eta_x{}^k \cdot \xi_{s'}{}^l$, whence $\eta_s{}^k = \eta_x{}^k$ and $\xi_{s'}{}^l = \xi_x{}^l$. Setting $\eta = \xi$ we have $\Delta(\xi^k, \xi^l) = \xi_x{}^{k+l}$. Hence

(37) *If $R_{k,l}$ is one of the determinantal forms of the resultant R and if $R_{k,l}$ be bordered to the right of, and under, the matrix $| \alpha_1, \cdot \cdot \cdot, \alpha_j |$ with respectively variables ξ^k and ξ^l, then the resulting bordered determinant $\Delta(\xi^k, \xi^l)$ is when $R = 0$ the $(k + l)$-th power of ξ_x where x is the common solution.*

We have then a rational integral expression for the common solution x when it exists.

Morley has pointed out that for any j given forms, $\alpha_{1y}{}^{m_i}$, there is a unique apolar form of class $k + l = \sum m_i - j$, say the form, $a_\xi{}^{k+l}$ i. e. a form for which the j polar forms, $(\alpha_i a)^{m_i} a_\xi{}^{k+l-m_i}$, vanish identically. Evidently, if $R = 0$, and x is the common solution then $\xi_x{}^{k+l}$ is the common apolar form. Since thus $\Delta(\xi^k, \xi^l)$ is the common apolar form when $R = 0$, and is of lower degree than R in the coefficients of each of the forms $\alpha_{iy}{}^{m_i}$ then $\Delta(\xi^k, \xi^l)$ must be the unique apolar form when $R \neq 0$. Hence

The unique form of order $\sum m_i - j$ apolar to one of the sets of j forms listed in the introduction is given by any one of the bordered resultants $\Delta(\xi^k, \xi^l)$ which pertain to the given set.

6. *A New Version.* We shall obtain some of the above results in another way, in line with Morley's memoir. We consider forms of the same degree. When we wish to pass to those of different degrees we have merely to include symbolic factors. Take $d+2$ numbers $a, b, c, \cdots$ and associate with each a member $\alpha, \beta, \gamma, \cdots$. We are concerned with the sum of homogeneous products or complete symmetric function of degree k,

$$\sum a^k + \sum a^{k-1}b + \sum a^{k-2}b^2 + \cdots$$

which (*loc. cit.*) was called H^k (as in Salmon, *Higher Algebra*, note, p. 290 of the third edition, 1876). We will write it more explicitly as $H_k(a, b, c, \cdots)$; when $a = 0$, it becomes $H_k(b, c\cdots)$.

Then we have the formula, for any positive integer k

$$(1) \qquad \sum \alpha H_k(b, c, \cdots) = 0$$

provided

$$(2) \qquad \sum \alpha = 0 \text{ and } \sum \alpha a = 0.$$

We define H_0 to be 1, and H_{-k} to be 0, so that (1) is true for all integers.

We see that $H_k(a, b, c, \cdots) - a^m H_{k-m}(a, b, c, \cdots)$ contains a only to the power a^{m-1}. Hence $H_k(abc\cdots) - \sum a^m H_{k-m}(abc\cdots)$ loses all powers as high as the mth, provided $2m > k$. Write this truncated form

$$H_k{}^m(a, b, c, \cdots)$$

Write also σ_m for

$$a^m + b^m + c^m + \cdots.$$

Then $\quad H_k{}^m(b, c, \cdots) - \sum(\sigma_m - a^m)\, H_{k-m}(b, c, \cdots) = H_k{}^m(b, c, \cdots)$
whence, so long as $2m > k$,

$$(3) \qquad \sum \alpha H_k{}^m(b, c, \cdots) = \sum \alpha a^m H_{k-m}(b, c, \cdots).$$

We now write $\qquad (\alpha x)\,|\,\beta\gamma\cdots\,| \quad \text{for } \alpha, \cdots$

and $\qquad (\alpha y)/(\alpha x) \qquad \text{for } a, \cdots;$

these obeying the relations (2).

Take the case of 5 cubic surfaces; these are denoted by $(\alpha x)^3 \cdots (\epsilon x)^3$.

There are 5 jacobians, for instance

$$J(\beta\gamma\delta\epsilon) = |\,\beta\gamma\delta\epsilon\,|\, (\beta x)^2 (\gamma x)^2 (\delta x)^2 (\epsilon x)^2.$$

$$H_1(bcde) \text{ is } (\beta y)/(\beta x) + (\gamma y)/(\gamma x) + (\delta y)/(\delta x) + (\epsilon y)/(\epsilon x)$$

so that $$J\,H_1 = J_1.$$
Similarly $$J\,H_2 = J_2.$$
But $$J\,H_3{}^3 = J_3.$$

For in forming J_3 we are to take all jacobians of degree 3 in y formed from

$$\begin{array}{cccc} (\beta x)^3, & (\gamma x)^3, & (\delta x)^3, & (\epsilon x)^3 \\ (\beta x)^2(\beta y), & (\gamma x)^2(\gamma y), & \cdots & \\ (\beta x)(\beta y)^2, & (\gamma x)(\gamma y)^2, & \cdots & \\ (\beta y)^3, & (\gamma y)^3, & \cdots & \end{array}$$

and the last row contributes no jacobian. So

$$\begin{aligned} J\,H_4{}^3 &= J_4 \\ J\,H_5{}^3 &= J_5 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ J\,H_8{}^3 &= J_8. \end{aligned}$$

Thus J_n is a form in x and y of degree x in y, $8-n$ in x, which is a sum of homogeneous products, but is truncated when $n > 2$. We have then from (1)

$$\sum (\alpha x)\,|\,\beta\gamma\delta\epsilon\,|\cdot H_k\left(\frac{(\beta y)}{(\beta x)},\ \frac{(\gamma y)}{(\gamma x)},\ \frac{(\delta y)}{(\delta x)},\ \frac{(\epsilon y)}{(\epsilon x)}\right) = 0$$

or multiplying by $(\alpha x)^2(\beta x)^2(\gamma x)^2(\delta x)^2(\epsilon x)^2$

$$\begin{aligned} &\sum (\alpha x)^3\,J(\beta\gamma\delta\epsilon) = 0 &&\text{when } k = 0 \\ &\sum (\alpha x)^3 J_1(\beta\gamma\delta\epsilon) = 0 &&\text{when } k = 1 \\ &\sum (\alpha x)^3 J_2(\beta\gamma\delta\epsilon) = 0 &&\text{when } k = 2 \end{aligned}$$

and we have from (3)

$$\begin{aligned} &\sum (\alpha x)^3 J_3(\beta\gamma\delta\epsilon) = \sum (\alpha y)^3\,J(\beta\gamma\delta\epsilon) &&\text{when } k = 3 \\ &\sum (\alpha x)^3 J_4(\beta\gamma\delta\epsilon) = \sum (\alpha y)^3 J_1(\beta\gamma\delta\epsilon) &&\text{when } k = 4 \\ &\sum (\alpha x)^3 J_5(\beta\gamma\delta\epsilon) = \sum (\alpha y)^3 J_2(\beta\gamma\delta\epsilon) &&\text{when } k = 5 \end{aligned}$$

and 3 other formulae, which are these with x, y interchanged. Generally then,

$$\sum (\alpha x)^3 J_n(\beta\gamma\delta\epsilon) = \sum (\alpha y)^3 J_{n-3}(\beta\gamma\delta\epsilon)$$

if $J_0 = J$, and $J_{-1} = 0$, $J_{-2} = 0 \cdot \cdot \cdot$.

The complete list of linear syzygies so obtained may be clearer in another notation. Denote the 5 cubics by

$$Ax^3 \cdot \cdot \cdot Ex^3$$

the jacobians by

$$\begin{array}{lll} & & BCDEx^8 \cdots \\ J_1(\beta\gamma\delta\epsilon) & \text{by} & BCDEx^7y \\ J_2(\beta\gamma\delta\epsilon) & \text{by} & BCDEx^6y^2 \end{array}$$

and so on. Then the 12 syzygies are

$$\begin{array}{lcl} \sum Ax^3 \cdot BCDEx^8 & = & 0 \\ \sum Ax^3 \cdot BCDEx^7y & = & 0 \\ \sum Ax^3 \cdot BCDEx^6y^2 & = & 0 \\ \sum Ax^3 \cdot BCDEx^5y^3 & = & \sum Ay^3 \cdot BCDEx^8 \\ \sum Ax^3 \cdot BCDEx^4y^4 & = & \sum Ay^3 \cdot BCDEx^7y \\ \sum Ax^3 \cdot BCDEx^3y^5 & = & \sum Ay^3 \cdot BCDEx^6y^2 \\ \sum Ax^3 \cdot BCDEx^2y^6 & = & \sum Ay^3 \cdot BCDEx^5y^3 \\ \sum Ax^3 \cdot BCDExy^7 & = & \sum Ay^3 \cdot BCDEx^4y^4 \\ \sum Ax^3 \cdot BCDEy^8 & = & \sum Ay^3 \cdot BCDEx^3y^5 \\ 0 & = & \sum Ay^3 \cdot BCDEx^2y^6 \\ 0 & = & \sum Ay^3 \cdot BCDExy^7 \\ 0 & = & \sum Ay^3 \cdot BCDEy^8. \end{array}$$

Since x and y may be interchanged these amount to six only. But for the application to elimination it is important to have the full scheme.

In this application we suppose x to be a common point of By^3, Cy^3, Dy^3, Ey^3, but not on Ay^3. That is we set

$$Bx^3 = Cx^3 = Dx^3 = Ex^3 = 0.$$

We have then

$$(1) \qquad BCDEx^8 = 0$$

$$(2) \qquad BCDEx^7y = 0$$

$$(3) \qquad BCDEx^6y^2 = 0$$

$$(4) \qquad Ax^3 \cdot BCDEx^5y^3 = \sum^4 By^3 \cdot CDEAx^8$$

$$(5) \qquad Ax^3 \cdot BCDEx^4y^4 = \sum^4 By^3 \cdot CDEAx^7y$$

$$(6) \qquad Ax^3 \cdot BCDEx^3y^5 = \sum^4 By^3 \cdot CDEAx^6y_2.$$

We note that hereby $J, J_1, \cdots, J_5$ are contained in the modulus defined by the 4 curves when these curves have a common point. We call the J_i the adjoint linear forms or simply the *adjoints* of the given forms.

To effect the elimination, we may take (4), (5) or (6). With (4), we have the $\binom{8}{3}$ or 56 unknowns $x_1^5, x_1^4x_2, \cdots, x_4^5$, and the 4 unknowns such as $(CDEAx^8)/Ax^3$.

We have since y is arbitrary $\binom{6}{3}$ or 20 equations, and also the 4×10 equations got by raising Bx^3, Cx^3, Dx^3, Ex^3 to quintics. That is we have 60 equations in 60 unknowns.

With (5) we have the 35 unknowns $x_1^4, x_1^3x_2, \cdots, x_4^4$ and the 4×4 coefficients of y in $(CDEAx^7y)/Ax^3$. And we have 35 equations from (5) and 4×4 by raising the cubics to quartics. That is we have 51 equations in 51 unknowns.

With (6) we have $56 + 4$ equations in $20 + 4 \times 10$ unknowns, namely $x_1^3, \cdots, x_4^3$ and the 10 coefficients of y^2 in each of $(CDEAx^6y^2)/Ax^3$.

Numerical Functions of Multipartite Integers and Compound Partitions.

BY E. T. BELL.

The multipartite integers and certain of their functions introduced here have partial arithmetics abstractly identical with that section of rational arithmetic (or of the theory of ideals) which is independent of order relations. They have immediate interpretations in terms of the algebra of compound partitions and their functions, here defined apparently for the first time, and they give rise to an extensive new class of properties of such partitions, which in turn can be replaced by sets of restricted diophantine equations. In particular, as a very special instance of the general theorem concerning a new type of arithmetical invariants in § 7, the simpler algebraic aspects of compound partitions are related through these hyper-complex integers to the algebra of the elliptic and theta functions in a manner abstractly identical with the like for a single rational integer; they also exhibit properties which are abstractly identical with the algebra of the numerous arithmetical functions of divisors, such as the totient, Möbius' function, Liouville's function, that originate in the unique factorization law. In short, the new properties of compound partitions unite the consequences of elliptic and Dirichlet processes in rational arithmetic. The more general relations mentioned have no analogues in the usual theory of partitions. The content of the theory is purely algebraic, although the elliptic case furnishes many formulas in a shape well adapted to asymptotic evaluations, to which I hope to return on another occasion.

In this paper we give only the simple general theorems underlying the whole subject. Applications to special instances, such as the elliptic case, are indicated only in sufficient detail to clarify the definitions. The entire theory of elliptic functions can be transposed into sets of relations between what are here called compositions and separations of functions, both of which are purely arithmetical processes, and this is but the simplest instance of the general theory.

The new technical terms necessary have all been chosen so as to recall the corresponding ones to which they are abstractly identical in rational arithmetic.

1. *M-numbers.* Let $(e_1, e_2, e_3, \cdots)$ be a one-rowed matrix formed from the reduced basis $e_1, e_2, e_3, \cdots$ of an abelian group G of any finite or

infinite order. Any element e of G other than the identity is of the form $e_1^{a_1}e_2^{a_2}e_3^{a_3}\cdots$, in which $a_1, a_2, a_3, \cdots$ are rational integers $\geqq 0$ not all zero, and e uniquely determines the matrix $(a_1, a_2, a_3, \cdots)$, which we shall call the *exponent* of e. To the identity we assign the exponent $(0, 0, 0, \cdots)$. We shall be concerned with a G whose elements are identical with their exponents and hence $(e_1, e_2, e_3, \cdots)$ may be ignored.

If c is any finite complex number the symbol $-\infty$ will be defined by the postulates that $-\infty + c = c - \infty = -\infty$, and $-(-\infty)$ does not exist, the last of which is imposed to make a certain type of division meaningless when the divisor is the zero of the set.

We consider the set S of all hypercomplex numbers

$$\alpha_i = (a_{1i}, a_{2i}, \cdots, a_{ki}) \qquad (i = 1, 2, \cdots),$$

in which k is constant but not necessarily finite, and the *coordinates* a_{ri} $(r = 1, \cdots, k)$ are either all finite complex numbers or all equal to $-\infty$, defined by the postulates (Π_j) $(j = 1, 2)$, of which the second is a mere definition.

(Π_1) If α_i, α_j are in S, $\alpha_i = \alpha_j$ only when $a_{ri} = a_{rj}$ $(r = 1, \cdots, k)$.

(Π_2) The *product* $\alpha_i\alpha_j$ of any elements α_i, α_j in S is the element

$$(a_{1i} + a_{1j},\ a_{2i} + a_{2j}, \cdots, a_{ki} + a_{kj}) \text{ of } S.$$

The *special elements* ζ, ϵ of S,

$$\zeta = (-\infty, -\infty, \cdots, -\infty), \quad \epsilon = (0, 0, \cdots, 0)$$

are such that $\zeta\alpha_i = \zeta$, $\epsilon\alpha_i = \epsilon$ for every α_i in S; they are the only elements of S having these respective properties, and hence they are called the *zero*, *unity* of S.

From the definition of $-\infty$ and (Π_2) it follows that if and only if $\alpha_i \neq \zeta$, there exists in S a unique solution

$$\alpha_x = (a_{1j} - a_{1i},\ a_{2j} - a_{2i}, \cdots, a_{kj} - a_{ki})$$

of $\alpha_i\alpha_x = \alpha_j$, where α_j is any element of S. We write $\alpha_x = \alpha_j/\alpha_i$ and call it the *quotient of* α_j *by* α_i. Hence *if* S' *be* S *with* ζ *omitted,* S' *is an abelian group under multiplication as defined in* (Π_2).

The product of n equal factors α_j in S is written α_j^n.

The elements of S are called *multipartite numbers of order* k or, since k is constant in a given context, briefly *M-numbers*.

We segregate M-numbers into classes according to the customary classification of complex numbers. An M-number is said to be *rational* if each of its coordinates is a rational integer $\geqq 0$; otherwise it is *irrational.* A rational M-number is called an *M-integer* if each of its coordinates is $\geqq 0$; otherwise it is an *M-fraction.* An irrational M-number is *real* or *imaginary* according as none or at least one of its coordinates is of the form $a + \pi i/2$, where a is real and π, i here (but not later) have their usual meanings. This classification is merely a restatement that M-numbers initially are exponents.

We shall be concerned here only with M-integers, that is with the set of all $(a_1, \cdots, a_k)$ subject to (Π_j) $(j = 1, 2)$, where the a_r $(r = 1, \cdots, k)$ range independently over all rational integers $\geqq 0$.

If α_i, α_j are M-integers, the quotient α_i/α_j is in general not an M-integer. The distinction thus introduced between algebraic and arithmetic divisibility is the source of interest of the theory.

2. *Divisibility for M-integers.* We say that the M-integer α *divides* the M-integer β, and write $\alpha \mid \beta$, if and only if an M-integer γ exists such that $\beta = \alpha\gamma$. If $\alpha = (a_1, \cdots, a_k)$, $\beta = (b_1, \cdots, b_k)$, the necessary and sufficient conditions that $\alpha \mid \beta$ are $b_r \geqq a_r$ $(r = 1, \cdots, k)$.

If $\delta \mid \alpha$ and $\delta \mid \beta$, δ is a *common divisor* of α, β. If δ is a common divisor of α, β, and if every common divisor of α, β divides δ, δ is called a G.C.D. of α, β. If $\alpha \mid \mu$, $\beta \mid \mu$, μ is a *common multiple* of α, β. If μ is a common multiple of α, β, and if every common multiple of α, β is a multiple of μ, μ is called a L.C.M. of α, β. It follows from these definitions that α, β have precisely one G.C.D. and precisely one L.C.M., and that if α, β are as above, the G.C.D. of α, β is $(d_1, \cdots, d_k)$ where $d_r =$ the lesser of a_r, b_r (or either of these if $a_r = b_r$) for $r = 1, \cdots, k$, and the L.C.M. is $(m_1, \cdots, m_k)$ where m_r is the greater of a_r, b_r $(r = 1, \cdots, k)$.

If the G.C.D. of a set of M-integers is the unity, ϵ, the members of the set are called *coprime.* An M-integer $\pi \neq \epsilon$ whose only divisors are π, ϵ is called an *M-prime.* Hence all the M-primes are the k M-integers.

$$(1, 0, 0, \cdots, 0),\ (0, 1, 0, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1),$$

which will be denoted by $\pi_1, \pi_2, \cdots, \pi_k$ respectively. An arbitrary M-prime will be denoted by π.

If $\pi \mid \alpha\beta$ then at least one of $\pi \mid \alpha$, $\pi \mid \beta$ is true. From this and the uniqueness of the G.C.D. it follows as usual that *an M-integer* $\alpha \neq \epsilon$ *is uniquely the product of M-primes,* except for unit factors (which may be ignored). If $\alpha = (a_1, a_2, \cdots, a_k)$ the resolution of α into its prime M-factors is

$$\alpha = \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_k^{a_k}, \tag{1}$$

and α has precisely the $\nu(\alpha) = (a_1 + 1) \cdots (a_k + 1)$ divisors

$$(d_1, \cdots, d_k), \quad 0 \leqq d_j \leqq a_j \qquad (j = 1, \cdots, k).$$

From the resolutions of several M-integers into their prime factors we can write down the G.C.D. and L.C.M. of the set as in rational arithmetic. Evidently we might have started from (1) and thence have obtained all the preceding definitions concerning M-integers, but it is of interest to follow the abstract identity of M-integers and rational integers directly from the postulates.

The connection with compound partitions is as follows. Let $\mu, \alpha, \beta, \cdots, \gamma$ be M-integers $\neq \epsilon$ such that

$$\mu = \alpha^a \beta^b \cdots \gamma^c \tag{2}$$

where $a, b, \cdots, c$ are rational integers $\geqq 0$, and

$$\alpha = (a_1, \cdots, a_k),\ \beta = (b_1, \cdots, b_k), \cdots, \gamma = (c_1, \cdots, c_k),$$
$$\mu = (m_1, \cdots, m_k).$$

Then (2) is equivalent to

$$m_j = aa_j + bb_j + \cdots + cc_j \qquad (j = 1, \cdots, k),$$

which is a compound partition of the set of integers m_j $(j = 1, \cdots, k)$ if these be regarded as constants. The significance of unique factorization of M-integers in terms of compound partitions will appear as we proceed.

3. *Separations mod C of M-integers.* Let C be a set of rational integers > 0; C may contain any finite number or an infinity of distinct elements (rational integers), but no element may occur more than once in C. The last is not a restriction on the generality of the processes next considered, as will be evidenced later. If C contains the integer $n > 0$, we write $C \mid n$.

Corresponding to the partitions in the usual sense of a single rational integer we shall be concerned here with the separations modulo C of M-integers, where C is a given set as above, which are defined as follows. Let α_i $(i = 1, \cdots, s)$ be any s M-integers all different from ϵ such that, for α a given M-integer,

$$\alpha = \alpha_1^{c_1} \alpha_2^{c_2} \cdots \alpha_s^{c_s}$$

where $C \mid c_i$ $(i = 1, \cdots, s)$, *and no two of the c_i are equal.* Indicating the set C which contains the c_i we write the above in the form

$$\alpha = \alpha_1^{c_1} \alpha_2^{c_2} \cdots \alpha_s^{c_s} \qquad (\text{mod } C) \tag{3}$$

when necessary. The right of (3) is called a *separation mod C of order s of α*, and the $s!$ separations obtained by permuting the factors $\alpha_i^{c_i}$ $(i = 1, \cdots, s)$ in all possible ways are defined to be identical. When α is given and s is constant the number of separations mod C of α of order s is finite, and the total number of separations of all orders $s = 1, 2, 3, \cdots$ mod C of α is finite. The set of all separations mod C of α of all orders is called the *total separation mod C of α*. For every C we assign by convention to ϵ the total separation ϵ. An example of (3) is given near the end of § 5. It does not follow that for C, α given a separation of α (mod C) exists.

To state the equivalent of (3) in terms of compound partitions, disregard the trivial case $\alpha = \epsilon$, and let

$$\alpha_i = \pi_1^{a_{1i}} \pi_2^{a_{2i}} \cdots \pi_k^{a_{ki}} \qquad (i = 1, \cdots, s)$$

be the resolutions of the α_i into their M-prime factors. Since in (3) $\alpha_i \neq \epsilon$ $(i = 1, \cdots, s)$, the α_{ji} are ks rational integers $\geqq 0$ such that

$$a_{1i} + a_{2i} + \cdots + a_{ki} > 0 \qquad (i = 1, \cdots, s).$$

Let α be as in (1). Then, comparing exponents of π_j in (3) we see that (3) is equivalent to the restricted system of k diophantine equations

$$a_i = \sum_{j=1}^{s} c_j a_{ij}, \quad C \mid c_j \qquad (i = 1, \cdots, k),$$

where the a_i $(i = 1, \cdots, k)$ are any given k integers $\geqq 0$ whose sum is > 0, s is constant, and the c_j, a_{ij} are to be determined subject to the stated conditions.

4. *Numerical functions of M-integers.* If $f(\alpha)$ takes a single finite value when α is any given M-integer, and if $f(\epsilon) = 1$, we call $f(\alpha)$ a *numerical function* of x. The *value* of $f(\alpha)$ is to be understood in the usual sense as a definite real or complex number. Hence the value of $f(\alpha)$ is not an M-number, and we operate simultaneously in two domains, that of M-integers and that of real or complex numbers.

The following instances of numerical functions will be useful later. Let $a, b, \cdots, c$ be κ distinct integers chosen from $1, 2, \cdots, k$, and let

$$\delta = \pi_a^p \pi_b^q \cdots \pi_c^r \qquad (p, q, \cdots, r > 0)$$

be the resolution of δ into its M-prime factors. Write

$$\rho(\delta) = p + q + \cdots + r, \quad \gamma(\delta) = \kappa; \tag{4}$$

$$\lambda(\delta) = (-1)^{\rho(\delta)}, \quad \lambda(\epsilon) = 1; \tag{5}$$

$$\mu(\epsilon) = 1, \quad \mu(\delta) = 0 \text{ if at least one of } p, q, \cdots, r > 1, \tag{6}$$

and otherwise $= (-1)^{\gamma(\delta)}$. Then $\lambda(\alpha)$, $\mu(\alpha)$ are numerical functions of α, while $\rho(\alpha)$ is not, since $\rho(\epsilon) = 0$.

If $f(\alpha\beta) = f(\alpha) f(\beta)$ for every pair of coprime M-integers α, β, we call $f(\alpha)$ a *normal* function of α. Hence, *if f is normal and δ as above,*

$$f(\delta) = f(\pi_a{}^p) f(\pi_b{}^q) \cdots f(\pi_c{}^r).$$

If $f(\alpha) = g(\alpha)$ for all M-integers α, we say that $f(\alpha)$, $g(\alpha)$ are *equal*, and write $f = g$.

5. *Separations mod C of numerical functions.* Let f be any numerical function and α a constant M-integer. Form the product $f(\alpha_1) f(\alpha_2) \cdots f(\alpha_s)$ with respect to the separation (3), and indicate by $\sum_s$ a sum with respect to all orders $s = 1, 2, 3, \cdots$, that is, over the total separation of α. Then the finite sum $f'(\alpha)$ defined by

$$f'(\alpha) = \sum\nolimits_s f(\alpha_1) f(\alpha_2) \cdots f(\alpha_s) \tag{7}$$

is a numerical function of α, which we shall call the *separation mod C of* $f(\alpha)$ or, when C is understood, the *separation* of $f(\alpha)$. If necessary to indicate C we may write $f'(\alpha)$ (mod C). A separation will always be indicated by an accent, as in the left of (7).

If $\alpha = \pi^n$, $n > 0$, where as always π is any M-prime, $f'(\alpha)$ (mod C) is connected as follows with certain of the partitions of n, as is seen at once from the definitions. Let

$$n = j_1 n_1 + j_2 n_2 + \cdots + j_s n_s$$

be any partition of n into s unequal integers n_r $(r = 1, \cdots, s)$ in C, repeated $j_r > 0$ times respectively. As in the classic theory of partitions the sequence of the parts in a given partition is immaterial; thus $7 + 35$ and $35 + 7$ are the same partition of 42 in the set of all integer multiples > 0 of 7. As in a frequent notation we write the above partition in the form

$$n = (n_1{}^{j_1} n_2{}^{j_2} \cdots n_s{}^{j_s}) \qquad (\text{mod } C), \tag{8}$$

where mod C refers to $n_1, n_2, \cdots, n_s$. We call (8) a *partition mod C of order s of n.* Let $\sum_s$ indicate a sum with respect to all partitions mod C of all orders $s = 1, 2, 3, \cdots$ of n. Then, the separation (7) when $\alpha = \pi^n$ is easily seen to be

$$f'(\pi^n) = \sum\nolimits_s f(\pi^{j_1}) f(\pi^{j_2}) \cdots f(\pi^{j_s}). \tag{9}$$

By convention, for every mod C, we assign to 0 the unique partition 0, and take

$$f'(\pi^0) = f'(\epsilon) = f(\epsilon) = 1.$$

If f is normal, and is as in (1),

$$f'(\alpha) = f'(\pi_1^{a_1})\, f'(\pi_2^{a_2}) \cdots f'(\pi_k^{a_k}), \tag{10}$$

which can be computed by (9).

As a numerical example, let C for the moment be the set of all rational integers > 0, and consider the separation $f'(p^2q^3)$ (mod C) of $f(p^2q^3)$, where p, q are M-primes. Then $\alpha = p^2q^3$ has only the six separations, as in (3),

$$\begin{gathered}(p)^2(q)^3, \quad (pq)^2(q)^1, \quad (p^2q^3)^1, \\ (p^2)^1(q)^3, \quad (p^2q)^1(q)^2, \quad (p)^2(q^3)^1,\end{gathered}$$

of the respective orders 2, 2, 1, 2, 2, 2, the M-integers in parentheses being the $\alpha_1, \alpha_2, \cdots$ of (3) and the exponents outside parentheses the $c_1, c_2, \cdots$. Hence, by (7) the value of $f'(p^2q^3)$ (mod C) is

$$\begin{aligned}f(p)f(q) + f(pq)f(q) + f(p^2q^3) + f(p^2)f(q)& \\ + f(p^2q)f(q) + f(p)f(q^3)&.\end{aligned}$$

If in particular f is normal (but not in general otherwise),

$$\begin{gathered}f(pq)f(q) = f(p)f^2(q), \quad f(p^2q^3) = f(p^2)f(q^3), \\ f(p^2q)f(q) = f(p^2)f^2(q),\end{gathered}$$

where $f^2(q) = [f(q)]^2$, and so in all like cases. Hence, for normal f, we have from the above value of $f'(pq)$ for any f,

$$f'(p^2q^3) = [f(p) + f(p^2)]\,[f(q) + f^2(q) + f(q^3)].$$

Again, corresponding to (8) the partitions mod C for the exponents 2, 3 in p^2q^3 are

$$2 = (2^1) = (1^2), \quad 3 = (3^1) = (2^1 1^1) = (1^3),$$

of the respective orders 1, 1, 1, 2, 1. Hence, by (9), we have the separations mod C

$$f'(p^2) = f(p) + f(p^2), \quad f'(q^3) = f(q) + f^2(q) + f(q^3).$$

Thus, if f is normal,

$$f'(p^2q^3) = f'(p^2)\, f'(q^3) \quad \text{mod } C,$$

in accordance with the theorem, and we might have computed $f'(p^2q^3)$ for normal f directly from $f'(p^2)$, $f'(q^3)$ with less labor. It is clear that for any other C than that of the above examples the separations will differ from those just found, also it is evident that the separations $f'(p^2)$, $f'(q^3)$ (mod C) do not enable us to calculate $f'(p^2q^3)$ mod C unless f is normal, and so in all cases. Separations may exist for one mod C but not for another; for any mod C there obviously are an infinite number of $f(\alpha)$ which have separations.

6. *Compositions of numerical functions.* Let α_j $(j=1, \cdots, r)$ be any r M-integers whose product is the constant M-integer α,

$$\alpha=\alpha_1\alpha_2\cdots\alpha_r. \tag{11}$$

Note that we have not excluded the case, which will always be relevant in what follows, when certain of the α_j may be ϵ. Attending to the arrangement of the factors $\alpha_1, \alpha_2, \cdots, \alpha_r$, we call the right of (11) a *composition of order r of n*. Thus, if $\alpha=\alpha_1\alpha_2$, and $\alpha_1\neq\alpha_2$, the compositions $\alpha_1\alpha_2$, $\alpha_2\alpha_1$ of order 2 of α are distinct, as also are $\epsilon\alpha$ and $\alpha\epsilon$.

Let f, f_i $(i=1, \cdots, r)$ be any $r+1$ numerical functions. For α, r constant, the finite sum

$$\sum f_1(\alpha_1)\,f_2(\alpha_2)\cdots f_r(\alpha_r) \tag{12}$$

taken over all compositions of α of order r, is evidently a numerical function of the single M-integer α, say $f(\alpha)$. We call this $f(\alpha)$ a *composition of order r*. Clearly $f(\alpha)$ is uniquely known when α and f_i $(i=1, \cdots, r)$ are assigned. Conversely, as will appear presently, if f be given it is possible in an infinity of ways to determine, for r constant, r numerical functions f_i $(i=1, \cdots, r)$ such that, for every α, $f(\alpha)$ is equal to the sum (12). Each such determination of the f_i $(i=1, \cdots, r)$ for a given f is called a *composition of order r of f*, and we write

$$f(\alpha)=\sum f_1(\alpha_1)\,f_2(\alpha_2)\cdots f_r(\alpha_r), \tag{13}$$

or, omitting all arguments and the sign of summation,

$$f=f_1f_2\cdots f_r, \tag{14}$$

as the symbolic equivalent of this composition. The $f_1, f_2, \cdots, f_r$ are called *factors* of f, and (14) defines a species of multiplication, concerning which we have the following theorem.

Composition as in (14) *is associative and commutative.*

Let $e=e(\alpha)$ be that numerical function which is defined by

$$e(\epsilon)=1, \quad e(\alpha)=0, \quad \alpha\neq\epsilon. \tag{15}$$

We call e the *unit function of composition,* or briefly the *unit function,* since $ef=f$ for every numerical function f.

If g is any numerical function there exists a unique numerical function, denoted by g^{-1}, and called the reciprocal of g, such that $gg^{-1}=e$.

The proof, abstractly identical with that of a former paper * for functions of rational integers, incidentally furnishes the explicit form of $g^{-1}(\alpha)$ in terms of functions g with the appropriate arguments determined by α. It need not be reproduced.

From the existence of reciprocals that of an infinity of compositions (14) of a given f as already stated is obvious.

The set of all numerical functions is an abelian group under composition, the identity of the group being e.

As a detail of notation it is necessary to distinguish between the composition of r equal functions and their algebraic product, which also can be taken as a single numerical function (when the arguments of all the factors are identical) and hence can occur as a factor in a composition. If in (14) $f_i=g$ $(i=1,\cdots,r)$, we shall write (14) as $f=g^r$. If on the other hand the algebraic product $[g(\alpha)]^r$ is to be represented with the argument α omitted, and hence in a form suitable for composition as in (14), we shall write the product as (g^r). For example, to illustrate both, the composition $h=f^2(g^3)$ signifies, for every M-integer α, the equality

$$h(\alpha)=\sum f(\alpha_1)\,f(\alpha_2)\,g(\alpha_3)\,g(\alpha_3)\,g(\alpha_3),$$

where $\sum$ is taken over all sets $(\alpha_1,\alpha_2,\alpha_3)$ of solutions $\alpha_1,\alpha_2,\alpha_3$ of $\alpha=\alpha_1\alpha_2\alpha_3$.

7. *Separation mod C and composition of numerical functions.* Combination of the stated processes leads to the most interesting properties of M-integers, particularly when the theorems, abstractly identical with those properties of rational integers or of ideals which depend only upon the unique factorization law of arithmetic, are restated as their equivalents in terms of compound partitions. All of the properties of compound partitions thus obtained appear to be new.

Abstractly, what follows is equivalent algebraically to the simultaneous performance of Dirichlet multiplication and Dedekind inversion in an enlarged sense, and the expansion of infinite products whose simplest instances are those occurring in elliptic functions. It thus combines, in its simplest instance, the multiplicative processes of algebraic arithmetic as given by operations upon Dirichlet series and those of the additive that are contained in relations between elliptic and theta functions. To composition correspond the Dirichlet processes, to separation the elliptic and their generalizations; the following combines them.

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* *Tohoku Mathematical Journal,* Vol. 17 (1920), pp. 221-231.

Let $f = f_1 f_2 \cdots f_r$ *be any composition of order* r *of* f. *Then*

$$f' = (f_1 f_2 \cdots f_r)' = f_1' f_2' \cdots f_r'; \tag{16}$$

that is, the separation of a composition is the composition of the separations of the several factors of the composition. Further, the relation (16), *implied by* $f = f_1 f_2 \cdots f_r$, *is the same for all sets* C, *and hence is a species of invariant of* C. This will be evident from the next section.

Thus far we have considered separations with respect to a single set C. Frequently, however, it is necessary in applications to discuss simultaneously separations with respect to several sets. Let C_j $(j = 0, 1, 2, \cdots)$ be any sets defined in the same way as C, and write $\phi_j'(\alpha)$ for the separation $\phi'(\alpha)$ (mod C_j), where $\phi(\alpha)$ is any numerical function of α. Then ϕ'_j $(j = 0, 1, \cdots)$ being numerical functions, the ϕ_j' can be taken as factors in compositions, and the entire preceding theory of composition of numerical functions applies to them. Similarly, if ψ, χ, $\cdots$ are any numerical functions, ψ_j' χ_j' $\cdots$ $(j = 0, 1, \cdots)$ are defined, and all of ϕ_j', ψ_j', χ_j', $\cdots$ $(j = 0, 1, \cdots)$, or any subset of them, are subject to composition. We thus operate simultaneously in sets of sets as moduli, in analogy with the familiar modular systems of algebraic numbers. If all of the ϕ_j', ψ_j', χ_j', $\cdots$ considered in a given context are normal, there is the obvious extension of (10) available for calculations. Examples will be found in § 11.

8. *Generators.* It is assumed as usual that indeterminates can be combined according to the processes of an abstract field, and that equality of polynomials or power series in several indeterminates signifies only that the coefficients of like power products of indeterminates are equal. As in any use of indeterminates in the literature of arithmetic, their introduction here is merely a convenience which can be avoided if desired. The suppression of the indeterminates however contributes nothing to the logic of the situation.

If t is a parameter, π any M-prime, α any M-integer, we treat as indeterminates the powers π^t, α^t which, by hypothesis, are subject to all the laws of common algebra. In particular, if α, β are any M-integers, $\alpha^t\beta^t = (\alpha\beta)^t$; $(\alpha^t)^n = \alpha^{nt}$, where $n > 0$ is a rational integer;

$$(\alpha^t)^0 = \alpha^0 = \epsilon.$$

For convenience, we now write $\pi^t = z$, $\alpha^t = w$, where π, α are the symbols of a general M-prime and a general M-integer; $\sum_\alpha$ refers to a sum over all M-integers, $\prod_\pi$ to a product over all M-primes, and $\prod_n$ (mod C) to a product over all n such that $C \mid n$, where C is any given set. The result of replacing t by nt is the same as that of changing z, w to z^n, w^n respectively.

If $f(\alpha)$, $g(\alpha)$ are any numerical functions, an equation

$$\textstyle\sum_\alpha f(\alpha)w = \sum_\alpha g(\alpha)w$$

is equivalent to $f(\alpha) = g(\alpha)$ for all M-integers α, and hence to $f = g$. It is convenient always to take as the first term of any $\sum_\alpha$ that in which $\alpha = \epsilon$; the sequence of the remaining terms is entirely arbitrary.

Let several such series $\sum_\alpha$ be multiplied together. Regroup the distributed product as a new $\sum_\alpha$, that is, as a series in which the total coefficient of a given M-integer α is collected as that of the corresponding $w = \alpha^t$. The product evidently is unique. Hence, *the set of all* $\sum_\alpha f(\alpha)w$, *as f runs through the set of all numerical functions, is closed under such multiplication of series* $\sum_\alpha$.

Let $f = f_1f_2 \cdots f_r$ as in (14). Then

$$(17) \qquad \textstyle\sum_\alpha f(\alpha)w = \sum_\alpha f_1(\alpha)w \times \sum_\alpha f_2(\alpha)w \times \cdots \times \sum_\alpha f_r(\alpha)w.$$

Again, if g be any numerical function, we see that

$$(18) \qquad \textstyle\prod_n [\sum_\alpha g(\alpha)w^n] = \sum_\alpha g'(\alpha)w \quad (\text{mod } C),$$

where as always $g'(\alpha)$ is the separation (mod C) of $g(\alpha)$.

If h be any numerical function, then

$$(19) \qquad \textstyle\prod_\pi [1 + h(\pi)z + h(\pi^2)z^2 + \cdots] = \sum_\alpha h(\alpha)w.$$

The series in [ ] on the left may be finite or infinite.

If g, g^{-1} are reciprocals, then

$$(20) \qquad \textstyle\sum_\alpha g(\alpha)w \times \sum_\alpha g^{-1}(\alpha)w = 1.$$

If h is as in (19), then

$$(21)\qquad [1+h(\pi)z+h(\pi^2)z^2+\cdots]\\ [1+h^{-1}(\pi)z+h^{-1}(\pi^2)z^2+\cdots]=1,$$

in which, as always, h^{-1} is the reciprocal of h as defined in § 6.

In (17) replace w by w^n, where $C \mid n$. Then letting n range over all elements of C, we have

$$\Pi_n\,[\Sigma_\alpha f(\alpha)w^n]=\Pi_n\,[\Sigma_\alpha f_1(\alpha)w]\times\cdots\times\Pi_n\,[\Sigma_\alpha f_r(\alpha)w^n],$$

which is equivalent to

$$\Sigma_\alpha f'(\alpha)w=\Sigma_\alpha f_1'(\alpha)w\times\cdots\times\Sigma_\alpha f_r'(\alpha)w.$$

Hence, if for the moment we write $k=f_1'f_2'\cdots f_r'$, we have

$$\Sigma_\alpha f'(\alpha)w=\Sigma_\alpha k(\alpha)w,$$

which is equivalent to the theorem in § 7.

Returning now to (19) and h normal as there, we shall call the typical factor

$$(22)\qquad H(z)=1+h(\pi)z+h(\pi^2)z^2+\cdots$$

the *generator* of $h(\alpha)$, or of h. Hence *the generator of the composition of any number of normal numerical functions is the product of the generators of the several functions; from* (21), *the generator of the reciprocal of a normal numerical function is the reciprocal of the generator of the function.*

From (19), (22) we have

$$\Pi_n\,[\Pi_\pi H(z^n)]=\Pi_\pi\,[\Pi_n H(z^n)]=\Sigma_\alpha h'(\alpha)w\ \ (\text{mod } C);$$

and hence, if

$$\Pi_n H(z)=1+g(\pi)z+g(\pi^2)z^2+\cdots\equiv G(z),$$

we get $g=h'$, and $G(z)$ is the generator of h'. If $H(z)$ contains more than 2 terms the explicit determination of $g(\pi^j)$ $(j=1,2,\cdots)$ may be difficult. When $H(z)$ contains only 2 terms, the least number possible, we can apply the known elliptic products in a great many instances.

If as in (17) none of $f_1,\cdots,f_r$ are necessarily normal, the generator of $f_j(\alpha)$ is defined to be $\Sigma_\alpha f_j(\alpha)w$, and we have precisely the same conclusion as in the foregoing theorem. The case of normal functions is however that of greater interest.

In the definition of numerical functions in § 4 it is not necessary to impose the condition $f(\epsilon)=1$, which was done merely for simplicity; it is sufficient to take $f(\epsilon)\neq 0$. The consequent modifications of all that precedes can be readily made.

Finally, as in § 7, end, all of this section has an immediate extension to separations with respect to any set of sets C_j $(j=0, 1, \cdots)$. The generators corresponding to different sets $C_0, C_1, \cdots$ may be combined by multiplications and divisions, in several ways. Thus, as the simplest, only those generators appertaining to a given C_j may be combined. Alternatives will be noticed in § 12 after the examples in § 11.

9. *Classes of normal functions and M-integers.* Normal functions are now classified as algebraic or transcendental, in a technical sense, according to their generators considered as functions of two independent indeterminates π, z. The normal h is called *algebraic* or *transcendental* according as its generator (22) is an algebraic or a transcendental function of π, z.

If algebraic, h is defined to be of the same species as its generator, for example rational, rational integral, etc. The generator of a rational algebraic h has a *canonical form* $P(\pi, z)/Q(\pi, z)$, in which P, Q are polynomials in π, z with no common factor other than 1, and the leading coefficients in P, Q are both 1; the case in which one but not both of P, Q is 1 is included.

If $Q=1$ in the above, the normal h generated by $P(\pi, z)$ is called *prime* or *composite* according as $P(\pi, z)$ is irreducible or reducible in the rational domain. Hence, *an algebraic h is uniquely of the form*

$$g_1g_2\cdots g_r \quad k_1k_2\cdots k_s \tag{23}$$

in which g_i $(i=1, \cdots, r)$ are prime functions, k_j $(j=1, \cdots, s)$ are reciprocals of prime functions, no k_j is the reciprocal of any g_i, and either all the g_i or all the k_j may be absent. We call (23) the *prime composition* of h; it is significant here only if h be algebraic as defined, although it may obviously be extended to certain transcendental h.

To apprehend readily the structure of a given normal function it is useful to separate all M-integers into two classes, simple and non-simple. Exactly the same principle applies to any set of elements having a unique factorization law, and hence in particular to the rational integers. In the latter case we may indeed take $\pi_k=$ the kth natural prime; M-integers then become rational integers >0.

An M-integer is called *simple* if and only if it is divisible by the square of no M-integer other than the unity ϵ. We shall exclude ϵ from the simple

M-integers, which therefore are of the form $(j_1, j_2, \cdots, j_k)$ where each of j_i $(i = 1, \cdots, k)$ is a definite one of 0, 1, and not all are 0. There are thus $2^k - 1$ simple M-integers if k be finite. A set of simple M-integers every pair of which are coprime are called *distinct.*

From the definitions it follows that *any M-integer* $\alpha \neq \epsilon$ *has a unique resolution into a product of powers of distinct simple M-integers* σ_j $(j = 1, \cdots, r)$ *of the type*

$$\alpha = \sigma_1^{s_1} \sigma_2^{s_2} \cdots \sigma_r^{s_r}, \tag{24}$$

in which the s_j $(j = 1, \cdots, r)$ *are rational integers* > 0 *no two of which are equal.*

In the generator (22) of any normal h certain of the coefficients $h(\pi^j)$ $(j = 1, 2, \cdots)$ may be identically zero. Retaining only non-vanishing coefficients we rewrite the generator of h as

$$1 + h(\pi^{j_1}) z^{j_1} + h(\pi^{j_2}) z^{j_2} + \cdots, \tag{25}$$

which may be either a finite or an infinite series; if h is a prime function, (25) is a polynomial. The h generated by (25) is completely defined by the following properties. Let (24) be the resolution of α into distinct simple factors. Then, if any one of s_i $(i = 1, \cdots, r)$ is not a member of the sequence $j_1, j_2, \cdots$, the function $h(\beta)$ vanishes for $\beta = \alpha$. To state the value of $h(\alpha)$ in the contrary case, let

$$\sigma_i = \pi_{i1} \pi_{i2} \cdots \pi_{im}, \quad m \equiv k_i \qquad (i = 1, \cdots, r).$$

Then, by (24), the $k_1 + k_2 + \cdots + k_r$ M-primes π_{ij} are all different. Write

$$H(\sigma_i) = h(\pi_{i1}{}^{s_1}) \, h(\pi_{i2}{}^{s_2}) \cdots h(\pi_{im}{}^{s_m}) \qquad (i = 1, \cdots, r).$$

Then the required value is

$$h(\alpha) = H(\sigma_1) \, H(\sigma_2) \cdots H(\sigma_r).$$

When $h(\alpha)$ is a prime algebraic function its generator (25) contains only a finite number n of distinct powers z^{j_i} $(i = 1, \cdots, n)$, and likewise for the product of any finite number of equal or distinct prime functions. The number of distinct powers of z in such a finite generator is called its *extent;* the extent of a function (when significant) is defined to be the *extent of its generator.* The simplest algebraic numerical functions are those of extent 1 and their reciprocals.

To investigate the algebraic relations between a given set of algebraic numerical functions we decompose their respective generators into their irreducible factors, thereby obtaining the corresponding resolutions (23), and from the latter, by multiplications and divisions as in any abelian group, write down any number of relations between compositions of functions, in abstract identity with the set of all relations derived by multiplications and divisions in the set of all rational numbers. From these relations we can pass at once as in § 7 (16) to abstractly identical relations between separations. Finally, as noted in § 3, any such relation is equivalent to a theorem in compound partitions. All the relations in the foregoing are evidently implicit in the prime resolutions (23). By obvious modifications we get similar results when the functions are transcendental. Irrational algebraic numerical functions are treated in an evident way by means of their generators regarded as power series in z.

10. *Addition and subtraction.* As in all arithmetical theories (rational arithmetic, ideals, etc.), there is here a sharp break between the properties of multiplication for M-numbers and those of addition as next defined. It seems in fact as if arithmetic is essentially two distinct theories, and that only adventitious circumstances (in rational arithmetic, but not strictly in the theory of ideals or even in that of algebraic integers without ideals) permit the two theories to be joined into one. That the nature of these circumstances is not fully understood seems fairly clear from the inordinate difficulties encountered in attempts to prove such simple conjectures as that which asserts every even integer to be the sum of two primes. The properties of M-integers as so far investigated have grown out of multiplication and its inverse division. From the above remarks it is reasonable to expect nothing natural when we introduce addition, which is now done merely for completeness. It is openly *ad hoc* (like most attempts to define addition for ideals).

Addition and its inverse subtraction for M-numbers are defined by the postulate that polynomials in M-numbers with coefficients in any field are combined as if the M-numbers were indeterminates. Hence equality of functions of M-numbers taken over any field is defined. Those processes are of but slight interest; the next are more significant.

Since the values of numerical functions are ordinary real or complex numbers, the values form a field. Let $f(\alpha)$ be any numerical function of the M-integer α, and write $f''(\alpha)$ for its value. Then if $a, b, \cdots, c$ are any real or complex numbers, $f(\alpha), g(\alpha), \cdots, h(\alpha)$ any numerical functions,

$$F''(\alpha) \equiv af''(\alpha) + bg''(\alpha) + \cdots + ch''(\alpha)$$

is defined as α runs through all M-integers. We say that $F''(\alpha)$ is the *value* of the *extended numerical function*

$$F(\alpha) \equiv af(\alpha) + bg(\alpha) + \cdots + ch(\alpha),$$

or, dropping the argument α as before, we define the extended numerical function F by

$$F = af + bg + \cdots + ch,$$

and this F is merely the symbol for the function which takes the values $F''(\alpha)$ as α runs through all M-integers α. It is clear that F is not in general a numerical function as defined in § 4, as is seen from the example

$$a + b + \cdots + c = 0, \quad f = g = \cdots = h.$$

On the other hand extended numerical functions include numerical functions as a special case; it suffices to take

$$a = 1, \quad b = \cdots = c = 0.$$

Let

$$F = af + bg + \cdots + ch, \quad G = p\phi + q\psi + \cdots + r\chi$$

be any extended numerical functions, $f, g, \cdots, h, \phi, \psi, \cdots, \chi$ being numerical functions, and $a, b, \cdots, c,\ p, q, \cdots, r$ real or complex numbers. The *product* FG of F, G is the extended numerical function

$$\begin{aligned} &apf\phi + aqf\psi + \cdots + arf\chi \\ &+ bpg\phi + bqg\psi + \cdots + brg\chi \\ &+ \cdots\cdots \\ &+ cph\phi + cqh\psi + \cdots + crh\chi, \end{aligned}$$

in which $f\phi, f\psi, \cdots, h\chi$ are compositions of numerical functions as defined in § 6. Thus composition of numerical functions is a special case of multiplication (as above defined) of extended numerical functions. The *sum* $F + G$ of F, G is the extended numerical function

$$af + \cdots + ch + p\phi + \cdots + r\chi.$$

Hence, *the set of all extended numerical functions is closed under addition and multiplication as above, and these operations are abstractly identical with addition and multiplication in a field.*

As in § 8 we next define the generator of the F above,

$$\begin{aligned}\textstyle\sum_a F(\alpha)w &= \textstyle\sum_a [f(\alpha) + g(\alpha) + \cdots + h(\alpha)]w,\\ &= \textstyle\sum_a f(\alpha)w + \sum_a g(\alpha)w + \cdots + \sum_a h(\alpha)w.\end{aligned}$$

It follows without difficulty (on comparing coefficients of like M-integers w after distribution of the left of the next equation and rearrangement of the result into a $\sum_a$) that if $\sum_a F(\alpha)w$, $\sum_a H(\alpha)w$ are given generators as just defined of extended numerical functions, there exists a unique generator $\sum_a G(\alpha)w$ such that

$$\textstyle\sum_a F(\alpha)w \times \sum_a G(\alpha)w = \sum_a H(\alpha)w$$

when and only when $F(\epsilon) \neq 0$. Provided $F(\epsilon) \neq 0$, the extended numerical function whose generator is $\sum_a G(\alpha)w$ is called the *quotient* of that whose generator is $\sum_a H(\alpha)w$ by that whose generator is $\sum_a F(\alpha)w$, and the function generated by $\sum_a F(\alpha)w$ is called *regular* or *irregular* according as $F(\epsilon) \neq 0$ or $F(\epsilon) = 0$.

The set of all extended numerical functions is an irregular field[*] *in which the irregular elements are the irregular functions and the four fundamental operations are as above defined.*

11. *Functions of extent* 1 *in the elliptic case.* When all the prime functions occurring in a given application of the preceding theory are of extent 1, the relations between compositions and separations of numerical functions are particularly simple and interesting, as they are abstractly identical with the algebra of the majority of arithmetical functions of rational integers depending upon the unique factorization law that occurs in the literature. Included in this very special case is another, in which the theory is abstractly identical also with the algebra of the elliptic theta functions and their quotients. To illustrate several of the preceding definitions and theorems we shall give a few examples of the simplest relations in the last instance, the elliptic case.

The theorems selected can be most simply stated in terms of the following classes of M-integers. If each of the coordinates a_j $(j = 1, \cdots, k)$ of the M-integer $\alpha = (a_1, \cdots, a_k)$ is represented in the form $\phi \equiv \phi\ (x, y, \cdots)$, in which $x, y, \cdots$ are indeterminates, we say that α is of *class* ϕ. For example, if each a_j is a square, necessarily of a rational integer, α is of class x^2; if each a_j is a triangular number, α is of class $x(x+1)/2$; if each a_j is a pentagonal number, α is of class $(3x^2 \pm x)/2$; if each a_j is the sum of 2 squares, α is of class $x^2 + y^2$, etc. In each of these examples ϕ is a quadratic form with rational number coefficients. This characterizes the elliptic case above described; the forms may be in any number of indeterminates.

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[*] For the postulates of such a field see *Annals of Mathematics*, Vol. 27 (1925), p. 512. The only distinction between a regular and an irregular field is the exclusion of an infinity of irregular divisors in the latter instead of but one in the former.

Let α be any M-integer, π any M-prime, and x a real or complex number. A uniform function of x which is also a numerical function of α may be designated by $f(x, \alpha)$. If however we wish to emphasize $f(x, \alpha)$ quâ function of α, or quâ function of x, we shall write $f(x, \alpha)$ in either of the equivalent forms $f_x(\alpha)$, $f_\alpha(x)$, so that

$$f(x, \alpha) \equiv f_x(\alpha) \equiv f_\alpha(x).$$

This device also simplifies printing.

Let $U(x, \alpha)$, $P(x, \alpha)$ be normal functions of α,

$$U(x, \alpha) = U_\alpha(x) = U_x(\alpha), \quad U_x(\epsilon) = 1,$$
$$P(x, \alpha) = P_\alpha(x) = P_x(\alpha), \quad P_x(\epsilon) = 1,$$

defined by the properties: $P_x(\pi^n) = x^n$; $U_x(\pi^n) = 0$ or x according as $n > 1$ or $n = 1$, where n is a rational integer. Then (see § 8) we have the generating identities

$$\Pi_\pi (1 + xz) = \Sigma_\alpha U_x(\alpha) w, \quad \Pi_\pi (1 - xz)^{-1} = \Sigma_\alpha P_x(\alpha) w,$$

and therefore

$$U_x P_{-x} = e, \quad P_x = U_{-x}^{-1}, \quad U_x = P_{-x}^{-1};$$

that is, (see § 6) U_x, P_{-x} are reciprocals and each is a prime function. Referring to § 4, (4)-(6), we see that

$$U_x(\alpha) = \mu^2(\alpha) x^{\gamma(\alpha)}, \quad P_x(\alpha) = x^{\rho(\alpha)},$$

and hence in particular,

$$U_{-1} = \mu, \quad P_{-1} = \lambda, \quad U_1 = (\mu^2), \quad P_1 = 1;$$

the significance of the parenthesis in (μ^2) is explained in § 6, end. Hence

$$\mu P_1 = \lambda(\mu^2) = e,$$

which in non-symbolic form state that each of the sums

$$\sum \mu(\tau), \quad \sum \lambda(\tau) \mu^2 (\alpha/\tau),$$

where $\sum$ refers to all divisors τ of α, has the value 1 or 0 according as $\alpha = \epsilon$ or $\alpha \neq \epsilon$. These are abstractly identical with the corresponding theorems concerning rational integers for the functions usually denoted by μ, λ. By the theory already developed the like extension to M-integers holds for any of the uniform functions of divisors of rational integers.

The separations U_x', P_x', are now known from the definitions, but they may be defined independently from the generators. The two definitions are of course identical. Consider then

$$\Pi_n (1 + xz^n), \qquad \Pi_n (1 - xz^n)^{-1} \qquad (\text{mod } C),$$

in which the coefficients of z^n are respectively (after distribution of the products)

$$U_x'(\pi^n), \qquad P_x'(\pi^n) \qquad (\text{mod } C).$$

Let $q_i(n)$ be the number of partitions of n into precisely i parts, no two of which are equal, and all of which are in C. Then, $\sum_i$ indicating a sum extending to $i = 1, 2, 3, \cdots$, we have

$$U_x'(\pi^n) = \sum_i q_i(n) x^i.$$

For $n = 1, \cdots, 6$ and C the set of all rational integers > 0, we thus have

$$\begin{aligned} U_x'(\pi) &= U_x'(\pi^2) = x, \\ U_x'(\pi^3) &= U_x'(\pi^4) = x^2 + x, \\ U_x'(\pi^5) &= U_x'(\pi^6) = 2x^2 + x. \end{aligned}$$

The equality of successive pairs is not general. Hereafter, unless otherwise stated, C is any set as before. Similarly, if $\sum_r$ refers to all sets of rational integer solutions $r_1, r_2, r_3, \cdots, \geqq 0$ of

$$n = r_1 n_1 + r_2 n_2 + r_3 n_3 + \cdots,$$

where no two of $n_1, n_2, n_3, \cdots$ are equal and all are in C, we have

$$P_x'(\pi^n) = \sum_r x^{r_1+r_2+r_3+\cdots}.$$

Since U_x, P_x are normal, $U_x'(\alpha)$, $P_x'(\alpha)$ (mod C) are now known by § 4, end, for all M-integers α. Note that π, α play the part of indeterminates, disappearing from the final forms. Thus if

$$\alpha = \pi_1^{a_1} \pi_2^{a_2} \cdots \pi_k^{a_k}$$

is the resolution of α into powers of M-primes,

$$U_x'(\alpha) = \prod_{j=1}^{k} [\sum_i q_i(a_j) x^i],$$

and similarly for the other.

Having found these explicit forms we may examine now the meaning in terms of rational integers of $U_x' P_{-x}' = e$. Let α be as above. Then all the divisors δ of α are

$$\delta = \pi_1^{d_1} \pi_2^{d_2} \cdots \pi_k^{d_k}, \quad 0 \leqq d_j \leqq a_j \qquad (j = 1, \cdots, k),$$

and the divisor δ' conjugate to that just written is

$$\delta' = \pi_1^{a_1-d_1} \pi_2^{a_2-d_2} \cdots \pi_k^{a_k-d_k}.$$

Form $U_x'(\delta)$, $P_{-x}'(\delta')$ as above for every pair (δ, δ'), take the product of the results and sum it with respect to all pairs (δ, δ'). Then this sum has the value 1 or 0 according as the given α is or is not ϵ or, what is the same, according as $(a_1, \cdots, a_k)$ is or is not $(0, \cdots, 0)$. This obviously is a property of compound partitions alone. The like is true, by the theory developed, for every theorem concerning compositions and separations of numerical functions.

By giving to x special values, and by properly choosing the set C which has been taken as the modulus in constructing $U'(\alpha)$, $P'(\alpha)$, we can apply the known product developments of the elliptic theta functions and constants to a great variety of separations of numerical functions. The following will be sufficient as examples. In addition to the functions λ, μ, ρ already defined we shall need $u(\alpha)$, which has the value 1 for all M-integers α; $\theta(\alpha)$, = the number of decompositions of the M-integer α into a pair of coprime M-integer factors. In the last, if $\alpha = \beta\gamma$ is one such decomposition, and $\beta \neq \gamma$, $\gamma\beta$ is counted as another.

As in § 7, end, we consider several sets: C_0 = the set of all integers > 0; C_1 = the set of all odd integers > 0; C_2 = the set of all even integers > 0. By the notation already explained in § 7, if ϕ is any numerical function, ϕ_j' is the separation ϕ' (mod C_j). Hence λ_j', μ_j', θ_j', $\cdots$ $(j = 0, 1, 2)$ are defined. For clearness we recall the necessary generators,

$$\begin{array}{rcl}
\mu, & \text{generator} & 1 - z; \\
\lambda, & \text{“} & (1 + z)^{-1}; \\
(\mu^2), & \text{“} & (1 + z); \\
\theta, & \text{“} & (1 + z)(1 - z)^{-1}; \\
(\theta\lambda), & \text{“} & (1 - z)(1 + z)^{-1}; \\
\nu, & \text{“} & (1 - z)^{-2}; \\
u, & \text{“} & (1 - z)^{-1};
\end{array}$$

$\nu(\alpha)$, as in § 2, is the number of divisors of α. From these generators we see by inspection numerous compositions. For example $\lambda(\mu^2) = e$, so that λ, (μ^2) are reciprocals; $\theta = u(\mu^2)$; $\nu = u^2$; $\mu\nu = u$; etc.

The following illustrates the way in which the remaining examples are written down. We have the well-known product expansion from the theta functions,

$$\Pi\,(1-z^j)(1+z^j)^{-1}=1+2\sum(-1)^j z^{j^2}, \qquad (j=1,2,\cdots),$$

and hence the theorem: the separation $(\theta\lambda)_0'$ vanishes for all arguments $\alpha\neq\epsilon$ not in class x^2 (see beginning of this section for definition of class); if the argument is the M-integer $(a_1^2, a_2^2, \cdots, a_k^2)$, where $a_j\geqq 0$ $(j=1, \cdots, k)$ and $a_1+a_2+\cdots+a_k>0$, the separation has the value $2(-1)^{a_1+a_2+\cdots+a_k}$; if the argument is ϵ the value of the separation is 1. The significance of the parentheses in $(\theta\lambda)$ is as in § 6, end. Note that $(\theta\lambda)'$, for any given C, is not equal to $\theta'\lambda'$.

The theta expansions used in writing down the following are all classical and easily recognized; there is no need to transcribe them. Write for the moment $\phi\equiv\mu^3$. Then the separation ϕ_0' has the value zero if its argument $\alpha\neq\epsilon$ is not in class $x\,(3x\pm 1)/2$; if $\alpha=\epsilon$ the value is 1; if

$$\alpha=(a_1(3a_1\pm 1)/2, \cdots, a_k(3a_k\pm 1)/2)$$

the value is $(-1)^{a_1+\cdots+a_k}$. Next, write $\phi\equiv\mu(\mu^2)$. Then ϕ_0' vanishes for all arguments not in class $x(x-1)/2$, while for such arguments it has the value 1. The next illustrates the occurrence of two sets in one relation. Write $\phi\equiv(\mu^2)^2$. Then the separation (which is a composition of two separations, or a separation of a composition of two functions) $\mu_2'\,\phi_1'$ vanishes if its argument $\alpha\neq\epsilon$ is not in class x^2; if the argument is $(a_1^2, \cdots, a_k^2)$, where $a_j\geqq 0$ $(j=1, \cdots, k)$ and $a_1+\cdots+a_k>0$, the value is 2. Such results can be multiplied indefinitely.

The next is one of a large class of similar results. Let $E(n)$ have its usual meaning as a binary quadratic class number function; write $\chi\equiv\mu^3$, $\psi\equiv(\mu^2)^6$. Then the separation $\chi_2'\ \psi_1'$ for the argument $\alpha=(a_1, \cdots, a_k)$ has the value $(12(-1)^{a_1}E(a_1), \cdots, 12(-1)^{a_k}E(a_k))$. To recall the meaning of $\chi\psi$ in this instance, let $\alpha=\alpha_1\alpha_2\cdots\alpha_9$ be any decomposition of α into a product of nine factors. Then the value of $\chi\psi$ for the argument α is

$$\sum\{\mu(\alpha_1)\mu(\alpha_2)\mu(\alpha_3)\,\mu^2(\alpha_4)\mu^2(\alpha_5)\cdots\mu^2(\alpha_9)\},$$

the sum extending to all sets $\alpha_1, \alpha_2, \cdots, \alpha_9$. These examples, which can be continued indefinitely, are enough to show the meanings of previous definitions and theorems, which is all that is necessary here. If instead of specializing x to be ± 1 in $U_x(\alpha)$, $P_x(\alpha)$ as in these examples for $\lambda, \mu, \cdots$, we take $x=\exp(\pm iy)$, where i is the imaginary unit, and combine results as directed by the product expansions of elliptic and theta functions of y, we reach theorems concerning compositions of separations, or vice-versa, which involve sines and cosines of integer multiples of y. These results can then be paraphrased, as in previous papers, into separation-composition theorems containing functions wholly arbitrary except as to parity. All theorems such as the above concerning specific functions λ, μ, $\cdots$ are special cases of such general paraphrased separation-compositions.

12. *Addition and subtraction theorems.* Several of the examples in the preceding section illustrate a general situation concerning separations with respect to sets that are either the logical sum or product, or the logical difference of given sets. If the set C contains ($\equiv$ includes) the set C_j, we write $C \mid C_j$, as in the usual generalization of arithmetical division in the theory of ideals. Sets having no element common are called *coprime*; the *sum* $C_i + C_j$ of two sets is their logical sum; their *product* C_iC_j is their logical product. Similarly for any number of sets. If C_j $(j = 1, \cdots, s)$ are such that every pair of them are coprime, the s sets are called coprime. The condition that C_j $(j = 1, \cdots, s)$ be coprime is $C_1C_2 \cdots C_s =$ the null set.

Let C_j $(j = 1, \cdots, s)$ be s coprime sets, and write $C = C_1 + C_2 + \cdots + C_s$. Let ϕ' be the separation of ϕ (mod C), and ϕ_j' that of ϕ (mod C_j) $(j = 1, \cdots, s)$. Then, from a consideration of the respective generators, it follows at once that

$$\phi' = \phi_1' \phi_2' \cdots \phi_s'; \tag{26}$$

that is, *the separation of any numerical function modulo the sum of s coprime sets is equal to the composition of the separations of s functions, each equal to the given function, taken to the s moduli composing the sum.*

Let $C_a, C_b, \cdots, C_t$ be any distinct $s - r$ sets, $r > 0$, chosen from the sets C_j $(j = 1, \cdots, s)$, and let the remaining r sets be $C_\alpha, C_\beta, \cdots, C_\tau$, so that in some order the $C_a, \cdots, C_t, C_\alpha, \cdots, C_\tau$ are identical with the $C_1, \cdots, C_s$. Let ϕ' be as before, and let ϕ_h' be the separation (mod C_h) of ϕ $(h = a, b, \cdots, t, \alpha, \beta, \cdots, \tau)$; also write ψ_h for the reciprocal of ϕ_h, and let ψ_h' be the separation of ψ_h. Write $\Gamma = C - C_\alpha - C_\beta - \cdots - C_\tau$; namely, Γ is the set which remains when from C are deleted all elements of $C_\alpha + C_\beta + \cdots + C_\tau$, and let ϕ'' be the separation of ϕ (mod Γ). Then

$$\phi'' = \phi_a' \phi_b' \cdots \phi_c' = \phi' \psi_\alpha' \psi_\beta' \cdots \psi_\gamma'. \tag{27}$$

On the basis of such considerations we can apply the arithmetic of logic developed in a forthcoming paper to construct an arithmetic (including the theory of congruences) of separations with reference to their sets.

Lagrange Resolvents in Euclidean Geometry.

By Leonard M. Blumenthal.

Introduction.

1. *An Invariant and its Geometrical Interpretation.* Consider the combination

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4)$$

of the four points x_1, x_2, x_3, x_4, in the complex plane. Here $\bar{x}_i$ is the conjugate of x_i. It is seen at once that this function is invariant under the translation $y = x + a$, and also under the rotation $y = tx$ where t is a turn, a complex quantity with unit modulus. For

$$\begin{aligned}(y_1 - y_3)(\bar{y}_2 - \bar{y}_4) &= (tx_1 + a - tx_3 - a)(\bar{x}_2/t + a - \bar{x}_4/t - a) \\ &= (x_1 - x_3)(\bar{x}_2 - \bar{x}_4) .\end{aligned}$$

The four points x_1, x_2, x_3, x_4 determine an ordered quadrangle. Denoting by δ_{13} and δ_{24} the lengths of the strokes $x_1 - x_3$ and $x_2 - x_4$ respectively, we may write

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = \delta_{13}\delta_{24}e^{i\theta} = \delta_{13}\delta_{24}\cos\theta + i\,\delta_{13}\delta_{24}\sin\theta,$$

where θ is the angle which $x_1 - x_3$ makes with $x_2 - x_4$. If x_0 is the point of intersection of $x_1 - x_3$ and $x_2 - x_4$,

$$\delta_{13} = \delta_{01} + \delta_{03}, \qquad \delta_{24} = \delta_{02} + \delta_{04},$$

and

$$\delta_{13}\delta_{24}\cos\theta = \tfrac{1}{2}(\delta_{12}^2 - \delta_{23}^2 + \delta_{34}^2 - \delta_{41}^2) = 2N.$$

We shall speak of N as the *norm* of the quadrangle.

Also we have

$$\delta_{13}\delta_{24}\sin\theta = 2A,$$

where A denotes the area of the quadrangle. Hence we may write

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = 2(N + iA);$$

i. e., the invariant we set out to examine is seen to express the area and norm of the quadrangle determined by the points x_i $(i=1, 2, 3, 4)$. We shall refer to the expression $2(N+iA)$ as the *norm-area* of the polygon.

If we write $x_1 - x_3 = v$ and $x_2 - x_4 = u$, the invariant takes the form

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) = v\bar{u};$$

and since each of v and u is invariant under translations we have the following

THEOREM. *The norm-area of a quadrangle is unaltered under a translation of either diagonal.*

2. *The Lagrange Resolvent.* In a memoir* devoted to the fundamental principles of the solutions of cubic and quartic equations, Lagrange introduced the resolvent

$$v = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \cdots + \epsilon^{n-1} x_n$$

where ϵ is a complex nth root of unity. By this notation we shall further understand ϵ to be a primitive root.

It may be observed at once that the resolvent v is invariant under translations $y = x + a$, since

$$\epsilon^{n-1} + \epsilon^{n-2} + \cdots + \epsilon + 1 = 0$$

and further, that combinations like $v\bar{u}$ (u being a similar resolvent) are invariant under rotations $y = tx$.

An extension of the example in § 1 suggests itself. We seek to know when it is possible to express the norm-area of an ordered polygon in terms of the Lagrange resolvents (and their conjugates) of polygons into which the given polygon is decomposed. Viewed from another angle, we wish to know "how much" of a polygon may be translated without altering its norm-area.

The investigations presented here prove that the norm-area of a $2n$-gon may be expressed in terms of the Lagrange resolvents of the two n-gons of which it is composd. The coefficients in the expression are given explicitly as very simple functions of n.

It is shown that such an expression in terms of the diagonal strokes and their conjugates is possible only in the case of the quadrangle.

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* Memoirs of Berlin Academy, 1769; reprinted in *Oeuvres de Lagrange* (Paris, 1868), Vol. 3, p. 207.

It is further proved that the areas of polygons of mn sides $(m \neq 2)$ are not expressible in terms of the Lagrange resolvents of the m n-gons. It may be noted that the areas of polygons are obtained without resort to the "triangulation" process; that is, the areas of polygons are obtained as the imaginary part of the expression for the norm-area of the polygons.

It is finally shown that the combination $v_i\bar{u}_i$ is the only invariant combination of the Lagrange resolvents of the polygons that is unaltered by the group of substitutions $(1, 3, 5, \cdots, 2n-1)(2, 4, 6, \cdots, 2n)$. The expression for norm-area may be referred to as a *lineo-linear* invariant of two n-gons.

Part One. Polygons of 2n Sides.

3. *The Hexagon as Two Triangles.* As indicated in Fig. 1, the hexagon is considered as being formed by the vertices of the two triangles $x_1x_3x_5$ and $x_4x_6x_2$.

Now the determinant*

$$\begin{vmatrix} x_1 & x_3 & x_5 \\ \bar{x}_4 & \bar{x}_6 & \bar{x}_2 \\ 1 & 1 & 1 \end{vmatrix}$$

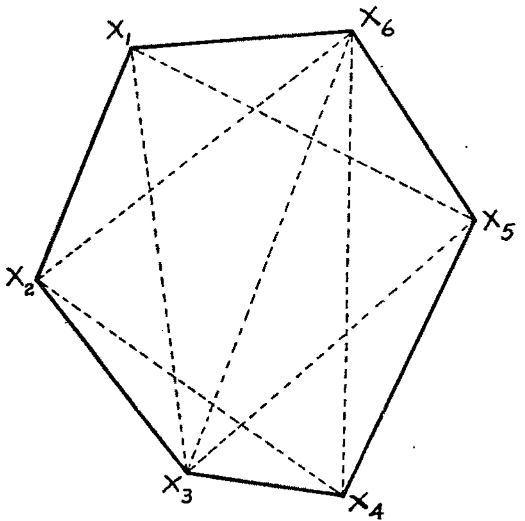

Fig. 1.

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* The determinant set equal to zero is the condition that the two triangles be negatively similar.

is seen to be unaltered by the translation $y = x + a$ and also by the rotation $y = tx$. Denoting this determinant by Δ, we have

$$\begin{aligned}-\Delta &= x_1\bar{x}_2 - \bar{x}_2x_3 + x_3\bar{x}_4 - \bar{x}_4x_5 + x_5\bar{x}_6 - \bar{x}_6x_1 \\ &= (x_1 - x_3)(\bar{x}_2 - \bar{x}_6) + (x_3 - x_5)(\bar{x}_4 - \bar{x}_6).\end{aligned}$$

We may conclude from § 1 that

$$\begin{aligned}(x_1 - x_3)(\bar{x}_2 - \bar{x}_6) &= \tfrac{1}{2}(\delta^2_{12} - \delta^2_{23} + \delta^2_{36} - \delta^2_{61}) + 2iA_1 \\ (x_3 - x_5)(\bar{x}_4 - \bar{x}_6) &= \tfrac{1}{2}(\delta^2_{34} - \delta^2_{45} + \delta^2_{56} - \delta^2_{36}) + 2iA_2\end{aligned}$$

where A_1 and A_2 are respectively the areas of quadrangles $x_1x_2x_3x_6$ and $x_3x_4x_5x_6$. Adding, we obtain

$$-\Delta = 2(N + iA)$$

where N and A are the norm and area, respectively, of the hexagon.

To express Δ in terms of the Lagrange resolvents of the two triangles,

$$\begin{aligned}v_1 &= x_1 + \omega x_3 + \omega^2 x_5; \quad & u_1 &= x_4 + \omega x_6 + \omega^2 x_2 \\ v_2 &= x_1 + \omega^2 x_3 + \omega x_5; \quad & u_2 &= x_4 + \omega^2 x_6 + \omega x_2\end{aligned}$$

where ω is an imaginary cube root of unity, we note that

$$\Delta = \begin{vmatrix} x_1 + \omega x_3 + \omega^2 x_5, & x_3 - x_5, & x_5 \\ \bar{x}_4 + \omega\bar{x}_6 + \omega^2\bar{x}_2, & \bar{x}_6 - \bar{x}_2, & \bar{x}_2 \\ 0 \quad , & 0 \quad , & 1 \end{vmatrix};$$

whence, multiplying the second column by $-\omega(1 - \omega)$ and adding to it the first column yields

$$\Delta = -[1/\omega(1 - \omega)]\,(v_1\bar{u}_1 - v_2\bar{u}_2)$$

and hence *

$$[1/\omega(1 - \omega)]\,(v_1\bar{u}_1 - v_2\bar{u}_2) = 2(N + iA).$$

Since the resolvents are invariant under translations, we have proved the

---

[*] Looking towards the later developpment of the paper we note here that, for

$$u_1 = x_2 + \omega x_4 + \omega^2 x_6 \text{ and } u_2 = x_2 + \omega^2 x_4 + \omega x_6,$$

we have

$$[1/(1 - \omega)]\,(v_1\bar{u}_1 - \omega v_2\bar{u}_2) = 2(N + iA),$$

which may be obtained from the expression above by substituting $\omega\bar{u}_1$ for $\bar{u}_1$ and $\omega^2\bar{u}_2$ for $\bar{u}_2$.

THEOREM. *The norm-area of a hexagon is unaltered by translating either of the component triangles.*

4. *The Decagon.* As the analysis to be employed in the case of the $2n$-gon can be well illustrated in the treatment of the decagon, we shall examine this case in some detail. Splitting the decagon into the four quadrangles indicated in the figure, we obtain

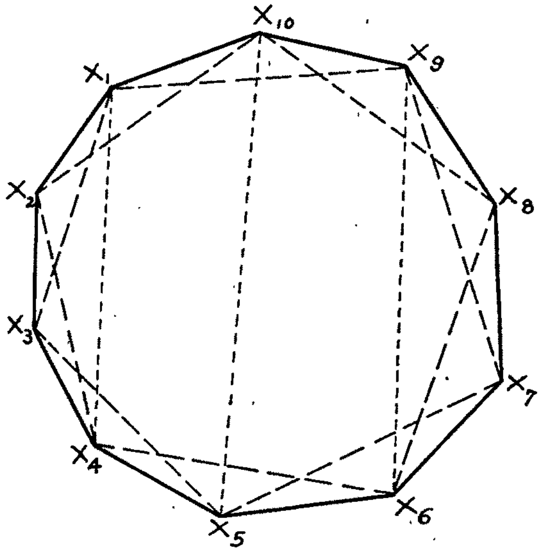

Fig. 2.

$$(x_1 - x_3)(\overline{x}_2 - \overline{x}_4) + (x_1 - x_5)(\overline{x}_4 - \overline{x}_{10}) + (x_5 - x_9)(\overline{x}_6 - \overline{x}_{10}) + (x_9 - x_7)(\overline{x}_6 - \overline{x}_8) =$$

$$\sum_{k=1}^{5} \overline{x}_{2k}(x_{2k-1} - x_{2k+1}) = 2(N + iA)$$

where N and A are the norm and area, respectively, of the decagon. The Lagrange resolvents considered are

$$v_1 = x_1 + \epsilon x_3 + \epsilon^2 x_5 + \epsilon^3 x_7 + \epsilon^4 x_9; \quad u_1 = x_2 + \epsilon x_4 + \epsilon^2 x_6 + \epsilon^3 x_8 + \epsilon^4 x_{10}$$

$$v_2 = x_1 + \epsilon^2 x_3 + \epsilon^4 x_5 + \epsilon x_7 + \epsilon^3 x_9; \quad u_2 = x_2 + \epsilon^2 x_4 + \epsilon^4 x_6 + \epsilon x_8 + \epsilon^3 x_{10}$$

$$v_3 = x_1 + \epsilon^3 x_3 + \epsilon x_5 + \epsilon^4 x_7 + \epsilon^2 x_9; \quad u_3 = x_2 + \epsilon^3 x_4 + \epsilon x_6 + \epsilon^4 x_8 + \epsilon^2 x_{10}$$

$$v_4 = x_1 + \epsilon^4 x_3 + \epsilon^3 x_5 + \epsilon^2 x_7 + \epsilon x_9; \quad u_4 = x_2 + \epsilon^4 x_4 + \epsilon^3 x_6 + \epsilon^2 x_8 + \epsilon x_{10}$$

where ϵ is a primitive fifth root of unity.

The products $v_i\bar{u}_i$ $(i=1,2,3,4)$ can be best written in the matrix from

$$v_1\bar{u}_1 = \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & x_7 & x_9 \\ \hline \bar{x}_2 & 1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\ \bar{x}_4 & \epsilon^4 & 1 & \epsilon & \epsilon^2 & \epsilon^3 \\ \bar{x}_6 & \epsilon^3 & \epsilon^4 & 1 & \epsilon & \epsilon^2 \\ \bar{x}_8 & \epsilon^2 & \epsilon^3 & \epsilon^4 & 1 & \epsilon \\ \bar{x}_{10} & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 & 1 \end{array}$$

Noting that the determinant of the matrix is a circulant,* we write symbolically,

$$v_1\bar{u}_1 = C(1\,\epsilon\,\epsilon^2\,\epsilon^3\,\epsilon^4); \qquad v_3\bar{u}_3 = C(1\,\epsilon^3\,\epsilon\,\epsilon^4\,\epsilon^2);$$
$$v_2\bar{u}_2 = C(1\,\epsilon^2\,\epsilon^4\,\epsilon\,\epsilon^3); \qquad v_4\bar{u}_4 = C(1\,\epsilon^4\,\epsilon^3\,\epsilon^2\,\epsilon).$$

Writing the function of the vertices $x_1, x_2, \cdots, x_{10}$ obtained as representing the norm-area of the decagon, in matrix form

$$\text{(I)} \qquad \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & x_7 & x_9 \\ \hline \bar{x}_2 & 1 & -1 & 0 & 0 & 0 \\ \bar{x}_4 & 0 & 1 & -1 & 0 & 0 \\ \bar{x}_6 & 0 & 0 & 1 & -1 & 0 \\ \bar{x}_8 & 0 & 0 & 0 & 1 & -1 \\ \bar{x}_{10} & -1 & 0 & 0 & 0 & 1 \end{array}$$

it is clear that our problem resolves itself into finding multipliers A_1, A_2, A_3, A_4 such that the matrix

$$A_1v_1\bar{u}_1 + A_2v_2\bar{u}_2 + A_3v_3\bar{u}_3 + A_4v_4\bar{u}_4$$

will be identical with the matrix (I). Equating coefficients leads to only four independent equations:

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 &= 1, \\ \epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \epsilon^4 A_4 &= -1, \\ \epsilon^2 A_1 + \epsilon^4 A_2 + \epsilon A_3 + \epsilon^3 A_4 &= 0, \\ \epsilon^3 A_1 + \epsilon A_2 + \epsilon^4 A_3 + \epsilon^2 A_4 &= 0; \end{aligned}$$

---

* We should expect the products $v_i\bar{u}_i$ to be of this form, since, it is well-known that if $1, i_1, i_2, \cdots, i_n$ be the $n+1$ roots of $x^{n+1}-1=0$, then for any set $A, A_1, \cdots, A_n$ we have (Spottiswoode, *Crelle's Journal für Mathematik*, Vol. 51, pp. 209-271)

$$\begin{vmatrix} A & A_1 & A_2 \cdots & A_n \\ A_n & A & A_1 \cdots & A_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ A_1 & A_2 & A_3 \cdots & A \end{vmatrix} = (-1)^{n(n-1)/2}(A+A_1+\cdots+A_n)(A+i_1A_1+i_1^2A_2+\cdots+i_1^nA_n)\cdots(A+i_nA_1+i_n^2A_2+\cdots+i_n^nA_n).$$

whence

$$5A_1 = 1 - \epsilon^4, \quad 5A_2 = 1 - \epsilon^3, \quad 5A_3 = 1 - \epsilon^2, \quad 5A_4 = 1 - \epsilon.$$

Hence

$$(1/5)\left[(1-\epsilon^4)v_1\bar{u}_1 + (1-\epsilon^3)v_2\bar{u}_2 + (1-\epsilon^2)v_3\bar{u}_3 + (1-\epsilon)v_4\bar{u}_4\right] = 2(N + iA)$$

is the desired expression for the norm-area of the decagon in terms of the Lagrange resolvents of the two pentagons whose vertices form the decagon. Since the resolvents are invariants under translations, we have proved the

THEOREM. *The norm-area of a decagon is unaltered by translating either of its component pentagons.*

5. *General Case. The 2n-gon.* We write the resolvents of each n-gon in the compact form

$$v_j = \sum_{k=0}^{n-1} \epsilon^{jk} x_{2k+1} \qquad (j = 1, 2, \cdots, n-1)$$

$$\bar{u}_j = \sum_{k=0}^{n-1} \epsilon^{j(n-k)} \bar{x}_{2(k+1)} \qquad (j = 1, 2, \cdots, n-1)$$

Now the $2n$-gon may be divided into $(n-1)$ quadrangles whose vertices are given in the rows of

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2n & 1 & 4 & 5 \\
2n-1 & 2n & 5 & 6 \\
2n-2 & 2n-1 & 6 & 7 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

and the areas of which exhaust the area of the $2n$-gon. We have

$$x_1\bar{x}_2 - \bar{x}_2x_3 + x_3\bar{x}_4 - \cdots + x_{2n-1}\bar{x}_{2n} - \bar{x}_{2n}x_1 =$$

$$\sum_{k=1}^{n} \bar{x}_{2k}(x_{2k-1} - x_{2k+1}) = 2(N + iA).$$

In matrix form, this is

$$\text{(II)} \qquad \begin{array}{c|cccccccc}
 & x_1 & x_3 & x_5 & x_7 & \cdots & x_{2n-3} & x_{2n-1} \\
\hline
\bar{x}_2 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
\bar{x}_4 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\
\bar{x}_6 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\bar{x}_8 & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\bar{x}_{2n-2} & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\
\bar{x}_{2n} & -1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}$$

The products $v_i\bar{u}_i$ we write, symbolically, as

$$
\begin{aligned}
v_1\bar{u}_1 &= C(1\ \epsilon\ \epsilon^2 \cdots \epsilon^{n-1}) \\
v_2\bar{u}_2 &= C(1\ \epsilon^2\ \epsilon^4 \cdots \epsilon^{(2n-1)}) \\
&\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\
v_{n-1}\bar{u}_{n-1} &= C(1\ \epsilon^{n-1}\ \epsilon^{2(n-1)} \cdots \epsilon^{(n-1)(n-1)})
\end{aligned}
$$

Again our problem is to find multipliers $A_i\quad (i = 1, 2, \cdots, n-1)$ so that the matrix

$$\sum_{i=1}^{n-1} A_i v_i \bar{u}_i$$

will be identical with the matrix (II).

We note that ϵ being a primitive root of $x^n - 1 = 0$ corresponding elements of corresponding rows of the matrices $v_i\bar{u}_i$ are

$$1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$$

in some order. The desired multipliers A_i are determined by the $n-1$ linear, non-homogeneous equations;

$$
\begin{aligned}
&\epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \cdots + \epsilon^{n-1} A_{n-1} = -1 \\
&\epsilon^2 A_1 + \epsilon^4 A_2 + \epsilon^6 A_3 + \cdots + \epsilon^{2(n-1)} A_{n-1} = 0 \\
&\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\
&\epsilon^{n-1} A_1 + \epsilon^{2(n-1)} A_2 + \epsilon^{3(n-1)} A_3 + \cdots\ \epsilon^{(n-1)(n-1)} A_{n-1} = 0.
\end{aligned}
$$

These equations possess a unique solution, for the determinant of the coefficients may be shown to be different from zero (indeed, evaluated) in the following manner.*

$$
\begin{vmatrix}
\epsilon & \epsilon^2 & \epsilon^3 & \cdots & \epsilon^{n-1} \\
\epsilon^2 & \epsilon^4 & \epsilon^6 & \cdots & \epsilon^{2(n-1)} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\epsilon^{n-1} & \epsilon^{2(n-1)} & \cdots & & \epsilon^{(n-1)(n-1)}
\end{vmatrix}
= \epsilon\ \epsilon^2\ \epsilon^3 \cdots \epsilon^{n-1}\ \zeta^{1/2}(\epsilon, \epsilon^2, \cdots, \epsilon^{n-1})
$$

the symbol on the right hand side being Sylvester's notation for a continued product of the differences $(\epsilon - \epsilon^2)\ (\epsilon - \epsilon^3) \cdots$. The function partakes of the nature of a square root, having two values corresponding to various permutations of $\epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$.

Since

$$\epsilon\ \epsilon^2\ \epsilon^3 \cdots \epsilon^{n-1} = \epsilon^{n(n-1)/2} = 1$$

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* *American Mathematical Monthly*, Vol. 32 (1925), p. 522.

we have the determinant of the coefficients equal to the zeta-ic function.

The square of this function being the discriminant of the equation

$$x^{n-1}+x^{n-2}+\cdots+x+1=0 \tag{1}$$

we have

$$\zeta(\epsilon,\epsilon^2,\cdots,\epsilon^{n-1})=\begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-2} \\ s_1 & s_2 & s_3 & \cdots & s_{n-1} \\ s_2 & s_3 & s_4 & \cdots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-2} & s_{n-1} & s_n & \cdots & s_{2(n-2)} \end{vmatrix}$$

where $s_i=$ sum of the ith powers of the roots of (1).

Since $\epsilon,\epsilon^2,\cdots,\epsilon^{n-1}$ are roots of (1)

$$s_0=n-1,\ s_1=s_2=\cdots=s_{n-1}=-1,\ s_n=n-1,\ s_{n+1}=-1,\cdots.$$

Hence

$$\zeta(\epsilon,\epsilon^2,\cdots,\epsilon^{n-1})=\begin{vmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 \\ -1 & -1 & -1 & \cdots & n-1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & n-1 & \cdots & -1 & -1 \end{vmatrix}$$

This determinant can be evaluated readily, yielding

$$\zeta(\epsilon,\epsilon^2,\cdots,\epsilon^{n-1})=(-1)^{(n-1)(n-2)/2}n^{n-2}.$$

We proceed to solve the set of $n-1$ equations in the A's.

Evidently, the product of A_i by $\zeta^{1/2}(\epsilon,\epsilon^2,\cdots,\epsilon^{n-1})$ is equal to the original determinant where the elements of the ith column have been replaced by $-1, 0, 0,\cdots, 0$, or, after some reductions

$$A_i=(1-\epsilon^{n-i})/\prod_{i=1}^{n-1}(1-\epsilon^i).$$

But, since ϵ is a primitive root of $x^n-1=0$, we have

$$(x-\epsilon)(x-\epsilon^2)(x-\epsilon^3)\cdots(x-\epsilon^{n-1})\equiv x^{n-1}+x^{n-2}+\cdots+x+1.$$

Letting $x=1$, we obtain

$$(1-\epsilon)(1-\epsilon^2)\cdots(1-\epsilon^{n-1})=n.$$

Hence

$$A_i=(1/n)(1-\epsilon^{n-i}) \qquad (i=1,2,\cdots,n-1)$$

are the solutions of the sets of equations, and therefore are the multipliers we set out to find. Whence the

THEOREM. *The norm-area of a 2n-gon expressed in terms of the Lagrange resolvents of the two component n-gons is given by*

$$(1/n)\sum_{i=1}^{n-1}(1-\epsilon^{n-i})v_i\bar{u}_i.$$

Since the resolvents are invariant under translations it follows immediately that

THEOREM. *The norm-area of a 2n-gon is unaltered by translating either of the component n-gons.*

Part Two. Strokes.

6. In the previous sections a $2n$-gon was considered as being composed of two n-gons, and expressions were found giving the norm-area in terms of the Lagrange resolvents of each n-gon.

In this part, the $2n$-gon is thought of as being composed of n 2-gons (i. e., n strokes), and we are concerned in ascertaining whether or not the norm-area of the $2n$-gon can be expressed in terms of the n strokes and their conjugates. As before, this is the problem of determining if the norm-area of a $2n$-gon is invariant under a translation of one of its component strokes.

We have already seen in § 1 that the norm-area of a quadrangle is expressible in terms of its diagonal strokes. We take up the case of the hexagon.

We have from § 3

$$(1)\qquad x_1\bar{x}_2-\bar{x}_2x_3+x_3\bar{x}_4-\bar{x}_4x_5+x_5\bar{x}_6-\bar{x}_6x_1=2(N+iA).$$

Let

$$u_1=x_1-x_4;\quad u_2=x_2-x_5;\quad u_3=x_3-x_6$$
$$U_1=x_1+x_4;\quad U_2=x_2+x_5;\quad U_3=x_3+x_6$$

whence

$$x_1=\tfrac{1}{2}(u_1+U_1),\quad x_2=\tfrac{1}{2}(u_2+U_2),\quad x_3=\tfrac{1}{2}(u_3+U_3)$$
$$x_4=\tfrac{1}{2}(U_1-u_1),\quad x_5=\tfrac{1}{2}(U_2-u_2),\quad x_6=\tfrac{1}{2}(U_3-u_3)$$

and substitution in (1) yields *

$$(u_2-u_3+U_2-U_3)(\bar{u}_2+\bar{u}_1-\bar{U}_2+\bar{U}_1)$$
$$+(u_2+u_1+U_2-U_1)(\bar{u}_3-\bar{u}_2+\bar{U}_2-\bar{U}_3)$$
$$=2(N+iA).$$

---

* Note that though none of U_1, U_2, U_3 is invariant under translations, the combinations U_2-U_3, U_2-U_1 etc. in which they enter the equation *are* invariant. Hence, of course, N and A are unchanged by translating or rotating the entire hexagon.

The terms containing U_1, U_2, U_3 will not vanish upon expanding the left-hand side of the last equation and hence we have the following

THEOREM. *The norm-area of a hexagon is not expressible in terms of its three diagonal strokes.*

COROLLARY. *The norm-area of a hexagon is not invariant under a translation of one of its diagonal strokes.*

Consider next the case of the octagon. Letting

$$\begin{aligned} u_1 &= x_1 - x_5, & u_2 &= x_2 - x_6, & u_3 &= x_3 - x_7, & u_4 &= x_4 - x_8 \\ U_1 &= x_1 + x_5, & U_2 &= x_2 + x_6, & U_3 &= x_3 + x_7, & U_4 &= x_4 + x_8 \end{aligned}$$

we get

$$\begin{gathered} x_1 = \tfrac{1}{2}(u_1 + U_1), \quad x_2 = \tfrac{1}{2}(u_2 + U_2), \quad x_3 = \tfrac{1}{2}(u_3 + U_3) \\ x_4 = \tfrac{1}{2}(u_4 + U_4), \quad x_5 = \tfrac{1}{2}(U_1 - u_1), \quad x_6 = \tfrac{1}{2}(U_2 - u_2) \\ x_7 = \tfrac{1}{2}(U_3 - u_3), \quad x_8 = \tfrac{1}{2}(U_4 - u_4). \end{gathered}$$

Substituting these values in

$$(x_1 - x_3)(\bar{x}_2 - \bar{x}_4) + (x_1 - x_5)\ (\bar{x}_4 - \bar{x}_8) + (x_5 - x_7)\ (\bar{x}_6 - \bar{x}_8) = 2(N + iA)$$

we obtain

$$(U_1 - U_3)(\bar{U}_2 - \bar{U}_4) + (u_1 - u_3)(\bar{u}_2 - \bar{u}_4) + 2u_1\bar{u}_4 = 4(N + iA).$$

As in the case of the hexagon, the terms in U_1, U_2, U_3, U_4 are present and hence the norm-area of the octagon is not invariant under a translation of a single diagonal stroke.

Note, however, that if the same translations are applied to the two strokes $x_1 - x_5$ and $x_3 - x_7$, the rest of the figure being left unaltered, the norm-area of the octagon *is* invariant (since the difference $U_1 - U_3$ is invariant). The same statement may be made with regard to the strokes $x_2 - x_6$ and $x_4 - x_8$.

For the case of the $2n$-gon, we give the formula

$$\sum_{k=1}^{n} (\bar{u}_{2k} + \bar{U}_{2k})\ [(u_{2k-1} + U_{2k-1}) - (u_{2k+1} + U_{2k+1})] = 2(N + iA)$$

where $u_{n+i} \equiv - u_i$; $U_{n+i} \equiv U_i$ $(i = 1, 2, \cdots, n)$ and all subscripts are first to be reduced (mod $2n$), and hence conclude that only in the case of the quadrangle is the norm-area expressible in terms of the diagonal strokes alone.

7. *The mn-gon.* We give the general formula for the area of the mn-gon

$$\sum_{i=0}^{n-1} (\delta_{i,j} v_{i,j} \bar{v}_{i,j+1} - \delta_{i,j} \bar{v}_{i,j} v_{i,j+1}) = 4niA \qquad (j = 1, 2, \cdots, m)$$

where $\delta_{im} \equiv \epsilon^{i}$ $(i = 1, 2, \cdots, n-1)$; $\delta_{i,j} \equiv 1$ $(j \neq m)$ and

$$v_{ij} = x_j + \epsilon^{i} x_{m+j} + \epsilon^{2i} x_{2m+j} + \cdots + \epsilon^{i(n-1)} x_{(n-1)m+j}$$

and conclude that only in the case of the $2n$-gon is the area expressible in terms of the Lagrange resolvents of the m n-gons. Hence, the area of an mn-gon is not invariant under a translation on one of its component n-gons for $m \neq 2$.

8. We have seen that

$$\text{(I)} \qquad (1/n) \sum_{i=1}^{n-1} (1 - \epsilon^{n-i}) v_i \bar{u}_i = 2(N + iA)$$

is an invariant under translations and rotations of the two polygons with vertices $x_1, x_3, \cdots, x_{2n-1}$ and $x_2, x_4, \cdots, x_{2n}$. It is easily seen that (I) is also invariant under the substitutions $(1\,3\,5 \cdots 2n-1) \cdot (2\,4\,6 \cdots 2n)$ applied simultaneously to each polygon.

Moreover, the combination $v_i \bar{u}_i$ is the *only* invariant combination of the Lagrange resolvents of the polygon that is unaltered by the above substitutions for

$$v_i \bar{u}_j = (x_1 + \epsilon^{i} x_3 + \epsilon^{2i} x_5 + \cdots + \epsilon^{(n-1)i} x_{2n-1})\,(\bar{x}_2 + \epsilon^{j(n-1)} \bar{x}_4 + \cdots + \epsilon^{j} \bar{x}_{2n}).$$

Apply the substitution and denote by $V_i \bar{U}_j$ the transformed $v_i \bar{u}_j$.

We have

$$\begin{aligned} V_i \bar{U}_j &= (x_3 + \epsilon^{i} x_5 + \epsilon^{2i} x_7 + \cdots + \epsilon^{(n-1)i} x_1)\,(\bar{x}_4 + \epsilon^{j(n-1)} \bar{x}_6 + \cdots + \epsilon^{j} \bar{x}_2) \\ &= \epsilon^{(n-1)i} (x_1 + \epsilon^{i} x_3 + \epsilon^{2i} x_5 + \cdots + \epsilon^{(n-1)i} x_{2n-1}) \cdot \epsilon^{j} (\bar{x}_2 + \epsilon^{j(nn)} x_n + \cdots + \epsilon^{j} \\ &= \epsilon^{j-i} v_i \bar{u}_j \end{aligned}$$

and thus we see that only for $j = i$ is the combination absolutely invariant.

Another invariant of the two polygons, *independent of* (I) *is*

$$\sum_{i=1}^{n-1} v_i \bar{u}_i = \begin{array}{c|ccccc} & x_1 & x_3 & x_5 & \cdots & x_{2n-1} \\ \hline \bar{x}_2 & n-1 & -1 & -1 & \cdots & -1 \\ \bar{x}_4 & -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{x}_{2n} & -1 & -1 & -1 & \cdots & n-1 \end{array}$$

The Ovals of the Plane Sextic Curve.

By J. H. McDonald.

The maximum number of ovals that a plane sextic can have is eleven. The statement was made without proof by Hilbert that the ovals cannot be external to each other. The proof of this theorem by Wright (*American Journal of Mathematics,* Vol. 29, 1907, p. 305) is complicated by the introduction of considerations which are unnecessary; in particular the discussions relating to the shrinkage of an oval and to the approach of the variable sextic and cubic may be omitted or replaced by a simpler treatment. There is a lack of rigor throughout; the existence of the limit curve with eleven acnodes on which the conclusion depends is not clearly demonstrated. It seems desirable to complete the proof of this theorem which is of some interest not only in itself but on account of more general questions which it suggests. An example of this is given at the end of this communication in the form of a theorem regarding the situation of the intersections of two nodal cubics.

It may be taken as proved that if a sextic with eleven ovals external to each other exists, an infinite number of sextics may be found having ten acnodes and an oval. Let these sextics have equations $S_n = 0$ and let them be designated by S_n. From each S_n a curve $S_n + \kappa C_1^n C_2^n = 0$ is formed as follows. Let A^n, B^n be two of the acnodes of S_n and C_1^n, C_2^n be cubics through all the acnodes except B^n, A^n respectively. The signs are taken so that $C_1^n C_2^n > 0$ for points on the oval of S_n. Then for small positive values of κ, $S_n + \kappa C_1^n C_2^n = 0$ is a sextic having eight acnodes, two small ovals near A^n and B^n lying in the part of the plane where $C_1^n C_2^n < 0$, and an oval lying within the oval of S_n. The process of reduction starts again with this curve and leads to S_{n+1}, and for sufficiently small values of k the ten acnodes of S_{n+1} are the original eight and two others lying within the ovals arising from A^n and B^n. The last oval is containually diminished and it may first be proved that as n increases the cubics C_1^n, C_2^n, cannot approach this oval. The cubics C_1^{n+1}, C_2^{n+1} pass through the nine points of intersection of C_1^n, C_2^n since they belong to the pencil of cubics through the eight fixed acnodes of the sextic and they also pass through the points A^{n+1}, B^{n+1} to which the ovals derived from A^n, B^n have shrunk. But these lie in the region where $C_1^nC_2^n < 0$ so that if $C_1^{n+1} = \lambda C_1^n + \mu C_2^n$ then λ and μ have the same sign and the entire course of $C_1^{n+1} = 0$ is where $C_1^nC_2^n < 0$ and the argument may be repeated for $C_2^{n+1} = 0$. Since the ovals of S_n, S_{n+1} lie in the region where $C_1^nC_2^n > 0$ the shortest distance from C_1^{n+1} or C_2^{n+1} to the ovals is greater than the shortest distance from C_1^n, C_2^n to the ovals.

The possibility must be considered that in the course of the reduction there may be a confluence of the acnodes of $S_n = 0$. Suppose first that the two variable acnodes converge to a point which is not one of the fixed acnodes. This can only occur if they are accessible to each other without crossing C_1^n, C_2^n. This convergence can be avoided as follows. Denote the variable acnodes by A and B and the fixed acnodes by $12\ldots8$, and determine two cubics by the points $12\ldots7AB$ and $12\ldots7A8$, and a second pair of cubics by the points $12\ldots7AB$ and $12\ldots7B8$, the pairs having a cubic in common. Then if the ovals arising from B and 8 by the use of the first pair of cubics are accessible to each other without crossing the cubics, those arising from A and 8 when the second pair is used are inaccessible to each other without crossing the pair of cubics used. This will certainly be true if the points A, B are sufficiently close which may be assumed since they are supposed to converge. This arrangement being adopted the convergence of A and B can henceforth, if it occurs, only be to one of the fixed acnodes. Suppose it to occur and denote the fixed acnode by 9. Then a cubic through six of the seven acnodes which are remote from the convergent set and through $A9B$ cannot have a double point at A, 9 or B, for that cubic would intersect the sextic in twenty points and so must constitute a part of it. This is impossible since the sextic consists of ten acnodes and an oval. The three convergent acnodes converge in such a way that the cubics through the acnodes $12\ldots7$ and through any two of A, 9, B must in these points make angles with each other approaching zero. A pair of these can always be taken so that the ovals arising from the points omitted in the determination and called A^n and B^n originally lie on opposite sides of the cubics.

To complete the proof it is necessary to arrive at a sextic having eleven acnodes. Let S_n, C_1^n, C_2^n be normalized so that the sum of the squares of the coefficients of each curve equals unity. Then for any curve $S_n + \kappa C_1^nC_2^n = 0$ there is a range of positive values of k such that the further reduction may be made. Let k_n be the upper limit of the values of k for which further reduction is possible, or if k has no upper limit let k_n denote a fixed positive number. With the increase of n the quantity k_n may approach zero. This might be the consequence of fulfilling any of the three following conditions:

first the double points which remain fixed must not cease to be acnodes, second the ovals resulting from A and B must not coalesce, third the last oval must not shrink to a point. In case of failure to fulfill any of these conditions the process of reduction cannot be continued. Taking the conditions in order; if an acnode became a crunode for values of k_n which tend to zero there must be a limiting form of the sextic with ten acnodes and an oval such that when the axes are properly placed the equation becomes $ay^2 + (\ )_3 + \cdots = 0$ where the terms not written are of the order indicated. The line $y = 0$ meets the curve at the origin in three points unless the group $(\ )_3$ vanishes with y. But since the curve has no odd branch there must be an even number of real intersections distinct from the origin, hence the group $(\ )_3$ contains y as a factor. But a unicursal curve of the form $y^2 + y(\ )_2 + \cdots = 0$ has two real branches passing through the origin, as may be seen from consideration of the parametric representation of the curve; hence the limiting curve of the form supposed is impossible, and therefore the value of k_n cannot tend to zero unless the second or third condition is unfulfilled. If the second condition is fulfilled but the value of k_n tends to zero the points A^n, B^n must approach and by the rearrangement of points described above this may be prevented. If the first and second conditions are fulfilled and k_n tends to zero on account of the third condition then the limit form of the curve has eleven acnodes. If all three conditions can be fulfilled it follows that $\kappa_n > g > 0$. But the assumptions that $\kappa_n > g > 0$ and that the last oval is not reduced are contradictory. For letting S be a limit curve of the set S_n the curve S will have an oval within the ovals of S_n and arbitrarily close to them for n sufficiently great, and the curve $S_n + \kappa C_1{}^n C_2{}^n$ can be made to pass through a point within the oval of S withotu requiring $k > g$; hence the limit curve S cannot have an oval. Therefore all three conditions cannot continue to be fulilled; and since the first two can be it follows that the limit of $k_n = 0$ and S the limit curve has eleven acnodes, which was to be proved.

It might happen that the reduction was prevented by the disappearance of the last oval before those due to A^n and B^n. Supposing this to occur let the remaining oval be regarded as the last one in place of the one previously so considered. Then there will be a limit curve with an oval whose greatest chord is less than that of any other. But this is impossible since the process always replaces the last oval by one of smaller greatest chord. Hence no sextic with eleven ovals external to each other can exist.

Throughout this discussion the existence of limit curves is assumed. There must be at least one limit curve, since if the coefficients of S_n are

On Isometric Systems of Curves and Surfaces.

BY C. E. WEATHERBURN.

In the first part of this paper some new theorems are proved concerning isometric orthogonal systems of curves drawn on any surface. The differential invariants employed in this connection are the two-parametric invariants introduced and discussed by the author in a recent paper entitled "On Differential Invariants in Geometry of Surfaces, with some Applications to Mathematical Physics."[*] In the second part some analogous properties of isometric systems of surfaces are examined, and an isometric congruence of curves is defined. The differential invariants in this portion of the paper are the ordinary invariants frequently used in applied mathematics.

Isometric Orthogonal Systems of Curves.

1. Consider first an isometric orthogonal system of curves on a given surface. These may be taken as parametric curves $u = \text{const.}$, $v = \text{const.}$ Let $\mathbf{r}$ be the position vector of the current point on the surface, and let suffixes 1, 2 denote partial differentiation with respect to u, v respectively. Then, with the usual notation,

$$E = (\partial \mathbf{r}/\partial u)^2 = \mathbf{r}_1^2; \qquad G = (\partial \mathbf{r}/\partial v)^2 = \mathbf{r}_2^2.$$

Now the necessary and sufficient condition that the parametric curves may be isometric is

$$\partial^2/\partial u \partial v \log E/G = 0. \tag{1}$$

We may write this as

$$\partial/\partial u(E_2/E) - \partial/\partial v(G_1/G) = 0$$

from which it follows that[†]

$$\operatorname{div} \left(\frac{E_2 \mathbf{r}_1}{E(EG)^{1/2}} - \frac{G_1 \mathbf{r}_2}{G(EG)^{1/2}} \right) = 0,$$

and therefore, if $\boldsymbol{a}$, $\boldsymbol{b}$ are the unit tangents to the parametric curves,

$$\operatorname{div} (\boldsymbol{a} \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \operatorname{div} \boldsymbol{a}) = 0. \tag{2}$$

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[*] *Quarterly Journal of Pure and Applied Mathematics*, Vol. 50, pp. 230-269. (Cambridge, 1925).

[†] *Loc. cit.*, § 4.

This equation may be transformed in different ways. First it may be expressed

$$\boldsymbol{a} \cdot \nabla \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \cdot \nabla \operatorname{div} \boldsymbol{a} = 0. \tag{3}$$

Now $-\operatorname{div} \boldsymbol{b}$ is the geodesic curvature * of the curve $v = \text{const.}$, and $\operatorname{div} \boldsymbol{a}$ is that of the curve $u = \text{const.}$ Also the above analysis is reversible; so that if (3) is satisfied the parametric curves form an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that, at any point, the sum of the derivatives of the geodesic curvatures of the curves, each in its own direction, be zero.

Again, the equation (2) may also be expressed

$$\operatorname{div} \{(\boldsymbol{a} \operatorname{div} \boldsymbol{a} + \boldsymbol{b} \operatorname{div} \boldsymbol{b}) \times \boldsymbol{n}\} = 0$$

where $\boldsymbol{n}$ is the unit normal to the surface. Writing

$$\boldsymbol{h} = -(\boldsymbol{a} \operatorname{div} \boldsymbol{a} + \boldsymbol{b} \operatorname{div} \boldsymbol{b})$$

and remembering that $\operatorname{curl} \boldsymbol{n} = 0$, we deduce that

$$\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{h} = 0. \tag{4}$$

Now we have shown elsewhere † that $\boldsymbol{h}$ is the tangential component of the vector curvature of the orthogonal system of parametric curves, and that its divergence is equal to the second (or specific) curvature, K, of the surface. Since $\boldsymbol{h}$ is itself tangential to the surface, and by (4) also $\operatorname{curl} \boldsymbol{h}$, it follows that $\boldsymbol{h}$ is the gradient of some scalar ‡ function of ϕ. Thus

$$\boldsymbol{h} = \nabla \phi$$

and consequently

$$K = \operatorname{div} \boldsymbol{h} = \nabla^2 \phi. \tag{5}$$

Again the above analysis is reversible, so that if (4) is satisfied the parametric curves constitute an isometric system. Hence the theorem:

A necessary and sufficient condition that an orthogonal system of curves on a surface may be isometric is that the tangential component of the vector

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* *Ibid.*, § 8.

† "Some New Theorems in Geometry of a Surface," § 5. *The Mathematical Gazette* (London), Vol. 13 (1926), pp. 1-6.

‡ *Quarterly Journal, loc. cit.*, p. 258.

curvature of the system be the gradient of some scalar point-function, ϕ, on the surface. The second curvature of the surface is then equal to $\nabla^2\phi$.

The value of this function, ϕ, when the condition (1) is satisfied, may be found as follows. The vector $\boldsymbol{h}$ is given by

$$-\boldsymbol{h} = \frac{G_1\boldsymbol{r}_1}{2EG} + \frac{E_2\boldsymbol{r}_2}{2EG} .$$

Now, with the usua notation, E and G may be expressed *

$$E = \lambda U, \qquad G = \lambda V$$

where U is a function of u only, and V a function of v only. Consequently, on differentiation we have

$$E_2/E = G_2/G - V_2/V.$$

Thus the above expression for $-\boldsymbol{h}$ is equivalent to

$$\begin{aligned} -\boldsymbol{h} &= \tfrac{1}{2}(\nabla \log G - \nabla \log V) \\ &= \tfrac{1}{2}\nabla \log \lambda \end{aligned}$$

Thus

(6) $$\phi = -\tfrac{1}{2}\log\lambda$$

and the second curvature of the surface is given by

(7) $$K = -\tfrac{1}{2}\nabla^2 \log \lambda.$$

Cor. For any isometric orthogonal system on a developable surface,

$$\nabla^2 \log \lambda = 0.$$

2. Consider next an orthogonal system of curves cutting an isometric system at a constant angle. We have shown † that the vector curvatures of these two systems are the same. Hence the condition of isometry is satisfied by the new system also; so that:

Any orthogonal system of curves on a surface, cutting an isometric system at a constant angle, is also isometric.

A still more general theorem may be proved as follows. We have shown elsewhere ‡ that the vector curvatures of two orthogonal systems, cutting at a variable angle θ, differ by the tangential vector curl $\theta\boldsymbol{n}$ or $\nabla\theta \times \boldsymbol{n}$. If then the condition of isometry is satisfied by one of the systems, it will also be satisfied by the other provided $\nabla\theta \times \boldsymbol{n}$ is the gradient of some scalar function. This will be the case provided *

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* See the author's *Differential Geometry*, § 39 (Cambridge University Press, 1926).

† *The Mathematical Gazette, loc. cit.*, § 6.

‡ *Ibid.*, p. 6.

$$\boldsymbol{n} \cdot \operatorname{curl} (\nabla\theta \times \boldsymbol{n}) = 0$$

that is

$$\boldsymbol{n} \cdot \{\boldsymbol{n} \cdot \nabla\nabla\theta - \nabla\theta \cdot \nabla\boldsymbol{n} + (\nabla\theta) \operatorname{div} \boldsymbol{n} - \boldsymbol{n}\, \nabla^2\theta\} = 0.$$

The first three terms vanish identically, showing that the required condition is

$$\nabla^2\theta = 0.$$

Hence the theorem:

An orthogonal system of curves, cutting an isometric orthogonal system at a variable angle θ, will itself be isometric provided $\nabla^2\theta = 0$.

The preceding theorem is a particular case of this.

3. Another form may be found for the condition of isometry of an orthogonal system. If such a system be taken for parametric curves, with $\boldsymbol{a}$ and $\boldsymbol{b}$ as unit tangents, it may be verified that

$$\boldsymbol{b} \cdot \nabla^2\boldsymbol{a} = \frac{1}{(EG)^{1/2}} \left(-JM - \tfrac{1}{2} \frac{\partial^2}{\partial u \partial v} \log \frac{E}{G} \right)$$

where L, M, N are the second order magnitudes for the surface, and J is the first curvature, commonly called the mean curvature. Now $M/(EG)^{1/2}$ has the value $\boldsymbol{a} \cdot \operatorname{curl} \boldsymbol{a}$, being equal to the geodesic torsion of the parametric curve $v = \text{const.}$ Hence the above equation shows that the orthogonal system will be isometric provided

$$\boldsymbol{b} \cdot \nabla^2\boldsymbol{a} + J\, \boldsymbol{a} \cdot \operatorname{curl} \boldsymbol{a} = 0.$$

Since $\boldsymbol{b} = \boldsymbol{n} \times \boldsymbol{a}$ we may state the theorem:

A necessary and sufficient condition, that a family of curves with unit tangent $\boldsymbol{t}$ may form with their orthogonal trajectories an isometric system on the surface, is that $\boldsymbol{n} \times \boldsymbol{t} \cdot \nabla^2\boldsymbol{t} + J\, \boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{t}$ vanish identically.

Now the second term of this expression is zero when the curves are lines of curvature on the surface. Therefore:

If $\boldsymbol{a}, \boldsymbol{b}$ are unit vectors in the principal directions, a necessary and sufficient condition that the lines of curvature may form an isometric system of curves is that $\boldsymbol{b} \cdot \nabla^2\boldsymbol{a}$ vanish identically.

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* *Quarterly Journal, loc. cit.,* § 13.

When the lines of curvature form an isometric system, the surface is sometimes said to be *isothermic.* The equation $\boldsymbol{b}\cdot\nabla^2\boldsymbol{a}=0$ may therefore be regarded as the differential equation satisfied by an isothermic surface.

Isometric Families of Surfaces.

4. Consider a triply orthogonal system of surfaces

$$u(x,y,z)=\text{const.},\qquad v(x,y,z)=\text{const.},\qquad w(x,y,z)=\text{const.}$$

Then, in the space occupied by the system, the position vector $\boldsymbol{r}$ of a point is a function of the variables u, v, w. Let differentiation with respect to these three parameters be denoted by the suffixes 1, 2, 3 respectively, and let

$$a=\boldsymbol{r}_1^2,\qquad b=\boldsymbol{r}_2^2,\qquad c=\boldsymbol{r}_3^2.$$

We may take as our starting point the known property that the family of surfaces $w=$ const. will be isometric provided $\nabla^2w/(\nabla w)^2$ is a function of w alone, the differential invariants being three-parametric. Now

$$\nabla^2w=\frac{1}{(abc)^{\frac{1}{2}}}\,\frac{\partial}{\partial w}\left(\frac{ab}{c}\right)^{\frac{1}{2}}$$

and

$$(\nabla w)^2=1/c,$$

so that

$$\frac{\nabla^2w}{(\nabla w)^2}=\left(\frac{c}{ab}\right)^{\frac{1}{2}}\frac{\partial}{\partial w}\left(\frac{ab}{c}\right)^{\frac{1}{2}}=\frac{\partial}{\partial w}\log\left(\frac{ab}{c}\right)^{\frac{1}{2}}$$

Thus the necessary and sufficient conditions that the family of surfaces $w=$ const. may be isometric are

$$(1)\qquad \frac{\partial^2}{\partial u\partial w}\log\frac{ab}{c}=0,\qquad \frac{\partial^2}{\partial v\partial w}\log\frac{ab}{c}=0.$$

We shall express these conditions in terms of differential invariants of the unit vector $\boldsymbol{n}$ normal to the surface of the family through the point considered. This unit normal is equal to $\boldsymbol{r}_3/(c)^{\frac{1}{2}}$, so that

$$\operatorname{curl}\boldsymbol{n}=\frac{1}{2c(ab)^{\frac{1}{2}}}\,(c_2\boldsymbol{r}_1-c_1\boldsymbol{r}_2)$$

and therefore

$$\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}=\tfrac{1}{2}\left(\frac{c_1\boldsymbol{r}_1}{ac}\right)+\tfrac{1}{2}\left(\frac{c_2\boldsymbol{r}_2}{bc}\right).$$

Also

$$\operatorname{div}\boldsymbol{n}=\frac{1}{(abc)^{\frac{1}{2}}}\,\frac{\partial}{\partial w}\,(ab)^{\frac{1}{2}}$$

and therefore

$$\operatorname{curl}\ (\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n})$$
$$=\frac{1}{2(abc)^{\frac{1}{2}}}\left(\boldsymbol{r}_1\frac{\partial^2}{\partial v\partial w}\log\frac{ab}{c}-\boldsymbol{r}_2\frac{\partial^2}{\partial u\partial w}\log\frac{ab}{c}\right).$$

which vanishes if the family of surfaces is isometric. Conversely, if this vector vanishes identically, the family of surfaces is isometric. Hence the theorem:

A necessary and sufficient condition that a family of surfaces with unit normal $\boldsymbol{n}$ *be isometric is*

$$(2)\qquad \operatorname{curl}\ (\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n})=0$$
or
$$\operatorname{curl}\ (\boldsymbol{n}\operatorname{div}\boldsymbol{n}-\boldsymbol{n}\cdot\nabla\boldsymbol{n})=0.$$

If this condition is satisfied, the vector $\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}$ is the gradient of some scalar function ϕ. Denoting the vector by $\boldsymbol{H}$, we may write $\boldsymbol{H}=\nabla\phi$. Also we have shown elsewhere * that the second curvature K of a surface of the family is given by

$$2K=\operatorname{div}\ (\boldsymbol{n}\operatorname{div}\boldsymbol{n}+\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}).$$

Therefore, when the family is isometric, it follows that

$$(3)\qquad 2K=\operatorname{div}\boldsymbol{H}=\nabla^2\phi$$

so that K is half the Laplacian of ϕ.

The value of the function ϕ, when the conditions (1) are satisfied, may be found as follows. The vector $\boldsymbol{H}$ may be expressed

$$\boldsymbol{H}=\left(\frac{1}{2c}\right)\left(\frac{a_3}{a}+\frac{b_3}{b}\right)\boldsymbol{r}_3+\left(\frac{1}{2c}\right)\left(\frac{c_1}{a}\boldsymbol{r}_1+\frac{c_2}{b}\boldsymbol{r}_2\right)$$
$$=\tfrac{1}{2}\left(\frac{\boldsymbol{r}_1}{a}\frac{\partial}{\partial u}\log c+\frac{\boldsymbol{r}_2}{b}\frac{\partial}{\partial v}\log c+\frac{\boldsymbol{r}_3}{c}\frac{\partial}{\partial w}\log c\right)$$
$$+\tfrac{1}{2}\frac{\boldsymbol{r}_3}{c}\left(\frac{a_3}{a}+\frac{b_3}{b}-\frac{c_3}{c}\right)$$
$$=\tfrac{1}{2}\nabla\log c+\tfrac{1}{2}\frac{\boldsymbol{r}_3}{c}\frac{\partial}{\partial w}\log\frac{ab}{c}\quad.$$

Now $\partial\log(ab/c)/\partial w$ is a function of w alone. Denote it by $f(w)$, and put

$$\int f(w)\,dw=\log\psi(w)$$

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* See §2 of a paper by the author "On Families of Curves and Surfaces," recently communicated to the *Messenger of Mathematics.*

so that

$$\psi(w) = e^{\int f(w)dw}.$$

Then

$$\begin{aligned} \boldsymbol{H} &= \tfrac{1}{2} \nabla \log c + \tfrac{1}{2} \nabla \log \psi \\ &= \tfrac{1}{2} \nabla \log (c\psi) \end{aligned} \tag{4}$$

and therefore

$$K = \tfrac{1}{4} \nabla^2 \log (c\psi). \tag{5}$$

5. Two consequences of the above theory may be noticed in passing. We have shown elsewhere * that, for a family of *parallel surfaces*, curl $\boldsymbol{n}$ vanishes identically. Hence, if a family of parallel surfaces is also isometric, the equation (2) shows that $\boldsymbol{n} \times \nabla J = 0$. From this it follows that ∇J is parallel to $\boldsymbol{n}$, so that J is constant over any one surface. Hence the theorem:

If a family of parallel surfaces is also isometric, the first curvature is constant over each surface of the family.

Next consider a triply orthogonal system of surfaces, and suppose that the families $v =$ const. and $w =$ const. are both isometric. Then we have the necessary relations

$$\left\{ \begin{array}{lll} & \dfrac{\partial^2}{\partial u \partial v} \log \dfrac{ac}{b} = 0, & \dfrac{\partial^2}{\partial w \partial v} \log \dfrac{ac}{b} = 0 \\ \text{and} & & \\ & \dfrac{\partial^2}{\partial u \partial w} \log \dfrac{ab}{c} = 0, & \dfrac{\partial^2}{\partial v \partial w} \log \dfrac{ab}{c} = 0 \end{array} \right. \tag{6}$$

From the second and fourth of these it follows that

$$\frac{\partial^2}{\partial v \partial w} \log a = 0, \qquad \frac{\partial^2}{\partial v \partial w} \log \frac{b}{c} = 0. \tag{7}$$

Now, on a surface $u =$ const., v and w are the current parameters; and the second equation (7) shows that the parametric curves on this surface, which are also lines of curvature by Dupin's theorem, form an isometric system. Hence the theorem:

If two families of a triply orthogonal system of surfaces are isometric, the lines of curvature on any surface of the other family constitute an isometric system of curves.

6. *Congruence of Curves.* We may define an *isometric* congruence of curves as a congruence which cuts orthogonally an isometric system of surfaces. It is therefore first of all a normal congruence, so that the unit tangent $\boldsymbol{t}$ to the congruence satisfies the equation

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* *Ibid.*, § 3.

$$\boldsymbol{t}\cdot\operatorname{curl}\boldsymbol{t}=0.$$

Then, since $\boldsymbol{t}$ is the unit normal to the isometric family of surfaces, it also satisfies the further condition

$$\operatorname{curl}\,(\boldsymbol{t}\operatorname{div}\boldsymbol{t}-\boldsymbol{t}\cdot\nabla\,\boldsymbol{t})=0.$$

These are the equations satisfied by the unit tangent to an isometric congruence.

The vector $(\boldsymbol{t}\operatorname{div}\boldsymbol{t}-\boldsymbol{t}\cdot\nabla\,\boldsymbol{t})$ plays an important part for any congruence of curves. We have elsewhere* defined the *limit surface* of any congruence, by analogy with that of a rectilinear congruence, and have found its equation in terms of oblique curvilinear coordinates. We here announce (without proof) for the first time the following theorem giving the equation of this surface in terms of differential invariants of the unit tangent:

For a congruence of curves with unit tangent $\boldsymbol{t}$ *the equation of the limit surface may be expressed*

$$\operatorname{div}\,(\boldsymbol{t}\operatorname{div}\boldsymbol{t}-\boldsymbol{t}\cdot\nabla\boldsymbol{t})=0$$

or

$$\operatorname{div}\,(\boldsymbol{t}\operatorname{div}\boldsymbol{t}+\boldsymbol{t}\times\operatorname{curl}\boldsymbol{t})=0.$$

Thus, in the case of a normal congruence of curves, the limit surface is the locus of the points at which the second curvature of the family of surfaces orthogonal to the congruence is zero. The *surface of striction* of the congruence is the locus of points at which the first curvature of the family is zero; for this surface is given by $\operatorname{div}\boldsymbol{t}=0$.

CHRISTCHURCH, N. Z.,
July, 1926.

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* "On Congruences of Curves," § 4, recently communicated to the *Tôhoku Mathematical Journal;* also a paper "On Triple Surfaces, etc.", *Proceedings of the Royal Society of Edinburgh,* Vol. 46 (1926), pp. 194-205.

A Characteristic Property of Certain Sets of Trigonometric Functions.

By M. H. Stone.

We shall discuss in this paper the nature of a set of functions $\{u_i\}$ characterised by certain conditions of orthogonality shared by well-known trigonometric sets, with the aim of showing that the functions u_i are necessarily trigonometric. Before writing down these conditions we state the conventions which we shall follow with regard to the notation for integrals. It is convenient to denote by $\int f$ the integral in the Lebesgue sense of a function $f(x)$ over the interval $(0,1)$; and only in cases where integrals over other ranges are considered will the limits be indicated explicitly. The functions u_i, $(i=1, 2, \cdots)$, are then restricted to satisfy the following conditions:

(1) the set $\{u_i\}$ is a closed normal orthogonal set of real functions of the real variable x on the interval $(0,1)$;

(2) the set $\{v_i\}$, where $v_i = u_i'$, is a set of orthogonal functions on the interval $(0,1)$;

(3) the functions u_i'' are summable with summable square;

(4) the functions u_i satisfy one of the five sets of boundary conditions

(a) $u_i(0) = u_i(1) = 0$,

(b) $v_i(0) = v_i(1) = 0$,

(c) $v_i(0) - v_i(1) = 0, \quad u_i(0) - u_i(1) = 0$,

(d) $v_i(0) + v_i(1) = 0, \quad u_i(0) + u_i(1) = 0$,

(e) $\alpha_1 v_i(0) - \beta_1 v_i(1) = 0, \; \alpha_2 u_i(0) - \beta_2 u_i(1) = 0, \; \alpha_1^2 - \beta_1^2 \neq 0.$

Our work is based on the following two lemmas.

LEMMA I. *The set of positive constants* ρ_i *defined by the equation* $\int v_i^2 = \rho_i^2$ *has* $+\infty$ *as its only limit point.*

Suppose that a sequence of values of i can be found such that $\rho_i \to \rho$, $0 \leq \rho < +\infty$, $\int v_i^2 \to \rho^2$. Then, by Schwarz's inequality,

$$| u_i(x) - u_i(y) | = | \int_y^x v_i | \leq (| x - y | \quad | \int_y^x v_i^2 |)^{1/2}$$
$$\leq (| x - y | \int v_i^2)^{1/2} \leq M \, | x - y |^{1/2}$$

for values of i in this sequence. Consequently, the set of functions u_i constitutes a uniformly bounded equicontinuous family. From this family it is possible to select a subsequence converging uniformly to a continuous function $\theta(x)$ on $(0, 1)$. For fixed x we find

$$\int_0^x u_i \to \int_0^x \theta, \qquad \int_0^x u_i \to 0$$

for functions in this subsequence, the last limit being a consequence of Bessel's inequality:

$$\int_0^x 1 \geq \sum_1^\infty (\int_0^x u_i)^2.$$

It follows that $\int_0^x \theta = 0$ whence $\theta \equiv 0$. This leads at once to a contradiction by virtue of the orthogonality of the set $\{u_i\}$. We let i be a value in the sequence $\{i\}$ under consideration; then

$$\int u_i^2 = 1, \qquad \int u_i^2 \to \int \theta^2 = 0 .$$

LEMMA II. *The functions*

$$\phi_i(x; f) = \int_0^x \sin \rho_i (x - y) \, f(y) \, dy,$$
$$\psi_i \, (x; f) = \int_0^x \cos \rho_i (x - y) \, f(y) \, dy,$$

where $f(x)$ *is summable on* $(0, 1)$, *converge uniformly to zero with* $1/i$.

We give the proof for the function ϕ_i, that for ψ_i being similar. Since

$$| \phi_i(x) - \phi_i(y) | = | \int_y^x \sin \rho_i (x - y) \, f(y) dy | \leq | \int_y^x | f | \, dy | < \epsilon$$

if $| x - y | < \delta$, where δ depends only on f and ϵ, the family of functions $\{\phi_i\}$ is equicontinuous. For each value of x on $(0, 1)$ we have

$$\phi_i = \sin \rho_i x \int_0^x \cos \rho_i y\, f(y)\, dy - \cos \rho_i x \int_0^x \sin \rho_i y\, f(y)\, dy = o(1)$$

by the fact that when ρ_i becomes infinite the two integrals in the middle term are $o(1)$. Because of the equicontinuity of the family $\{\phi_i\}$ it follows that $\phi_i(x) = o(1)$ uniformly on $(0, 1)$.

Lemma I shows that with a suitable renumbering of the set $\{u_i\}$ the inequalities $0 \leqq \rho_1 \leqq \rho_2 \leqq \cdots$ may be supposed valid. In our further reasoning such a renumbering will be assumed.

We can now prove the

THEOREM. *The set* $\{u_i\}$ *satisfying conditions* (1)–(4) *is one of the closed normal orthogonal sets of trigonometric functions defined by a self-adjoint differential system of the form*

$$u'' + \rho^2 u = 0, \quad \alpha_1 u^{(k_1)}(0) + \beta_1 u^{(k_1)}(1) = 0,$$
$$\alpha_2 u^{(k_2)}(0) + \beta_2 u^{(k_2)}(1) = 0, \quad 1 \geqq k_1 \geqq k_2 \geqq 0,$$

with linearly independent boundary conditions, and conversely.

It should be noticed that the differential system does not define a unique normal orthogonal set because of the fact that in any normal orthogonal set any of the functions may be replaced by its negative. Indeed, in the case corresponding to the boundary conditions (4c) and (4d) the situation is even more complicated.

We have

$$\rho_k^2 \int u_i u_k = \int u_i' u_k' = u_i(1) u_k'(1) - u_i(0) u_k'(0) - \int u_i u_k''$$

whence

$$\text{(A)} \qquad \int (u_k'' + \rho_k^2 u_k) u_i = u_i(1) u_k'(1) - u_i(0) u_k'(0) = a_{ik}.$$

This equation is fundamental in our succeeding work.

When the boundary conditions satisfied by the set $\{u_i\}$ are of types (4a), (4b), (4c), (4d), the constants a_{ik} in (A) all vanish. The closure of the set $\{u_i\}$ then requires that $u_k'' + \rho_k^2 u_k = 0$ almost everywhere on $(0, 1)$ for $k = 1, 2, \cdots$. Thus the set $\{u_i\}$ must be one of the normal orthogonal sets determined by the differential equation $u'' + \rho^2 u = 0$ with the boundary conditions (4a), (4b), (4c), or (4d). For special choices of k_1, k_2, α_1, β_1, the system written down in the theorem is equivalent to one or another of these systems. With (4a) we have $u_i = \pm 2^{1/2} \sin \pi i x$; with (4b), $u_1 = \pm$: $u_i = \pm 2^{1/2} \cos \pi (i-1) x$; with (4c),

$$u_1 = \pm 1,$$

$$u_{2l} = \lambda_l \cos 2\pi l x + \mu_l \sin 2\pi l x,$$

$$u_{2l+1} = \mu_l \cos 2\pi l x - \lambda_l \sin 2\pi l x, \quad \lambda_l^2 + \mu_l^2 = 2;$$

with (4d),

$$u_{2l-1} = \lambda_l \cos (2l-1)\pi x + \mu_l \sin (2l-1)\pi x,$$

$$u_{2l} = \mu_l \cos (2l-1)\pi x - \lambda_l \sin (2l-1)\pi x, \quad \lambda_l^2 + \mu_l^2 = 2;$$

When the set $\{u_i\}$ satisfies conditions (4e) the discussion is more diff cult. We divide it into three subcases.

Case I. $\beta_1 = 0$. We may assume that $u_k'(1)$ is different from zero fo $k = \alpha$; otherwise we fall back on the case in which the set $\{u_i\}$ satisfies th conditions (4b). From (3) and (A) we conclude that

$$\sum_{i=1}^{\infty} u_i^2(1) = \sum_{i=1}^{\infty} a_{i\alpha}^2 / u_\alpha'(1)$$

is a convergent series. The theorem of Riesz-Fischer proves the existence o a function f summable with summable square such that $\int f u_i = u_i(1)$; an the closure of the set $\{u_i\}$ enables us to write

$$u_k'' + \rho_k^2 u_k = u_k'(1) f$$

almost everywhere on $(0,1)$. The general solution of this equation is well known. The function u_k is found to be

$$u_k = a_k \cos \rho_k x + b_k \sin \rho_k x + u_k'(1) \phi_k (x; f) / \rho_k,$$

$$u_k' = -\rho_k a_k \sin \rho_k x + \rho_k b_k \cos \rho_k x + u_k'(1) \psi_k (x; f),$$

where a_k, b_k are constants depending upon k, and ϕ_k, ψ_k are the functions o Lemma II. The condition $u_k'(0) = 0$ yields $\rho_k b_k = 0$. Since only one of th functions u_i can reduce to a constant, we must have $\rho_k > 0$, $k \geq 2$, and $b_k = 0$ $k \geq 2$. Then from Lemma II

$$u_k'(1) = -a_k \rho_k \sin \rho_k + u_k'(1)\, o(1)$$

so that

$$u_k'(1) = -a_k\rho_k \sin \rho_k\,(1 + o(1))$$

and

$$u_k = a_k\,(\cos \rho_k x + o(1)).$$

When this value of u_k is substituted in the equation $\int u_k^2 = 1$, we find

$$\begin{aligned} 1 = a_k^2 \int (\cos \rho_k x + o(1))^2 &= a_k^2\,(\textstyle\int \cos^2 \rho_k x + o(1)) \\ &= a_k^2\,[\tfrac{1}{2} + (\sin 2\rho_k)/4\rho_k + o(1)] = a_k^2(\tfrac{1}{2} + o(1)). \end{aligned}$$

Hence $a_k^2 = 2 + o(1)$, $a_k = \pm 2^{1/2} + o(1)$, and

$$\begin{aligned} u_k &= \pm\, 2^{1/2}\,(\cos \rho_k x + o(1)), \\ u_k' &= \pm\, 2^{1/2}\,\rho_k\,(-\sin \rho_k x + o(1)). \end{aligned}$$

The condition that $\sum u^2_k(1)$ be convergent shows that $\cos \rho_k = o(1)$. We now consider the boundary condition $\alpha_2 u_k(0) + \beta_2 u_k(1) = 0$. On replacing $u_k(0)$, $u_k(1)$ by their asymptotic forms this condition becomes

$$\pm\, 2^{1/2}\,\alpha_2 + \alpha_2\, o(1) + \beta_2\, o(1) = 0$$

so that $\alpha_2 = 0$. Thus the boundary conditions (4e) must take the form $u'_k(0) = 0$, $u_k(1) = 0$. In consequence the constants a_{ik} vanish identically, with the result that

$$u_k'' + \rho_k^2\, u_k = 0$$

almost everywhere. The remainder of the reasoning is analogous to that of the first paragraph in the proof of the theorem. It should be noted that the functions v_k satisfy the self-adjoint differential system

$$\begin{aligned} v_k'' + \rho_k^2\, v_k &= 0, \\ v_k(0) = v_k'(1) &= 0, \end{aligned}$$

and therefore constitute an orthogonal set.

Case II. $\alpha_1 = 0$. By the transformation $x' = 1 - x$ this case becomes Case I.

Case III. $\alpha_1\beta_1 \neq 0$. We may assume that $u_k'(0)$ is different from zero for $k = \alpha$, since otherwise we should fall back on Case I. Since

$$\alpha_1 u_\alpha'(0) + \beta_1 u_\alpha'(1) = 0$$

we have

$$a_{i\alpha} = [\beta_1 u_i(0) + \alpha_1 u_i(1)]\, u_\alpha'(0)/\beta_1$$

so that

$$\sum_{i=1}^{\infty} [\beta_1 u_i(0) + \alpha_1 u_i(1)]^2/\beta_1^2 = \sum_{i=1}^{\infty} a_{i\alpha}^2/u_\alpha'^2(0) \qquad \text{converges.}$$

Hence there exists a function f summable with summable square such that

$$\int f\, u_i = [\beta_1 u_i(0) + \alpha_1 u_i(1)]/\beta_1 .$$

As a result

$$u_k'' + \rho_k^2 u_k = u_k'(0)\, f,$$

and

$$u_k = a_k \cos \rho_k + b_k \sin \rho_k x + u_k'(0)\, \phi_k(x;f)/\rho_k,$$
$$u_k' = -\rho_k a_k \sin \rho_k x + b_k \rho_k \cos \rho_k x + u_k'(0)\, \psi_k(x;f).$$

From these equations we find $u_k'(0) = \rho_k b_k$, and

$$u_k = a_k \cos \rho_k x + b_k\, (\sin \rho_k x + o(1)),$$
$$u_k = \rho_k\, [- a_k \sin \rho_k x + b_k\, (\cos \rho_k x + o(1))] .$$

The equation $\int u_k^2 = 1$ shows that

$$1 = a_k^2\, (\tfrac{1}{2} + o(1)) + a_k b_k\, o(1) + b_k^2\, (\tfrac{1}{2} + o(1)).$$

Thus a_k, b_k, which are real quantities in view of the fact that $a_k = u_k(0)$, $b_k = u_k'(0)/\rho_k$, must specify a point on the real trace of the conic C_k:

$$x^2\, (\tfrac{1}{2} + o(1)) + xy\, o(1) + y^2\, (\tfrac{1}{2} + o(1)) = 1.$$

For large values of k this is an ellipse lying in the neighborhood of the circle $x^2 + y^2 = 2$. Consequently $|\, a_k \,|$ and $|\, b_k \,|$ are bounded, and

$$\text{(B)} \qquad a_k^2 + b_k^2 = 2 + o(1).$$

By the convergence of the series $\sum\, (\beta_1 u_k(0) + \alpha_1 u_k(1))^2$ we have

$$\beta_1 u_k(0) + \alpha_1 u_k(1) = o(1).$$

The boundary condition of (4e) involving the first derivatives is

$$\alpha_1 u_k'(0) + \beta_1 u_k'(1) = 0.$$

In these equations we substitute the asymptotic values of the functions u_k and u_k', and, on dividing the second by ρ_k, obtain the relations

$$\text{(C)} \qquad \beta_1 a_k + \alpha_1 \ (a_k \cos \rho_k + b_k \sin \rho_k) = o(1),$$

$$\text{(D)} \qquad \alpha_1 b_k + \beta_1 \ (-a_k \sin \rho_k + b_k \cos \rho_k) = o(1).$$

We regard these as equations for a_k and b_k. In view of (B) their determinant must be $o(1)$, since otherwise we would conclude that $a_k = o(1)$, $b_k = o(1)$. Hence we find

$$-2\alpha_1\beta_1 - (\alpha_1^2 + \beta_1^2) \cos \rho_k = o(1),$$

$$\text{(E)} \qquad \cos \rho_k = -2\alpha_1\beta_1 / (\alpha_1^2 + \beta_1^2) + o(1).$$

From the boundary condition $\alpha_2 u_k(0) + \beta_2 u_k(1) = 0$ we derive the equation

$$\text{(F)} \qquad \alpha_2 a_k + \beta_2 (a_k \cos \rho_k + b_k \sin \rho_k) = o(1).$$

Taking (C) and (F) as simultaneous equations for a_k and b_k, we notice that the determinant of the system must be $o(1)$ by virtue of (B). This determinant is

$$\sin \rho_k \ (\alpha_1\alpha_2 - \beta_1\beta_2).$$

If $\sin \rho_k = o(1)$, then $\cos \rho_k = 1 + o(1)$ or $\cos \rho_k = -1 + o(1)$. In the first case (E) yields $2\alpha_1\beta_1/(\alpha_1^2 + \beta_1^2) = -1$, $\alpha_1 = -\beta_1$; in the second, $2\alpha_1\beta_1/(\alpha_1^2 + \beta_1^2) = 1$, $\alpha_1 = \beta_1$. These possibilities are excluded by the inequality $\alpha_1^2 - \beta_1^2 \neq 0$. Consequently $\sin \rho_k$ is not $o(1)$; and $\alpha_1\alpha_2 - \beta_1\beta_2 = 0$. We may take $\alpha_2 = \beta_1$, $\beta_2 = \alpha_1$, without loss of generality. Under these conditions the constants a_{ik} in (A) all vanish, so that

$$u_k'' + \rho_k^2 u_k = 0, \quad \alpha_1 u_k'(0) + \beta_1 u_k'(1) = 0,$$
$$\beta_1 u_k(0) + \alpha_1 u_k(1) = 0.$$

The determination of the set $\{u_i\}$ is complete. Since it is clear that the functions v_k satisfy the self-adjoint differential system

$$v_k'' + \rho_k^2 v_k = 0, \quad \beta_1 v_k'(0) + \alpha_1 v_k'(1) = 0,$$
$$\alpha_1 v_k(0) + \beta_1 v_k(1) = 0,$$

they constitute an orthogonal set.

COROLLARY. *Under conditions* (1)–(4), *the set* $\{v_i/\rho_i\}$ *is a closed normal orthogonal set except in the cases* (4b) *and* (4c).

It is evident that, if the boundary conditions (4) be relinquished, we can still obtain some information concerning the asymptotic character of the functions u_i, v_i and the constants ρ_i when i becomes infinite. In fact, by the consideration of proper subcases, we can use the equation (A) to obtain asymptotic forms under all circumstances. The principal result may be stated roughly by asserting that the set u_i and a properly chosen subset $\bar{u}_i$ of one of the normal orthogonal sets described in the theorem above have the relation $u_i - \bar{u}_i = o(1)$. The fact that we have been unable to apply the properties of the set $\{u_i\}$ in such a manner as to obtain more exact information makes it seem useless to give any account of these rather tedious investigations.

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The Normal Probability Function and General Frequency Functions.

By M. H. Stone.

Many years ago Pearson investigated the problem of representing a given frequency function as the sum of a number of normal probability functions or Gaussian frequency functions. The problem was suggested to him by Weldon, who had seen the possibility of resolving a frequency distribution encountered in certain biological researches into the sum of two normal distributions. Pearson gives a method for obtaining a "best" approximation to a given frequency function by a linear combination of n normal probability functions, but does not consider the convergence of the approximating sums when n becomes infinite. In the specific problem raised by Weldon Pearson shows that his method yields curves fitting the given frequency curves in a highly satisfactory manner.*

The object of the present paper is to discuss the approximation of an arbitrary function $f(x)$ on the interval $(-\infty, +\infty)$ by linear combinations of functions of the form $e^{-\lambda(x-\mu)^2}$, where λ and μ are constants, $\lambda > 0$. We obtain three theorems concerning the representation of an arbitrary function in terms of a *given* set of functions of this type. This set may be so chosen as to include the functions determined by Pearson's process for a given frequency function $f(x)$, when those functions are normal probability functions. We do not obtain any information about the convergence of Pearson's sums, however.

In the following pages all functions considered are real functions of the real variable x defined on the infinite interval $(-\infty, +\infty)$ and measurable in the sense of Lebesgue on that range. All integrals are taken in the sense of Lebesgue; and it should be particularly noted that a function is integrable in the sense of Lebesgue if and only if it is absolutely integrable in the sense of Lebesgue.

Our work is based on the Riesz-Fischer theorem for a set of functions normal and orthogonal on the infinite interval $-\infty < x < +\infty$,[*] and on certain theorems concerning the representation of an arbitrary function $f(x)$ by the integral

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* Karl Pearson, *Philosophical Transactions of the Royal Society*, Vol. 185 A (1894), Part I, pp. 71-111.

$$W_\nu(f;x) = (\nu/\pi)^{1/2} \int_{-\infty}^{+\infty} f(y)\, e^{-\nu(y-x)^2}\, dy$$

when ν becomes infinite.

We begin by establishing four lemmas. The first deals with the integral W_ν defined above, and may be stated as follows:

LEMMA I. *If $f(x)$ is an arbitrary function for which the integral $W_\nu(f;x)$ exists for every pair of values (ν, x), $\nu \geqq N > 0$, $-\infty < x < +\infty$, then*

$$\lim_{\nu\to\infty} W_\nu(f;x) = f(x)$$

almost everywhere.

Lebesgue has shown that if $f(x)$ is integrable on $(-\infty, +\infty)$ then

$$\lim_{\nu\to\infty} W_\nu(f;x) = f(x)$$

at every point at which $f(x)$ is the derivative of its indefinite integral; that is, almost everywhere. The convergence is uniform on any closed interval completely interior to any range on which $f(x)$ is continuous..[†] The hypothesis that $f(x)$ is summable on the infinite interval is unnecessary. We let $f(x)$ be a function for which the integral W_ν exists, as stated in the lemma. For instance a sufficient condition for the existence of this integral is that

$$|f(x)| = O(e^{K|x|^\alpha}),\ \alpha < 2,\ K < 0.$$

We choose any value of x for which $f(x)$ is the derivative of its indefinite integral; for it is clear that on any finite interval $f(y)$ is summable, being the product of the summable function $f(y)e^{-\nu(y-x)^2}$ and the continuous function $e^{\nu(y-x)^2}$. We choose also a fixed positive number σ and a value ν_0 greater than either of the numbers N and $1/4\sigma^4$. Then for $\nu \geqq \nu_0$ and for each value of y outside the interval $(x-\sigma, x+\sigma)$ the function $(\nu/\pi)^{1/2}e^{-\nu(y-x)^2}$ is a monotonely decreasing function of ν, as is readily seen by computing its derivative with respect to ν. Thus, by choosing a negative quantity A and a positive quantity B, both sufficiently large in absolute value, we can write the inequalities

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[*] Plancherel, *Rendiconti del Circolo Matematico di Palermo*, Vol. 30 (1910), pp. 290-297.

[†] H. Lebesgue, *Annales de la Faculté des Sciences de Toulouse*, ser. 3, Vol. 1 (1909), pp. 90-91.

$$\begin{aligned} \left| (\nu/\pi)^{1/2} \int_{-\infty}^{A} f(y) e^{-\nu(y-x)^2} dy \right| &\leq (\nu/\pi)^{1/2} \int_{-\infty}^{A} | f(y) | \, e^{-\nu(y-x)^2} dy \\ &\leq (\nu_0/\pi)^{1/2} \int_{-\infty}^{A} | f(y) | \, e^{-\nu_0(y-x)^2} dy \\ &< \epsilon/4, \end{aligned}$$

and, similarly,

$$\left| (\nu/\pi)^{1/2} \int_{B}^{+\infty} f(y) e^{-\nu(y-x)^2} dy \right| < \epsilon/4$$

where ϵ is a preassigned positive number. Then by Lebesgue's results

$$\left| (\nu/\pi)^{1/2} \int_{A}^{B} f(y) e^{-\nu(y-x)^2} dy - f(x) \right| < \epsilon/2$$

for ν sufficiently large. On adding the three inequalities we see that

$$| W_\nu(f;x) - f(x) | < \epsilon$$

if ν is sufficiently great. The lemma is therefore true.

We let $0 < \lambda_1, \lambda_2, \cdots$, indicate the numbers of a monotonely increasing sequence with limit point $+\infty$, and $\mu_1, \mu_2, \cdots$, the numbers of a countable set everywhere dense in the open continuum $-\infty < \mu < +\infty$. We employ henceforth the notations

$$\phi_{ik} = e^{-\lambda_i(x-\mu_k)^2}, \qquad \psi_{ik} = (x-\mu_k) e^{-\lambda_i(x-\mu_k)^2}.$$

It is then possible to prove

LEMMA II. *No finite set of the functions ϕ_{ik} or of the functions ψ_{ik} is linearly dependent.*

We are to prove that the identities

$$\sum c_{ik}\phi_{ik} \equiv 0, \qquad \sum d_{ik}\psi_{ik} \equiv 0, \qquad -\infty < x < +\infty,$$

in which the sums are finite sums, imply that the constant coefficients c and d all vanish.

Let $\sum c_{ik}\phi_{ik} \equiv 0$ be a finite sum in which none of the coefficients vanishes. We single out the least value of λ_i occurring in this sum and denote it by λ_0. Among the functions ϕ containing λ_0 there is one containing a greatest value of μ_k; we denote this function by ϕ_0, its constant coefficient by c_0, and the value of μ_k which it involves by μ_0. Then we have

$$1 + \sum{}' c_{ik}\phi_{ik}/c_0\phi_0 \equiv 0, \qquad c_0 \neq 0,$$

the sum being extended over all the functions except ϕ_0 appearing in the given identity. Now it is easily seen that the ratio ϕ_{ik}/ϕ_0 has the limit zero when x becomes positively infinite; for, either

$$\phi_{ik}/\phi_0 = e^{-\lambda_i(x-\mu_k)^2 + \lambda_0(x-\mu_0)^2} \to 0; \qquad \lambda_i > \lambda_0,$$

or

$$\phi_{ik}/\phi_0 = e^{2\lambda_0(\mu_k-\mu_0)x + \lambda_0(\mu_0^2-\mu_k^2)} \to 0, \qquad \mu_k < \mu_0.$$

As a result, we can obtain a contradiction by letting x become positively infinite in the second form of the identity. This proves the first part of the lemma.

Because of the fact that $\phi'_{ik} = -2\lambda_i\psi_{ik}$, an identity of the form $\sum d_{ik}\psi_{ik}$ implies an identity $\sum c_{ik}\phi_{ik} \equiv C$, where C is a constant and $c_{ik} = -d_{ik}/2\lambda_i$. On letting x become infinite we see that C must vanish, since each of the functions approaches zero with $1/x$. From the first part of the lemma it follows that the constants c, and therefore the constants d, must vanish.

LEMMA III. *If* $f(x)$ *is a function for which the integrals* $\int_{-\infty}^{+\infty} f\,\phi_{ik}dy$, $i, k = 1, 2, \cdots$, *exist, then the integral* W_ν *exists,* $\nu \geq \lambda_1$, $-\infty < x < \infty$, *and conversely. If* $f(x)$ *is a function for which* $\int_{-\infty}^{+\infty} f\,\phi_{ik}dy = 0$, $i, k = 1, 2, \cdots$, *then* $f(x) = 0$ *almost everywhere.*

To prove the existence of W_ν we use the fact that if two functions f_1, f_2 satisfy the inequality $|f_1| \leq |f_2|$ and if f_2 is summable on $(-\infty, +\infty)$ then f_1 is summable on the same infinite range. For any given value of x we can choose μ_l and μ_m so that $\mu_l < x < \mu_m$. Then it is easily verified that for $\nu \geq \lambda_1$

$$0 < e^{-\nu(y-x)^2} \leq e^{-\lambda_1(y-x)^2} < Ke^{-\lambda_1(y-\mu_l)^2}, \qquad -\infty < y \leq 0,$$
$$0 < e^{-\nu(y-x)^2} \leq e^{-\lambda_1(y-x)^2} < Ke^{-\lambda_1(y-\mu_m)^2}, \qquad 0 \leq y < +\infty.$$

The principle by which the third inequality in each line is obtained may be stated in graphical terms as follows: the curve representing a function of y is monotonely decreasing and asymptotic to the axis of y; if the curve is translated to the left it lies wholly below its original position. It follows that

$\int_{-\infty}^{+\infty} f(y)e^{-\nu(y-x)^2}dy$ exists. Thus the existence of W_ν is established. The converse is trivial.

Suppose that $\int_{-\infty}^{+\infty} f\phi_{ik}dy = 0$, $i, k = 1, 2, \cdots$. We choose x as any value for which $W_\nu(f; x) \to f(x)$, since this integral exists and has a limit almost everywhere; we choose μ_l and μ_m so that $\mu_l < x < \mu_m$; and we determine a sequence of values μ_k lying on the range $\mu_l < \mu < \mu_m$ and approaching x as a limit. Then the functions ϕ_{ik} corresponding to these values of μ_k satisfy the inequalities

$$0 < \phi_{ik} < \phi_{il}, \; -\infty < y \leqq \mu_l, \quad 0 < \phi_{ik} < \phi_{im}, \quad \mu_m \leqq y < +\infty.$$

We now choose a and b so that

$$\left| \int_{-\infty}^{a} f\phi_{ik}dy + \int_{b}^{+\infty} f\phi_{ik}dy \right| < \int_{-\infty}^{a} |f|\phi_{il}dy + \int_{b}^{+\infty} |f|\phi_{im}dy < \epsilon/3,$$

$$\left| -\int_{-\infty}^{a} f(y)e^{-\lambda_i(y-x)^2}dy - \int_{b}^{+\infty} f(y)e^{-\lambda_i(y-x)^2}dy \right| < \epsilon/3,$$

where ϵ is an arbitrary preassigned positive number. We notice that $f(x)$ is summable on (a, b), by reasoning like that used in a similar situation in Lemma I. Because $e^{-\lambda_i(y-\mu_k)^2}$ converges uniformly to $e^{-\lambda_i(y-x)^2}$ on (a, b) when μ_k approaches x we can take

$$\left| \int_{a}^{b} f(y)\phi_{ik}(y)dy - \int_{a}^{b} f(y)e^{-\lambda_i(y-x)^2}dy \right| < \epsilon/3$$

by choosing μ_k near enough to x. Adding the three inequalities we obtain

$$\left| \int_{-\infty}^{+\infty} f\phi_{ik}dy - \int_{-\infty}^{+\infty} f(y)e^{-\lambda_i(y-x)^2}dy \right| < \epsilon,$$

whence, by our hypothesis,

$$\left| \int_{-\infty}^{+\infty} f(y)e^{-\lambda_i(y-x)}dy \right| < \epsilon.$$

Since ϵ can be chosen arbitrarily small, i and x being fixed, it follows that for each pair of values (λ_i, x) the integral $W_{\lambda_i}(f; x)$ vanishes. On allowing λ_i to become infinite we find

$$W_{\lambda_i}(f; x) \to 0, \qquad W_{\lambda_i}(f; x) \to f(x), \qquad f(x) = 0.$$

Since in choosing x we had merely to avoid the points of a set of measure zero, we conclude that $f(x) = 0$ almost everywhere.

LEMMA IV. *If $f(x)$ is a function for which the integrals $\int_{-\infty}^{+\infty} f\psi_{ik}dy$, $i, k = 1, 2, \cdots$, exist, then the integral*

$$I_\nu(f;x) = (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{+\infty} f(y)(y-x)e^{-\nu(y-x)^2}dy$$

exists, and conversely; and $W_\nu(f;x)$ exists. If $f(x)$ is a function such that $\int_{-\infty}^{+\infty} f\psi_{ik}dy = 0$, $i, k = 1, 2, \cdots$, then $f(x) = c$, where c is a constant, almost everywhere.

The proof of the existence of the integral I_ν is analogous to that of the corresponding proof in Lemma III. The existence of W_ν can then be deduced almost immediately from the inequality

$$0 < e^{-\nu(y-x)^2} < |y-x|\, e^{-\nu(y-x)^2}, \qquad |y-x| > 1.$$

We now suppose $f(x)$ such that $\int_{-\infty}^{+\infty} f\psi_{ik}dy = 0$, $i, k = 1, 2, \cdots$. For an arbitrary value of x we can show that $I_\nu(f;x) = 0$, $\nu = \lambda_1, \lambda_2, \cdots$, by reasoning similar to that of the last part of the proof of Lemma III. We next choose a value a for which $W_\nu(f;a) \to f(a)$, and an arbitrary value x for which $W_\nu(f;x) \to f(x)$. If we can show that

$$0 = \int_x^a I_\nu(f;x)dx = W_\nu(f;x) - W_\nu(f;a), \qquad \nu = \lambda_1, \lambda_2, \cdots,$$

we have the results

$$W_{\lambda_i}(f;x) - W_{\lambda_i}(f;a) \to 0, W_{\lambda_i}(f;x) - W_{\lambda_i}(f;a) \to f(x) - f(a) = 0.$$

From the last equation we see that $f(x) = f(a)$ for almost all values of x.

In order to establish the last part of the lemma we wish to show, therefore, that the integral $\int_a^b I_\nu(f;x)dx$ may be evaluated by integrating under the sign of integration in I_ν, for fixed ν and arbitrary finite limits a and b. Now for $a \leq x \leq b$

$$\begin{aligned} |y-x|\, e^{-\nu(y-x)^2} &\leq |y-a|\, e^{-\nu(y-a)^2}, & -\infty < y \leq y_1, \\ |y-x|\, e^{-\nu(y-x)^2} &\leq |y-b|\, e^{-\nu(y-b)^2}, & y_2 \leq y < +\infty. \end{aligned}$$

Thus, for $a \leq x \leq b$,

$$\left| (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{A} f(y)\,(y-x)\,e^{-\nu(x-y)^2}dy + (2\nu^{3/2}/\pi^{1/2}) \int_{B}^{+\infty} f(y)\,(y-x)\,e^{-\nu(x-y)^2}dy \right|$$

$$\leq (2\nu^{3/2}/\pi^{1/2}) \int_{-\infty}^{A} |\, f(y)\,|\;|\, y-a\,|\; e^{-\nu(y-a)^2}dy$$

$$+ (2\nu^{3/2}/\pi^{1/2}) \int_{B}^{+\infty} |\, f(y)\,|\;|\, y-b\,|\; e^{-\nu(y-b)^2}\,dy$$

$$\leq \epsilon/2(b-a)$$

for A negative and B positive and both sufficiently great in absolute value. Similarly we have for $x=a$ and $x=b$

$$\left| (\nu/\pi)^{1/2} \int_{-\infty}^{A} f(y)\,e^{-\nu(y-x)^2}dy + (\nu/\pi)^{1/2} \int_{B}^{+\infty} f(y)\,e^{-\nu(y-x)^2}dy \right| < \epsilon/4$$

if A and B are properly chosen. It is well known that

$$(2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \int_{A}^{B} f(y)\,(y-x)\,e^{-\nu(y-x)^2}dy\,dx$$

$$= (\nu/\pi)^{1/2} \int_{A}^{B} f(y)\,(e^{-\nu(y-b)^2} - e^{-\nu(y-a)^2})\,dy,$$

the integration under the integral sign being permissible. Thus we have

$$\left| \int_{a}^{b} I_\nu(f;x)\,dx - W_\nu(f;b) - W_\nu(f;a) \right| =$$

$$\Bigg| (2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y)\,(y-x)\,e^{-\nu(y-x)^2}dy\,dx$$

$$- (\nu/\pi)^{1/2} \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y)\,(e^{-\nu(y-b)^2} - e^{-\nu(y-a)^2})\,dy \Bigg| \leq$$

$$(2\nu^{3/2}/\pi^{1/2}) \int_{a}^{b} \left| \left(\int_{-\infty}^{A} + \int_{B}^{+\infty} \right) f(y)\,(y-x)\,e^{-\nu(y-x)^2}dy \right| dx + (\epsilon/2)$$

$$< \epsilon.$$

The desired equality follows at once, since ϵ may be taken arbitrarily small.

We next prove

THEOREM I. *If $f(x)$ is summable on every finite interval, then on any finite interval (a, b) a sequence of linear combinations of functions of the type $e^{-\lambda(x-\mu)^2}$ can be determined converging almost everywhere to $f(x)$, the convergence being uniform on any closed interval completely interior to a range on which $f(x)$ is continuous.*

We choose an interval (A, B) including an arbitrary preassigned interval (a, b) in its interior. We let f_1 be identical with f on (A, B) and identically zero elsewhere. Then $W_\nu(f_1; x)$ converges to $f(x)$ almost everywhere on (a, b), the convergence being uniform on any closed interval completely interior to an interval on which $f(x)$ is continuous.*

We divide the interval (A, B) into n equal intervals $\Delta_1, \cdots, \Delta_n$. In each interval we choose a point $x = \mu_i$. Then for y in Δ_i and x in (a, b)

$$\begin{aligned} | e^{-\nu(y-x)^2} - e^{-\nu(x-\mu_i)^2} | &= 2\nu \, | \mu_i - x + \theta(y - \mu_i) | \, | e^{-\nu(\mu_i - x + \theta(y-\mu_i))^2} | \, | y - \mu_i | \\ &\leq \max | 2Xe^{-X^2} | \, \nu^{1/2} (B - A)/n \\ &< (C\nu^{1/2}/n), \qquad 0 < \theta < 1. \end{aligned}$$

by the law of the mean. Hence we can write

$$\begin{aligned} W_\nu(f_1; x) = \sum_{i=1}^{n} (\nu/\pi)^{1/2} \int_{\Delta_i} f(y)\, dy \cdot e^{-\nu(x-\mu_i)^2} \\ + \sum_{i=1}^{n} (\nu/\pi)^{1/2} \int_{\Delta_i} f(y) e^{-\nu(y-x)^2} - e^{-\nu(x-\mu_i)^2})\, dy. \end{aligned}$$

The first term is of the form $\sum_{i=1}^{n} C_i^{(n)} e^{-\nu(x-\mu_i)^2})$; the second is in absolute value less than $C\nu \int_A^B | f | \, dy/n\pi^{1/2}$. Hence if $\nu = n^\alpha$, $\alpha < 1$, the last term approaches zero on (a, b). In short

$$\sum_{i=1}^{n} C_i^{(n)} e^{-n^\alpha (x-\mu_i)^2} = W_{n^\alpha}(f_1; x) + O(n^{\alpha-1})$$

uniformly for $a \leq x \leq b$. The theorem results immediately.

THEOREM II. *If $f(x)$ is a function whose square is summable on the infinite interval $(-\infty, +\infty)$, then $f(x)$ can be represented by a sequence of sums of the form $\sum c_{ik}\phi_{ik}$ converging essentially uniformly to $f(x)$ on $(-\infty, +\infty)$.*

By Lemma II the set of functions ϕ_{ik} can be normalized and orthogonalized by Schmidt's process * on the interval $(-\infty, +\infty)$. The functions Φ_i so defined are linear combinations of the functions ϕ_{ik} and conversely. In view of this fact and of Lemma III the functions Φ_i constitute a normal orthogonal set closed with respect to the class of functions f for which the integrals $\int_{-\infty}^{+\infty} f\Phi_i dy$ exist, this class being the same as that for which the integrals $\int_{-\infty}^{+\infty} f\phi_{ik} dy$ exist. Any function f whose square is summable on $(-\infty, +\infty)$ can be represented by a sequence of sums selected from the set

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* Lebesgue, *loc. cit.*

† E. Schmidt, *Mathematische Annalen*, Vol. 63 (1907), pp. 442-444.

$$\sum_{i=1}^{n} c_i\Phi_i, \qquad c_i = \int_{-\infty}^{+\infty} f\Phi_i dy,$$

this sequence converging essentially uniformly to $f(x)$ on $(-\infty, +\infty)$, in accordance with the theorem of Riesz-Fischer in the form given to it by Weyl and Plancherel.* This establishes the theorem, since Φ_i is a linear combination of functions ϕ.

For the proof of the third theorem we employ a continuous function with the following properties

$$\begin{aligned} &p(x) > 0, && && \\ &x^{\alpha_1}p(x) \to c_1 > 0, && x \to +\infty, && \alpha_1 > \tfrac{1}{2}, \\ &x^{\alpha_2}p(x) \to c_2 > 0, && x \to -\infty, && \alpha_2 > \tfrac{1}{2}. \end{aligned}$$

Then p^2 is summable on $(-\infty, +\infty)$, and its integral over that range may be taken equal to unity. We denote by $g(x)$ any function expressible in the form

$$g(x) = \int_{-\infty}^{x} pf\,dy - A\int_{-\infty}^{x} p^2 dy, \qquad A = \int_{-\infty}^{+\infty} pf\,dy;$$

that is, in the form

$$g(x) = \int_{-\infty}^{x} pf_1 dy, \quad f_1 = f - Ap, \qquad \int_{-\infty}^{+\infty} pf_1 dy = 0.$$

where f and f_1 are functions whose squares are summable on $(-\infty, +\infty)$. If $g(x)$ is an indefinite integral, such that g and g' satisfy the conditions

$$\begin{aligned} &g(x) \to 0, && |x| \to \infty, && \\ &g'(x) = O(x^{\beta_1}), && x \to +\infty, && \alpha_1 + \beta_1 < -1, \\ &g'(x) = O(|x|^{\beta_2}), && x \to -\infty, && \alpha_2 + \beta_2 < -1 \end{aligned}$$

then $f_1 = g'/p$ is a function whose square is summable, and such that

$$g(x) = \int_{-\infty}^{x} pf_1 dy = \int_{-\infty}^{x} g' dy.$$

We can now prove

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[*] Plancherel, *loc. cit.*

THEOREM III. *Any function* $g(x)$ *can be represented by a sequence of sums of the form* $\sum c_{ik}\phi_{ik}$ *converging uniformly on the infinite range* $-\infty < x < +\infty$.

From Lemma II and the fact that ψ_{ik}/p is summable with summable square on $(-\infty, +\infty)$ we can form from the set of functions ψ_{ik}/p a normal orthogonal set Ψ_i/p. The functions Ψ are expressible as linear combinations of the functions ψ and conversely. Consequently the set of functions f for which the integrals $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy$ exist is identical with the set for which the integrals $\int_{-\infty}^{+\infty}(f\,\psi_{ik}/p)\,dy$ exist. Since the set of equations $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy = 0$ implies the set $\int_{-\infty}^{+\infty}(f\,\psi_{ik}/p)\,dy = 0$, it implies also that $f(x)$ reduces almost everywhere to a constant multiple of $p(x)$, by Lemma IV. The set of normal orthogonal functions composed of p and the functions Ψ_i/p is closed with respect to the class of all functions f for which the integrals $\int_{-\infty}^{+\infty} f\,p\,dy$, $\int_{-\infty}^{+\infty}(f\,\Psi_i/p)\,dy$ exist. We now employ the theorem of Riesz-Fischer to show that, if f_1 is a function with summable square for which $\int_{-\infty}^{+\infty} p\,f_1\,dy = 0$,

$$\lim_{\nu\to\infty} \int_{-\infty}^{+\infty}(f_1 - \sum_{i=1}^{n} c_i\Psi_i/p)^2 dy = 0,$$

where $c_i = \int_{-\infty}^{+\infty} f_1\Psi_i/p\,dy$. Thus, by Schwarz's inequality,

$$\left| \int_{-\infty}^{x} p[f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]\,dy \right| \leqq \left(\int_{-\infty}^{x} p^2 dy \cdot \int_{-\infty}^{x} [f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]^2 dy\right)^{1}$$
$$\leqq \left(\int_{-\infty}^{+\infty} [f_1 - \sum_{i=1}^{n} c_i(\Psi_i/p)]^2 dy\right)^{\frac{1}{2}} < \epsilon$$

for n sufficiently large. In other terms, the sequence

$$\sum_{i=1}^{n} c_i \int_{-\infty}^{x} (p\cdot\Psi_i/p)\,dy = \sum_{i=1}^{n} c_i \int_{-\infty}^{x} \Psi_i\,dy = \sum c_{ik}\phi_{ik}$$

converges uniformly to $g(x) = \int_{-\infty} pf_1 dy$ on the range $-\infty < x < \infty$.

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On (3, 3) and Higher Point Correspondences.*

BY TEMPLE RICE HOLLCROFT.

1. *Purpose and general definitions.* The classification of (2, 2) point correspondences between two planes was made in 1917 by F. R. Sharpe and Virgil Snyder.† Since then (2, 3) point correspondences have been classified by the same methods.‡ There are also some earlier papers that discuss certain correspondences but make no attempt at classification. References to these are given in the papers on (2, 3) point correspondences.

Point correspondences are said to be classified when all the algebraic curve systems defining such correspondences are found that are not reducible to each other by birational transformations.

In the present paper it is proposed to classify (3, 3), (2, 4), (3, 4) and (4, 4) algebraic point correspondences and give the principal features of each. The (3, 3) correspondences will be treated in some detail, but only the essentially different features of the others will be discussed.

Point correspondences between two planes, both multiple, may be either involutorial or not involutorial. The latter are called general correspondences, the former, compound involutions. In a general (3, 3) point correspondence, to any point P' of the plane (x') correspond three points P_1, P_2, P_3 of the plane (x). The image of P_1 is P' and two other points P_1' and P_1''; that of P_2 consists of P', P_2', P_2'' and that of P_3: P', P_3', P_3'', all distinct points. Going back to (x), to P_1' correspond P_1, P_4, P_5; to P_1'' correspond P_1, P_6, P_7, etc. The correspondence does not close up, that is, starting with an arbitrary point in either plane, any number of points in either plane can be obtained by repeating the transformation. In the case of a (3, 3) compound involution, however, the correspondence closes up after the second application. To a point P' of (x') correspond three points P_1, P_2, P_3 of (x) to each of which correspond the original point P' and two other points P_4', P_5', and the images of either P', P_4' or P_5' are the three points P_1, P_2, P_3 of (x). In this case, one point of either plane determines only three points of each plane.

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* Presented to the American Mathematical Society, February 26, 1927.

† F. R. Sharpe and V. Snyder, "Types of (2, 2) Point Correspondences Between Two Planes," *Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 409-414.

‡ T. R. Hollcroft, "A Classification of General (2, 3) Point Correspondences Between Two Planes," *American Journal of Mathematics*, Vol. 41 (1919), pp. 5-24; "On (2, 3) Compound Involutions," *American Journal of Mathematics*, Vol. 43 (1921), pp. 199-212.

2. *Properties of the general correspondence.* In the general $(3, 3)$ point correspondence, a type is distinguished and defined by two equations, each containing the coordinates of both planes. Independent types are those whose defining equations can not be reduced to those of another by any birational transformation. There are fifteen independent types of general $(3, 3)$ point correspondences.

The defining equations of a general point correspondence between two planes (x) and (x') may be written as two algebraic equations of the form

$$\sum u_k(x)\, u_k'(x') = 0$$
$$\sum v_k(x)\, v_k'(x') = 0.$$

For any set of values of the x_i' $[x_i]$ these equations represent two curves of (x) $[(x')]$ intersecting in three non-basic points which are the images of the point defined by the original set of values of x_i' $[x_i]$. In addition to the point correspondence, the defining equations establish a transformation between the two planes such that to a line C_1' of (x') $[C_1$ of $(x)]$ corresponds a curve $C_n(x)$ $[C_n'(x')]$ of order n and genus p through the basis points of (x) $[(x')]$.*

In any $(3, 3)$ point correspondence, the general features of both planes are the same, differing only because the defining equations of the two planes differ. In the remainder of this section, the general properties will be given for only one plane, but they will apply equally well to the other.

To a point P' of (x') correspond three points of (x) which describe the image curve $C_n(x)$ as P' describes the line C_n'. To the point of intersection of two lines of (x') correspond in (x) three non-basic intersections of the two curves which are the images of these lines. These three image points always lie at the intersections of the two curves of (x) given by the defining equations for the set of values of x_i' determined by the point P' of (x'). The images are distinct points except for points on certain fixed curves.

If two of the three image points of (x) coincide, the point P' lies on the branch-point curve of (x') which will be denoted by L'. The locus of the corresponding coincidences is the coincidence curve of (x) denoted by K. The locus of the third image point in (x) is the residual curve denoted by G. Both K and G are in $(1, 1)$ correspondence with L'. When all three image points coincide, K and G have a common tangent at the coincidence and P' is a cusp on L'. The image of L' is K^2G. Aside from basis points, the only singularities of L' are cusps. K and G each have the same genus as L', but, aside from basis points, their singularities are not further restricted.

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* For an explanation of basis points see Hollcroft, *loc. cit.*, (first paper), page 7.

The following relation exists among the intersections of the coincidence and residual curves with the respective branch-point curves of the two planes. The contacts of K and L increased by the intersections of G and L equal the contacts of K' and L' increased by the intersections of G' and L'.

If a line C_1 meets K and G in i and j points respectively, its image $C_n'(x)$ has i contacts and j intersections with L'. The image of $C_n'(x)$ is C_1 counted three times and a residual curve that cuts C_1 in $2d+i$ points corresponding to the d non-basic double points of C_n' and the i contacts of L' and C_n'.

3. *Types.* The fifteen independent types of general (3, 3) point correspondences will now be enumerated. Each type is established by a set of defining equations as described above. Special cases may be common to two or more types. In the following table, the types are denoted by Roman numerals and opposite each is given the description of the curves constituting the defining equations belonging to that type. The symbol $C_n; jP_i$ represents a curve of order n which has j basis points each of multiplicity i.

Type	$u_k(x)$	$u_k'(x')$	$v_k(x)$	$v_k'(x')$
I	C_1	C_1	C_3	C_3
II	C_1	C_3	C_3	C_1
III	C_1	$C_n; P_{n-3}$	C_3	$C_1; P_1$
IV	C_1	$C_2; P_1$	C_3	$C_2; P_1$
V	C_1	$C_3; 6P_1$	C_3	$C_3; 6P_1$
VI	C_1	$C_9; 8P_3$	C_3	$C_3; 8P_1$
VII	C_1	$C_2; 2P_1$	C_3	$C_3; P_1, P_2$
VIII	$C_1; P_1$	$C_1; P_1$	$C_n; P_{n-3}$	$C_{n'}; P_{n'-3}$
IX	$C_1; P_1$	$C_3; 6P_1$	$C_n; P_{n-3}$	$C_3; 6P_1$
X	$C_1; P_1$	$C_3; 8P_1$	$C_n; P_{n-3}$	$C_9; 8P_3$
XI	$C_2; P_1$	$C_2; P_1$	$C_2; P_1$	$C_2; P_1$
XII	$C_2; P_1$	$C_3; 6P_1$	$C_2; P_1$	$C_3; 6P_1$
XIII	$C_3; 6P_1$	$C_3; 6P_1$	$C_3; 6P_1$	$C_3; 6P_1$
XIV	$C_3; 6P_1$	$C_3; 8P_1$	$C_3; 6P_1$	$C_9; 8P_3$
XV	$C_3; 8P_1$	$C_3; 8P_1$	$C_9; 8P_3$	$C_9; 8P_3$

The methods by which the characteristic curves of each type are obtained are similar to those used in the paper on general (2, 3) point correspondences cited in section 1.

4. *Completeness of the classification.* In order to show that the classification is complete, it must be proved that the defining equations of any general (3, 3) point correspondence are birationally equivalent to the defining equations of some one of these fifteen types. To establish a general (3, 3) point correspondence, it is necessary and sufficient that in both planes the two curves given by the defining equations intersect in three and only three non-basic points, provided that in neither plane both the curves given by the defining equations belong to nets or pencils. The six curve systems employed in the above fifteen types are as follows: line and cubics; line pencil and curves of order n with a point of multiplicity $n - 3$ at the vertex of the pencil; conics with one basis point; cubics with six basis points; cubics with a double and a simple basis point through both of which pass conics; cubics with eight basis points and curves of order nine with triple points at each of them.

From these six curve systems can be obtained forty-two sets of defining equations of which twenty-seven are either reducible to the remaining fifteen or else do not define a general (3, 3) point correspondence. The latter is true of all those for which in either plane both the defining curves form nets or pencils since in these cases the correspondence is a compound involution. Also when the fact that one of the components of a defining equation forces the other component to be also a pencil, the correspondence established is usually a special case of some type. For examples of this and of sets reducible to other sets by means of birational transformations, see the paper on general (2, 3) point correspondences cited in section 1.

It now remains to be shown that any curve system that has three non-basic intersections is birationally equivalent to some combination in the above six systems. The proof for this is essentially the same as that for (1, 2) and (1, 3) plane involutions and was first used by Bertini * in reducing (1, 2) involutions to three types.

5. *Compound involutions.* Every (3, 3) compound involution can be expressed as a combination of two (1, 3) involutions and one birational transformation.†

Assume that a (1, 3) involution has been established between two planes (y) and (x). Then to a point P of (y) correspond three points P_1, P_2, P_3 of (x) and to each of these points of (x) corresponds the point P of (y).

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* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. 8 (1877), pp. 244-286.

† R. Baldus, "Zur Theorie den gegenzeitig mehrdeutigen algebraischen Ebenentransformationen," *Mathematische Annalen*, Vol. 72 (1912), p. 33.

Also to lines of (x) through P_1, P_2, or P_3 correspond rational curves of (y) forming a net and all passing through P. Assume further that a $(1, 3)$ involution has been established between the planes (y') and (x') such that to a point P' of (y') correspond the points P_1', P_2', P_3' of (x').

The planes (y) and (y') contain nets of rational image curves through the points P and P' respectively. Since any two rational curves are birationally equivalent, a $(1, 1)$ correspondence exists between (y') and (y) such that P corresponds to P' and P' to P and such that all curves in (y') that are images of rational curves of (x') and all curves in (y') that are images of rational curves of (x') are transformed reciprocally one into the other.

Then to each of the three points P_1', P_2', P_3' of (x') corresponds P' of (y') to which corresponds P of (y) to which correspond P_1, P_2, P_3 of (x). Reciprocally $P_1, P_2, P_3 \sim P \sim P' \sim P_1', P_2', P_3'$. Between the planes (x) and (x') there is thus established a $(3, 3)$ compound involution.

The curves of either plane are transformed into curves of the other by three transformations, two rational and one irrational. Since the two planes (y) and (y'), are birationally equivalent, they may be considered the same plane in which the two systems are rationally separable. Then the image in (x') of a curve of (x) is obtained by applying to its image in (y) considered as being also in (y') and as retaining its basis points as fixed points of (y') the transformation from (y') to (x'). A similar series of transformations relates (x') to (x).

If two of the three images of a point P' of (x') coincide, P' is on L', the branchpoint curve of (x'). The locus of the corresponding coincidences in (x) is the coincidence curve K and the locus of the third image point is the residual curve G. K is the Jacobian of the net of curves in (x) that are images of the lines of (y). The image of L' is K counted six times and G counted three times. The complete image of either K or G is L'. The curves L', K and G are not in $(1, 1)$ correspondence as in the case of general $(3, 3)$ point correspondences. The non-basic intersections of K and G are all contacts to each of which correspond three cusps of L'. The three images of each of these three cusps coincide at the point of contact of K and G. The same general relations hold for L, K' and G'.

The non-basic intersections of L and K, of L and G, of L' and K' and of L' and G' are all equal in number and are all contacts. To each contact of L and K correspond a contact of L' and K' and a contact of L' and G'. To the associated contact of L and G corresponds the same contacts of L' and K' and of L' and G' and the image of each of these is the original pair of contacts of L and K and of L and G.

6. *Types.* There are five independent types of $(1, 3)$ point correspondences.* These types are defined as follows:

Type I.	Lines and cubics of a net.
Type II.	Line pencil, vertex P and C_n with $(n-3)$-fold point P.
Type III.	Two nets of conics with one basis point.
Type IV.	Two nets of cubics with six basis points.
Type V.	Pencil of cubics and net of C_9 with $8P_3$ at basis points of pencil.

Fifteen types of $(3, 3)$ compound involutions are obtained by combining the types of $(1, 3)$ involutions in all possible ways. In the following table, the Roman numerals refer to the types of $(1, 3)$ involutions, and the Arabic numerals to the types of $(3, 3)$ compound involutions. Any type of $(3, 3)$ compound involutions is established by combining the $(1, 3)$ involutions in its row and column. The $(1, 3)$ involutions in the row are those relating the planes (x') and (y') and those in the column relate (x) and (y).

	I	II	III	IV	V
I	1	2	3	4	5
II		6	7	8	9
III			10	11	12
IV				13	14
V					15

The proof that these are all the possible independent types of $(3, 3)$ compound involutions follows immediately from the fact that every $(3, 3)$ compound involution can be established by two $(1, 3)$ involutions and that there are but five independent types of $(1, 3)$ involutions.

The characteristic curves of each type can be readily found by the general method outlined in the preceding section.

7. *Pencil cases.* The theorems proved for $(2, 2)$ and $(2, 3)$ compound involutions hold for all compound involutions and may be stated as follows:

I. *The necessary and sufficient condition that a multiple point correspondence be a compound involution is that the image curves in either plane form a net.*

II. *A sufficient condition that a multiple point correspondence be a com-*

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* A. M. Howe, "A Classification of Plane Involutions of Order Three," *American Journal of Mathematics*, Vol. 41 (1919), pp. 25-40.

pound involution is that in either plane both components of the curve system defining the image points form pencils.

In accordance with Theorem II, the fifteen types of general (3, 3) point correspondences become compound involutions when the defining equations each have but two homogeneous parameters. In the following tabulation, the pencil form of each general (3, 3) point correspondence type (Roman numeral) is a special case of the (3, 3) compound involution type (Arabic numeral) given with it.

I, II, III, IV, VII, VIII, XI	6
V, IX, XI	8
VI, X	9
XIII	13
XIV	14
XV	15

8 *Cyclic cases.* In Types II, IV and a special case of Type V of (1, 3) involutions, the three image points form a cyclic projectivity of period three.[*] Those (3, 3) compound involutions formed from these types retain this property of the image points in one or both planes, depending on whether the compound involution is established by one or two (1, 3) involutions of this kind. In a (1, 3) involution there is but one set of three points determined by two lines of the triple plane and forming a cyclic projectivity, but in a compound involution two lines of one plane determine several triads of points in the other plane, each triad forming a cyclic projectivity of period three. The following table shows the number of triads in each plane for a given type of (3, 3) compound involutions. Types involving Type V of (1, 3) involutions are not the most general (3, 3) compound involutions of those types. In other cases the types are entirely general.

Types	*Number of triads in*	
	(x) plane	*(x′) plane*
2, 4, 5, 11, 12		16
6	$2n'+1$	$2n+1$
7		$2n+1$
8	36	$2n+1$
9	63	$2n+1$
13	36	36
14	63	36
15	63	63

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[*] A. M. Howe, *loc. cit.*, pp. 39-49.

9. (2, 4) *Point Correspondences.* For the general correspondence, the defining equations are of the same form as those of section 2 except that now the curve system of (x) defines four and that of (x') two non-basic points. The essential difference between any two general point correspondences concerns the branch-point curve and its images, so these will be the only features discussed.

Given a pair of equations defining a general (2, 4) point correspondence between the two planes (x) and (x') such that to a point of (x') correspond four points of (x) and to a point of (x) correspond two points of (x'). The curve L is the locus of points of (x) whose two images in (x') coincide, and the complete image of L is the coincidence locus K' counted twice. The curve L' is the locus of points two of whose images in (x) coincide. The complete image of L' is K^2G, where K is the locus of the two coincident points and G is the locus of the two residual non-coincident points. The complete image of K is L' and the complete image of G is L'^2.

When three images of a point P' coincide, P' is a cusp of L' and the triple coincidence lies at a contact of K and G. When a pair of coincidences corresponds to P', P' is a node of L' and the two coincidences lie at two intersections of K and G such that the line joining them is tangent to G at both points of intersection. All four image points can not in general coincide since the curves of the system defining the four points can not in general have four point contact.

The intersections of L with K and G are all contacts and those of L' and K' are as many contacts as the contacts of L and K and as many intersections as the contacts of L and G.

A (2, 4) compound involution is equivalent to the involution product (2, 1) (1, 1) (1, 4). The discussion of these is similar to that previously given. In the classification of (1, 4) involutions, nine independent types were found* and these may be combined with the three types of (1, 2) involutions in twenty-seven ways, so there are twenty-seven independent types of (2, 4) compound involutions.

10. (3, 4) *Point Correspondences.* When the defining equations determine four non-basic points of (x) and three of (x') a general (3, 4) point

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* T. R. Hollcroft, "Plane Involutions of Order Four," *American Journal of Mathematics*, Vol. 44 (1922), pp. 163-171. Professor F. R. Sharpe has called my attention to the fact that in the (x) plane, the net of image curves of Type 9 of (1, 4) involutions is reducible to the net of image curves of Type 8 by quadric inversion. Type 9 should therefore be omitted and there remain nine independent types of (1, 4) plane involutions obtained by the method of this paper.

correspondence is established. The curve L' is now the locus of points two of whose images coincide on K and whose third image lies on G. The curve L is the locus of points which have two coincident images on K' and two distinct images on G'. Both nodes and cusps occur on L but L' has only cusps. Certain types have triads of points in (x') forming cyclic involutions of period three similar to those discussed in section 8.

A $(3, 4)$ compound involution is equivalent to the involution product $(3, 1)(1, 1)(1, 4)$. Since there are five types of $(1, 3)$ and nine types of $(1, 4)$ involutions, there are forty-five independent types of $(3, 4)$ compound involutions.

11. $(4, 4)$ *Point Correspondences.* When the defining equations determine four non-basic points in each plane, a general $(4, 4)$ point correspondence is established. Both L and L' now have properties similar to L of the two preceding sections and both K and K', G and G' now have properties similar to those of K' and G' of the two preceding sections.

A $(4, 4)$ compound involution is equivalent to the involution product $(4, 1)(1, 1)(1, 4)$. The nine birationally distinct types of $(1, 4)$ involutions, when combined each with itself and with each of the others, give forty-five independent types of $(4, 4)$ compound involutions.

12. *Types of general* $(2, 4)$, $(3, 4)$ *and* $(4, 4)$ *point correspondences.* There are, respectively, 30, 65 and 78 independent types of general $(2, 4)$, $(3, 4)$ and $(4, 4)$ algebraic point correspondences. These are exhibited in the following table.

In the uppermost row of the table are given the eleven birationally independent types of algebraic curve systems of the (x) plane that define four points. In the column at the left, the first three curve systems are three birationally independent curve systems of the (x') plane that define two points. (The Bertini curve system C_3, $8P_1$; C_6, $8P_2$, when combined with any curve system always defines an involutorial transformation. The system of two conics with two basis points is not an independent $(1, 2)$ involution type when combined with two lines, but, in general, it forms independent types when combined with curve systems defining two or more points.)

The independent types of general point correspondences are defined by equations of the form given in section 2 in which the u_k; v_k are the curve systems given in the row and u_k'; v_k' the curve systems given in the column. Each number denotes a type whose curve systems are given in the row and column in which the number is found. If the number has no subscript, the two curve systems occur in the defining equations in the order given, that is, the first parts of each curve system form the first defining equation and the second parts of each curve system, the second defining equation. For example, Type 1 of general (2, 4) point correspondences is established by two defining equations, the first formed by $C_1(x)$ and $C_1(x')$ and the second by $C_4(x)$ and $C_2(x')$. If the type number has the subscript *r*, the equations are formed by reversing the order of either set. For example, Type 2 of (2, 4) point correspondences (denoted by 2_r) is established by equations, the first formed by $C_1(x)$ and $C_2(x')$ and the second by $C_1(x')$ and $C_4(x)$. The combination corresponding to a space in which no number is given is birationally equivalent to some combination for which a number is given. All for which numbers are given are birationally independent of all others for which different numbers are given.

After the first three rows, the next six [last eleven] in the column at the left, contain the six [eleven] birationally independent algebraic curve systems that define three [four] points, and in the six [eleven] rows opposite these are found the type numbers corresponding to combinations of curve systems that define independent types of general (3, 4) [(4, 4)] point correspondences. The last curve system $C_2, 2P_1;\ C_4, P_3, P_1$ defining four points combines only with curve systems defining more than one point to form independent types of point correspondences.

(x) x')	$C_1; C_4$	C_n, P_{n-4} C_1, P_1;	$C_3, 5P_1$ $C_3, 5P_1$;	$C_2; C_2$	$C_3, 8P_1$; $C_{12}, 8P_4$	$C_3, 8P_1$; $C_4, 8P_1$	C_2, P_1; C_3, P_2	C_2, P_1; $C_3, 2P_1$	$C_2, 2P_1$; $C_4, 2P_2$	$C_3, 7P_1$; $C_4, P_2, 6P_1$	$C_2, 2P_1$; C_4, P_3, P_1
$C_1; C_2$	$1, 2_r$		3	4			$5, 6_r$	$7, 8_r$	$9, 10_r$	$11, 12_r$	$28, 29_r$
C_1, P_1; C_n, P_{n-2}	13_r	14	15		16	17		18_r		$19, 20_r$	
$C_2, 2P_1$; $C_2, 2P_1$	21		22	23			24	25	26	27	30
C_1, C_3	$1, 2_r$	3_r	4	5	6_r	7_r	$8, 9_r$	$10, 11_r$	$12, 13_r$	$14, 15_r$	$60, 61_r$
C_1, P_1; C_n, P_{n-3}	16_r	17	18		19	20		21_r		$22, 23_r$	
C_2, P_1; C_2, P_1	24		25	26			27	28	29	30	62
$C_3, 6P_1$; $C_3, 6P_1$	31	32	33	34	35	36	37	38	39	40	63
$C_3, 8P_1$; $C_9, 8P_3$	41_r	42	43		44	45		46_r		$47, 48_r$	
$C_2, 2P_1$; C_3, P_2, P_1	49		50	51			$52, 53_r$	$54, 55_r$	$56, 57_r$	$58, 59_r$	$64, 65_r$
$C_1; C_4$	$1, 2_r$	3_r	4	5	6_r	7_r	$8, 9_r$	$10, 11_r$	$12, 13_r$	$14, 15_r$	$65, 66_r$
C_1, P_1; C_n, P_{n-4}		16	17		18	19		20_r		$21, 22_r$	
$C_3, 5P_1$; $C_3, 5P_1$			23	24	25	26	27	28	29	30	67
$C_2; C_2$				31			32	33	34	35	68
$C_3, 8P_1$; $C_{12}, 8P_4$					36	37		38_r		$39, 40_r$	
$C_3, 8P_1$; $C_4, 8P_1$						41		42_r		$43, 44_r$	
C_2, P_1; C_3, P_2							$45, 46_r$	$47, 48_r$	$49, 50_r$	$51, 52_r$	$69, 70_r$
$C_2, 2P_1$; $C_3, 2P_1$								$53, 54_r$	$55, 56_r$	$57, 58_r$	$71, 72_r$
$C_2, 2P_1$; $C_4, 2P_2$									$59, 60_r$	$61, 62_r$	$73, 74_r$
$C_3, 7P_1$; $C_4, P_2, 6P_1$										$63, 64_r$	$75, 76_r$
$C_2, 2P_1$; C_4, P_3, P_1											$77, 78_r$

13. *Correspondences of multiplicity higher than four.* The chief distinguishing features of general (m, n) point correspondences lie in the singularities and the relations of the images of the branch-point curve, that is, are concerned with the various ways in which image points may coincide. Two general curves of a defining equation can not have more than three consecutive points in common nor more than two simple contacts. This is true because each pair of curves defining a set of points is determined by two independent parameters, the coordinates of the point of the other plane to which this set of points corresponds. These coincidences of image points occur when either m or n is four, so that for m or n greater than four, no new features of this kind are introduced.

Also, since no new features occur in $(1, n)$ involutions for $n > 4$, a $(4, 4)$ compound involution has all the characteristic features of an (m, n) compound involution for all values of m and n.

It has always been assumed that the curves of the defining equations of both general (m, n) point correspondences and $(1, n)$ involutions are the most general curves possible. In any of these, special curves can be chosen for the defining equations that will allow the coincidence of more image points. For example, in the $(1, 4)$ involution determined by lines and a net of quartics, quartics that possess undulations may be chosen to define the four image points, in which case all four image points may coincide.

The number of independent types of (m, n) point correspondences increases very rapidly as the multiplicity of one plane or each of the two planes increases. There is no further relation between the multiplicity and the number of types.

WELLS COLLEGE.

Contributions to the Theory of Conjugate Nets.

By Ernest P. Lane.

1. *Introduction.* The projective differential geometry of a conjugate net on a curved surface was studied by G. M. Green in two memoirs which appeared in the *American Journal of Mathematics* in 1915 and 1916.* In these Green took the conjugate net under discussion as the parametric net.

The recent researches of Fubini and others have shown the power of the methods of the absolute calculus in projective differential geometry. In the present paper we follow Green in taking a conjugate net as parametric, and obtain a new canonical form for Green's equations. This we are able to write with covariant derivatives, using the methods of the absolute calculus. Our canonical form is invariant under all transformations of conjugate nets on the surface.

The differential equations of a surface referred to its projective lines of curvature are obtained, as well as the equations of a surface referred to certain other covariantly defined conjugate nets. The invariants of such a net are really invariants of the surface, and we are enabled thus to define projectively certain classes of surfaces which seem to be of considerable interest.

Several integrals geometrically connected with a conjugate net are set up, and some methods of investigation indicated. The unsolved problem of interpreting these integrals geometrically is formulated.

Wilczynski discovered the two *directrices* associated with each point of a surface (now called by Italian geometers the *lines of Wilczynski*) when he was investigating the congruence of intersection of the osculating linear complexes of the two asymptotic curves through a surface point. In our concluding section we have carried out a similar investigation for the osculating linear complexes of the two curves of a conjugate net through a point of a surface.

2. *A Canonical Form of Green's Differential Equations.* Let the homogeneous coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point on a non-degenerate non-ruled surface be given as analytic functions of two independent variables u, v. If the parametric net is conjugate, the four functions y are solutions of a system of differential equations of the form *

$$
\begin{aligned}
y_{uu} &= a y_{vv} + b y_u + c y_v + d y, \\
y_{uv} &= \phantom{a y_{vv} +{}} b' y_u + c' y_v + d' y,
\end{aligned} \tag{1}
$$

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* Green, "Projective Differential Geometry of One-Parameter Families of Space Curves, etc.," *American Journal of Mathematics*, Vol. 37 (1915).

whose coefficients satisfy certain integrability conditions, of which the only one that we shall use is

$$(2)\qquad (b + 2c')_v = (2b' - c/a - a_v/a)_u.$$

Besides the invariant coefficient a, we shall have occasion to use also the following invariants:

$$(3)\qquad \begin{aligned} &H = d' + b'c' - b'_u, & &K = d' + b'c' - c'_v,\\ &W^{(u)} = 2c'_v + (c/a)_u, & &W^{(v)} = 2b'_u - b_v,\\ &\mathfrak{B}' = (1/8a)(4ab' + 2c - a_v), & &\mathfrak{C}' = (1/8a)(4ac' - 2ab + a_u),\\ &R = a\mathfrak{B}'^2 + \mathfrak{C}'^2, & &\mathfrak{D} = d' + ab'^2 - c'^2 + b'c + bc' + ab'_v - c'_u. \end{aligned}$$

These invariants are absolutely unchanged by the transformation

$$(4)\qquad y = \lambda\bar{y},$$

which transforms the coefficients of system (1) according to the formulas

$$(5)\qquad \begin{aligned} &\bar{a} = a, \quad \bar{b} = b - 2(\lambda_u/\lambda), \quad \bar{c} = c + 2a(\lambda_v/\lambda),\\ &\qquad\qquad \bar{d} = (1/\lambda)(-\lambda_{uu} + a\lambda_{vv} + b\lambda_u + c\lambda_v + d\lambda),\\ &\bar{b}' = b' - \lambda_v/\lambda, \quad \bar{c}' = c' - \lambda_u/\lambda,\\ &\qquad\qquad \bar{d}' = (1/\lambda)(-\lambda_{uv} + b'\lambda_u + c'\lambda_v + d'\lambda). \end{aligned}$$

The integrability condition (2) shows that there exists a function q such that

$$(6)\qquad q_u = b + 2c' + \tfrac{3}{2}a_u/a - 2R_u/R, \quad q_v = 2b' - c/a + \tfrac{1}{2}a_v/a - 2R_v/R;$$

and equations (5) show that the effect of the transformation (4) on these deriatives of q is given by

$$\bar{q}_u = q_u - 4\lambda_u/\lambda, \qquad \bar{q}_v = q_v - 4\lambda_v/\lambda.$$

Therefore, if we choose

$$\lambda = e^{\frac{1}{4}q},$$

we shall have $\bar{q}_u = \bar{q}_v = 0$. In this way we obtain a canonical form for system (1), whose coefficients, denoted by capital letters, may be calculated by means of equations (5) and (6). These coefficients are expressed in terms of the invariants of system (1) by the following formulas:

$$(7)\qquad \begin{aligned} &A = a, \quad B = -2\mathfrak{C}' - \tfrac{1}{2}a_u/a + R_u/R, \quad C = a(2\mathfrak{B}' + \tfrac{1}{2}a_v/a - R_v/R),\\ &\qquad B' = \mathfrak{B}' + \tfrac{1}{2}R_v/R, \qquad C' = \mathfrak{C}' - \tfrac{1}{2}a_u/a + \tfrac{1}{2}R_u/R,\\ &\qquad D = \mathfrak{D} - AB'^2 + C'^2 - AB'_v + C'_u - B'C - BC',\\ &\qquad D' = H - B'C' + B'_u = K - B'C' + C'_v. \end{aligned}$$

This canonical form is characterized by the conditions

$$B + 2C' + \tfrac{3}{2}A_u/A - 2R_u/R = 0, \quad 2B' - C/A + \tfrac{1}{2}A_v/A - 2R_v/R = 0. \tag{8}$$

The transformation

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v) \tag{9}$$

leaves the parametric net invariant, but changes the coefficients of system (1) according to the formulas

$$\begin{aligned} \bar{a} = (\psi_v{}^2/\phi_u{}^2)a, \quad \bar{b} &= (1/\phi_u)(b - \phi_{uu}/\phi_u) \\ \bar{c} &= (\psi_v/\phi_u{}^2)(c + a\psi_{vv}/\psi_v), \quad \bar{d} = (1/\phi_u{}^2)d, \\ \bar{b}' &= (1/\psi_v)b', \quad \bar{c}' = (1/\phi_u)c', \quad \bar{d}' = (1/\phi_u\psi_v)d', \end{aligned} \tag{10}$$

and changes the relative invariants (3) according to the formulas

$$\bar{\mathfrak{B}}' = (1/\psi_v)\mathfrak{B}', \quad \bar{\mathfrak{C}}' = (1/\phi_u)\mathfrak{C}', \quad \bar{R} = (1/\phi_u{}^2)R, \quad \bar{\mathfrak{D}} = (1/\phi_u\psi_v)\mathfrak{D}, \tag{11}$$

the invariants H, K, $W^{(u)}$, $W^{(v)}$ being cogredient with $\mathfrak{D}$. Since equations (8) are invariant under this transformation, it follows that our canonical form is undisturbed thereby. In fact *the most general transformation of the group*

$$y = \lambda\bar{y}, \quad \bar{u} = \phi(u), \quad \bar{v} = \psi(v)$$

which leaves our canonical form invariant is obtained by placing $\lambda =$ const.

The quadratic form

$$(R/A)(A\,du^2 + dv^2) \tag{12}$$

is absolutely invariant under the transformations (4) and (9) and vanishes for the asymptotic net. Calculating the Christoffel symbols of the second kind for this form, we find that the second covariant derivatives of a scalar y with respect thereto are given by the formulas

$$\begin{aligned} y_{11} &= y_{uu} - \tfrac{1}{2}y_u(\partial/\partial u)\log R + (A/2)y_v(\partial/\partial v)\log R, \\ y_{12} &= y_{uv} - \tfrac{1}{2}y_u(\partial/\partial v)\log R - \tfrac{1}{2}y_v(\partial/\partial u)\log(R/A), \\ y_{22} &= y_{vv} + (1/2A)y_u(\partial/\partial u)\log R/A - \tfrac{1}{2}y_v(\partial/\partial v)\log(R/A). \end{aligned} \tag{13}$$

Therefore, *when written with these covariant derivatives, our canonical form becomes*

$$\begin{aligned} y_{11} &= Ay_{22} - 2\mathfrak{C}'y_1 + 2A\mathfrak{B}'y_2 + Dy, \\ y_{12} &= \mathfrak{B}'y_1 + \mathfrak{C}'y_2 + D'y. \end{aligned} \tag{14}$$

It is not difficult to show that *the canonical form* (14) *is invariant under* *all transformations of curvilinear coordinates which leave a conjugate net parametric.* In fact, the most general transformation of this kind is

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

subject to the condition

$$\phi_u \psi_u + A \phi_v \psi_v = 0.$$

Carrying out this transformation, we find that system (14) becomes

$$\begin{aligned} \bar{y}_{11} &= \alpha \bar{y}_{22} + \beta \bar{y}_1 + \gamma \bar{y}_2 + \delta y, \\ \bar{y}_{12} &= \phantom{\alpha \bar{y}_{22} +} \beta' \bar{y}_1 + \gamma' \bar{y}_2 + \delta' y, \end{aligned}$$

wherein dashes indicate derivatives with respect to $\bar{u}$ and $\bar{v}$, and the coefficients are given by the following formulas:

$$\begin{aligned} &\phi_u^2 \alpha = \psi_v^2 A, \quad \Delta\delta = D(\phi_u^2 - A\phi_v^2) + 4AD'\phi_u\phi_v, \\ &\Delta\beta = 2[\mathfrak{C}'\phi_u(-\phi_u^2 + 3A\phi_v^2) + A\mathfrak{B}'\phi_v(3\phi_u^2 - A\psi_v^2)], \\ &\Delta\phi_u\gamma = 2A[\mathfrak{C}'\phi_v\psi_v(3\phi_u^2 - A\phi_v^2) + \mathfrak{B}'\phi_u\psi_v(\phi_u^2 - 3A\phi_v^2)], \\ (15) \quad &\Delta\psi_v\beta' = \mathfrak{C}'\phi_u\phi_v(3\phi_u^2 - A\phi_v^2) + \mathfrak{B}'\phi_u^2(\phi_u^2 - 3A\phi_v^2), \\ &\Delta\gamma' = \mathfrak{C}'\phi_u(\phi_u^2 - 3A\phi_v^2) + A\mathfrak{B}'\phi_v(-3\phi_u^2 + A\phi_v^2), \\ &\Delta\psi_v\delta' = -D\phi_u^2\phi_v + D'\phi_u(\phi_u^2 - A\phi_v^2), \quad \Delta = (\phi_u^2 + A\phi_v^2)^2. \end{aligned}$$

It is now easy to verify that

$$(16) \qquad \beta + 2\gamma' = 0, \qquad \gamma - 2\alpha\beta' = 0,$$

and *these conditions are characteristic of the canonical form* (14).

3. *The Projective Lines of Curvature and their Reciprocals.*

The cubic form

$$(17) \qquad (R/A)(A\mathfrak{C}'du^3 - 3A\mathfrak{B}'du^2dv - 3\mathfrak{C}'dudv^2 + \mathfrak{B}'dv^3)$$

is absolutely invariant, just as is the quadratic form (12), and vanishes * for the curves of Darboux. It is evident that $\mathfrak{B}' = 0$ if, and only if, the curves $u =$ const. are curves of Darboux, and $\mathfrak{C}' = 0$ if, and only if, the curves $v =$ const. are curves of Darboux.

The forms (12) and (17) are the forms ϕ_2 and ϕ_3 of Fubini, as may be verified by calculating the ratio † of the discriminant of (17) to the cube of the discriminant of (12). This ratio is a constant. Therefore *the choice of proportionality factor which leads to the canonical form* (14) *is precisely that which yields Fubini's normal coordinates.* It follows that the second differential parameter $\Delta_2 y$, given by

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* Lane, "Bundles and Pencils of Nets on a Surface," *Transactions of the American Mathematical Society,* Vol. 28 (1926), p. 163.

† Fubini and Čech, *Geometria proiettiva differenziale,* Zanichelli (1926), pp. 84-87.

$$\Delta_2 y = (1/R)(y_{11} + Ay_{22}),$$

represents a point on the projective normal n through the point y, while the first differential parameter $\Delta_1 y$, given by

$$\Delta_1 y = (1/R)(y_1{}^2 + Ay_2{}^2),$$

represents the points where the reciprocal n' of the projective normal with respect to the quadric of Lie crosses the asymptotic tangents. Therefore the points y_u and y_v are the points where n' crosses the parametric tangents.

Those curves on a surface in which the developables of the congruence of projective normals intersect the surface are called by Fubini the *projective lines of curvature.* The differential equation of these curves may be determined by the following method. The point ϕ defined by

$$\phi = y_{vv} + (1/A)(C' - 2\mathfrak{C}')y_u + (2\mathfrak{B}' - B' + \tfrac{1}{2}A_v/A)y_v + \lambda y$$

is any point (except P_y) on the projective normal n. As P_y varies along a curve C on the surface, the point ϕ describes a curve, and the point $d\phi$ is on the tangent of the latter. If C corresponds to a developable of the projective normal congruence and if P_ϕ is the corresponding focal point of n, then the point $d\phi$ lies on n, so that $d\phi$ is a linear combination of y and ϕ. So we obtain two equations containing λ and dv/du, and when we eliminate λ therefrom we obtain *the differential equation of the projective lines of curvature*

$$ALdu^2 - Mdudv - Ldv^2 = 0, \tag{18}$$

where

$$\begin{aligned} L &= D' + \mathfrak{C}'_v + \mathfrak{B}'_u + 2C'\mathfrak{B}' + 2B'\mathfrak{C}' - 4\mathfrak{B}'\mathfrak{C}', \\ M &= D + A(4B'\mathfrak{B}' - 4\mathfrak{B}'^2 + 2\mathfrak{B}'_v) + A_v\mathfrak{B}' - 4C'\mathfrak{C}' \\ &\quad + 4\mathfrak{C}'^2 - 2\mathfrak{C}'_u + \mathfrak{C}'(A_u/A). \end{aligned} \tag{19}$$

Therefore *the parametric conjugate net is the projective lines of curvature in case* $L = 0$, $M \neq 0$. *And the associate conjugate net of the parametric net is the projective lines of curvature in case* $M = 0$, $L \neq 0$.

We shall now determine the developables of the reciprocal of the projective normal congruence. The point ψ defined by

$$\psi = y_u + \mu y_v$$

is any point on the reciprocal n' of the projective normal (except the point y_v). As P_y moves along a curve, the point ψ also describes a curve, and the point $d\psi$ is on the tangent of the latter. Continuing the argument, which is the same as that used in the case of the projective normal congruence, we find that the differential equation of the curves corresponding to the developables of the reciprocal of the projective normal congruence, which we shall call *the reciprocal projective lines of curvature,* is

$$AD'du^2 - Ddudv - D'dv^2 = 0. \tag{20}$$

Moreover, if μ_1 and μ_2 are the roots of the equation

$$D'\mu^2 + D\mu - AD' = 0, \tag{21}$$

then the corresponding points ψ_1, ψ_2 are the focal points of n'.

The geometrical significance of the invariants D and D' now becomes evident. *The reciprocal projective lines of curvature are parametric in case* $D' = 0$, $D \neq 0$. Then the foci of n' are the points y_u and y_v, the focus y_u corresponding to the curve $u = \text{const.}$ And the *reciprocal projective lines of curvature are the associate net of the parametric net in case* $D = 0$, $D' \neq 0$. If $D = D' = 0$, the reciprocal projective lines of curvature are indeterminate and the lines n' all lie in a fixed plane. The class of surfaces for which this happens seems worthy of a more detailed study than we can make here.

The geometrical significance of the invariants B' and C' is easy to discover. The ray of P_y with respect to the parametric net crosses the parametric tangents at the points ρ, σ given by

$$\rho = y_u - C'y, \qquad \sigma = y_v - B'y.$$

The points ρ and y_u coincide if, and only if, $C' = 0$; similarly the points σ and y_v coincide in case $B' = 0$. *The ray of the parametric net and the line* n' *reciprocal to the projective normal coincide in case* $B' = C' = 0$.

In case $B' = C' = D' = 0$, $D \neq 0$, the surface is such that the ray of the reciprocal projective lines of curvature coincides with the reciprocal of the projective normal. Then we have $y_{uv} = 0$, $H = K = 0$. The ray-points ρ all lie on one curve, and the points σ on another. The developables of the reciprocal of the projective normal congruence consist of cones each having its vertex on one of these curves and passing through the other. Many special cases arise concerning these curves, but we shall not discuss them here.

If the associate of the parametric net is the reciprocal projective lines of curvature, and if the ray of the parametric net is the reciprocal of the projective normal, then $B' = C' = D = 0$, $D' \neq 0$. Therefore we have $H = K$

$= 0$, as we should expect, since the ray curves of the parametric net form a conjugate system. Moreover, we have $\mathfrak{D} = 0$, so that the parametric net is harmonic.

All of the preceding considerations may be dualized.

4. *Invariant Integrals.* The projective differential theory of a surface referred to its asymptotics has been enriched by the study of certain integrals geometrically connected with the surface. It is the purpose of this section to show how in a similar way to set up some integrals invariantly connected with a conjugate net and its sustaining surface, and to initiate an investigation thereof.

Let Φ be any one of the invariants

$$(Hv')^{1/2},\ (Kv')^{1/2},\ (W^{(u)}v')^{1/2},\ (W^{(v)}v')^{1/2},\ \ (D'v')^{1/2},\ (\mathfrak{D})^{1/2},\ (R)^{1/2},\ \mathfrak{C}',\ \mathfrak{B}'v',$$

where $v' = dv/du$. Then Φdu is an absolutely invariant differential, and the integral $\int \Phi du$ is an invariant integral which has a projective geometric relation to the paramteric conjugate net and its sustaining surface.

In particular let $\phi = (Hv')^{1/2}$. Then Euler's equation for the externals of the integral $\int \phi du$ is

$$v'' = (H_u/H)v' - (H_v/H)v'^2. \tag{22}$$

These externals contain the parametric net. They contain the asymptotic net if, and only if,

$$[H/(A)^{1/2}]_u = 0, \qquad [H/(A)^{1/2}]_v = 0.$$

We propose to find the envelope of the osculating planes at P_y of all the curves (22) which pass through P_y. Using the local tetrahedron with vertices at the points y, y_u, y_v, z, where, in the notation of system (1), we have placed

$$z = ay_{vv} + cy_v + dy,$$

we find that the equation of the osculating plane of any curve on the surface through P_y is

$$\begin{aligned}(1/a)(a + v'^2)(v'x_2 - x_3)& \\ + (2c'v' - bv' - (c/a)v'^2 - 2b'v'^2 + v'')x_4 &= 0.\end{aligned} \tag{23}$$

Therefore the coordinates of the osculating plane of a curve (22) are given by

$$\begin{gathered} u_1 = 0, \quad u_2 = (v'/a)(a + v'^2), \quad u_3 = -(1/a)(a + v'^2), \\ u_4 = v'P - v'^2Q, \end{gathered} \tag{24}$$

where

$$(25)\qquad P = 2c' - b + H_u/H, \qquad Q = c/a + 2b' + H_v/H.$$

And the equations of the required envelope in plane coordinates are

$$(26)\qquad u_1 = 0, \quad u_4(u_2^2 + au_3^2) - a(Pu_2u_3^2 + Qu_2^2u_3) = 0.$$

This is a cone of the third class touching the tangent plane along the asymptotic tangents and having three cusp-planes which intersect in the line joining P_y to the point whose coordinates are $(0, -P, -aQ, 4)$. The directions in which the three cusp-planes meet the tangent plane are given by

$$Pu_2^3 - 3aQu_2^2u_3 - 3aPu_2u_3^2 + a^2Qu_3^3 = 0.$$

These coincide with the directions of Segre, which are the conjugates of the directions of Darboux given by (17), in case $P : Q = \mathfrak{C}' : \mathfrak{B}'$, and coincide with the directions of Darboux themselves in case $P : aQ = -\mathfrak{B}' : \mathfrak{C}'$.

It is easy to determine the developables and focal surfaces of the congruence of cusp-axes of the cones (26) by the methods employed in section 3. And dual considerations lead immediately to a curve of order three in the tangent plane, which is the locus of the ray-points of the extremals (22). This curve has a double point at P_y, and a flex-ray. The congruence of flex-rays might be studied.

Finally, it would be desirable to have a geometric interpretation of each of our invariant integrals. It would be equally desirable to have a geometric interpretation for each of the forms Φdu similar to the interpretation given * by Wilczynski for the metric of Fubini, or the interpretations given by Bompiani and Čech for the linear projective element.

5. *Osculating Linear Complexes.* The osculating linear complex at a point P of a space curve is the limit of the linear complex determined by the tangent at P and four neighboring tangents, as each of the four tangents independently approaches the tangent at P. We shall now obtain the equations in line coordinates of the two linear complexes which osculate at a point P_y of our surface the two curves of the parametric conjugate net through P_y.

The coordinates of a point x near P_y and on the curve C_u, or $v = \text{const.}$, through P_y are given by an expansion of the form

$$(27)\qquad x = y + y_u\Delta u + \tfrac{1}{2}y_{uu}\Delta u^2 + \cdots,$$

of which we shall need the first six terms. By means of the operations of differentiation and elimination it is possible to express every derivative of y as a linear combination of y_{vv}, y_u, y_v, y by an equation of the form

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* Fubini and Čech, *Geometria proiettiva differenziale,* p. 138.

$$\frac{\partial^{h+k}y}{\partial u^h \partial u^k} = \alpha^{(h,k)} y_{vv} + \beta^{(h,k)} y_u + \gamma^{(h,k)} y_v + \delta^{(h,k)} y.$$

Placing $k = 0$, $h \geqq 2$, we may calculate the successive coefficients of (27) by means of the recursion formulas

$$\begin{aligned}
&\alpha^{(h,0)} = {\alpha_u}^{(h-1,0)} + \alpha^{(h-1,0)}\alpha^{(1,2)} + a\beta^{(h-1,0)},\\
(28)\qquad &\beta^{(h,0)} = {\beta_u}^{(h-1,0)} + \alpha^{(h-1,0)}\beta^{(1,2)} + b\beta^{(h-1,0)} + b'\gamma^{(h-1,0)} + \delta^{(h-1,0)},\\
&\gamma^{(h,0)} = {\gamma_u}^{(h-1,0)} + \alpha^{(h-1,0)}\gamma^{(1,2)} + c\beta^{(h-1,0)} + c'\gamma^{(h-1,0)},\\
&\delta^{(h,0)} = {\delta_u}^{(h-1,0)} + \alpha^{(h-1,0)}\delta^{(1,2)} + d\beta^{(h-1,0)} + d'\gamma^{(h-1,0)}.
\end{aligned}$$

Using the points y, y_u, y_v, y_{vv} as vertices of a local tetrahedron of reference, with suitably chosen unit point, we find that the local coordinates of P_x are represented by expansions of the form

$$\begin{aligned}
&x_1 = 1 \quad + \tfrac{1}{2}\, d\, \Delta u^2 + \tfrac{1}{6}\, \delta^{(3,0)} \Delta u^3 + \cdots,\\
(29)\qquad &x_2 = \Delta u + \tfrac{1}{2}\, b\, \Delta u^2 + \tfrac{1}{6}\, \beta^{(3,0)}) \Delta u^3 + \cdots,\\
&x_3 = \qquad \tfrac{1}{2}\, c\, \Delta u^2 + \tfrac{1}{6}\, \gamma^{(3,0)} \Delta u^3 + \cdots,\\
&x_4 = \qquad \tfrac{1}{2}\, a\, \Delta u^2 + \tfrac{1}{6}\, \alpha^{(3,0)} \Delta u^2 + \cdots .
\end{aligned}$$

The corresponding expansions for the coordinates of a point z, other than P_x, on the tangent of C_u at P_x, are obtained by differentiating each of the series (29) with respect to Δu. The Plückerian coordinates ω_{ik} of the tangent of C_u at P_x are defined by the usual formulas,

$$\omega_{ik} = x_i z_k - x_k z_i,$$

and are represented by the following expansions:

$$\begin{aligned}
&\omega_{12} = 1 + \cdots,\\
&\omega_{13} = c\,\Delta u + \tfrac{1}{2}\gamma^{(3,0)}\Delta u^2 + \tfrac{1}{6}\gamma^{(4,0)}\Delta u^3\\
&\qquad\qquad + \tfrac{1}{24}(\gamma^{(5,0)} + 2d\gamma^{(3,0)} - 2c\delta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{14} = a\,\Delta u + \tfrac{1}{2}\alpha^{(3,0)}\Delta u^2 + \tfrac{1}{6}\alpha^{(4,0)}\Delta u^3\\
&\qquad\qquad + \tfrac{1}{24}(\alpha^{(5,0)} + 2d\alpha^{(3,0)} - 2a\delta^{(3,0)})\Delta \mathrm{u}^4 + \cdots,\\
(30)\qquad &\omega_{23} = \tfrac{1}{2}c\,\Delta u^2 + \tfrac{1}{3}\gamma^{(3,0)}\Delta u^3 + \tfrac{1}{24}(3\gamma^{(4,0)} + 2b\gamma^{(3,0)} - 2c\beta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{42} = -\tfrac{1}{2}a\,\Delta u^2 - \tfrac{1}{3}\alpha^{(3,0)}\Delta u^3\\
&\qquad\qquad - \tfrac{1}{24}(3\alpha^{(4,0)} + 2b\alpha^{(3,0)} - 2a\beta^{(3,0)})\Delta u^4 + \cdots,\\
&\omega_{34} = \tfrac{1}{12}(c\,\alpha^{(3,0)} - a\gamma^{(3,0)})\Delta u^4 + \cdots,
\end{aligned}$$

in which we have written as many terms as we shall need. Writing the equation of a linear complex in the form $\sum a_{ik}\omega_{ik} = 0$, we determine the coefficients so that this equation may be satisfied by the series (30) identically in Δu as far as the terms in Δu^4. In this way we find

$$
\begin{aligned}
&a_{12} = 0, \quad a_{13} = 4aA^2, \quad a_{14} = -4cA^2, \\
&a_{23} = 2A(2A\alpha^{(3,0)} - aB'), \quad a_{42} = 2A(2A\gamma^{(3,0)} - cB'), \\
&a_{34} = 2A[C + 2A(d + \beta^{(3,0)}) + 3(\gamma^{(4,0)}\alpha^{(3,0)} \\
&\qquad\qquad - \alpha^{(4,0)}\gamma^{(3,0)}) - bB'] - 3B'^2,
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
&A = a^2(K + W^{(u)}), \\
&B' = A_u + A(b + 2c'), \\
&C = B_u + c'B + (c_u + a\gamma^{(1,2)})\alpha^{(4,0)} - a_u\gamma^{(4,0)}.
\end{aligned} \tag{32}
$$

The coefficients (31) are not all invariants because the vertices of the tetrahedron of reference are not all covariant. If we choose a new tetrahedron of reference with vertices at the points y, ρ, σ, τ for which

$$
\begin{aligned}
&\rho = y_u - c'y, \qquad \sigma = y_v - b'y, \\
&\tau = ay_{vv} + cy_v - (\tfrac{1}{2}\mathfrak{D} + c'^2 + c'_u - bc' - d)y,
\end{aligned}
$$

then the points ρ, σ are the ray points of P_y with respect to the parametric net, and the point τ is the harmonic conjugate of P_y with respect to the foci of the axis of P_y. Referred to this tetrahedron, *the equation of the osculating linear complex of the curve C_u is*

$$
\begin{aligned}
&4aA^2\omega_{13} - 2aAI\omega_{23} + 4A^3\omega_{42} \\
&\qquad + a[2AI_u + 16AI(\mathfrak{C}' - \tfrac{3}{8}a_u/a) - 3I^2 + 2A^2\mathfrak{D}]\omega_{34} = 0,
\end{aligned} \tag{33}
$$

where

$$
I = A_u + 4A(\mathfrak{C}' - \tfrac{5}{8}a_u/a). \tag{34}
$$

Similarly, it can be shown that *the equation of the osculating linear complex of the curve C_v, referred to the same tetrahedron $y\rho\sigma\tau$, is*

$$
\begin{aligned}
&4B^2\omega_{12} + 3BJ\omega_{23} - 4aB^3\omega_{34} \\
&\qquad - [2aBJ_v + 16aBJ(\mathfrak{B}' + \tfrac{1}{4}a_v/a) - 3aJ^2 - 2B^2\mathfrak{D}]\omega_{42} = 0,
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
&B = (1/a)(H + W^{(v)}), \\
&J = B_v + 4B(\mathfrak{B}' + \tfrac{3}{8}a_v/a).
\end{aligned} \tag{36}
$$

We shall refer to the complexes (33) and (35) as the first and second complexes respectively.

Some properties of these complexes may now be deduced. If the curve C_u is a plane curve, then $A = 0$ and the first complex is indeterminate. The invariant of the first complex $16aA^5$. Therefore *if C_u is not a plane curve, the first complex is not special.* Since the coefficients a_{12} and a_{14} of the first complex are zero, it follows that *this complex contains all the lines of the pencil with center at P_y and lying in the osculating plane of C_u.* In fact, P_y corresponds to the osculating plane of C_u in the null system of the first complex. Moreover, the point ρ corresponds to the plane

$$aIx_3 + 2A^2x_4 = 0.$$

This plane coincides with the tangent plane of the surface if, and only if, $I = 0$. In case $I = 0$, to P_σ corresponds the plane

$$2x_1 - \mathfrak{D}x_4 = 0,$$

and to P_τ corresponds the plane

$$Ax_2 - a\mathfrak{D}x_3 = 0.$$

If $I = \mathfrak{D} = 0$, to P_σ corresponds the plane $\rho\sigma\tau$, and to P_τ corresponds the osculating plane of C_v. Finally, the ray $\rho\sigma$ belongs to the first complex in case $I = 0$. Similar statements are true regarding the second complex.

In order to obtain the equations of the directrices of the linear congruence of intersection of the first and second complexes, it is sufficient to write down the conditions that the plane corresponding to a variable point in the null system of a special complex of the pencil determined by the two complexes may be indeterminate. Denoting the coefficients of the first and second complexes by a_{ik} and b_{ik} respectively, we obtain for the equations of the directrices

$$(37) \qquad \begin{gathered} b_{12}x_2 + \lambda_i a_{13}x_3 = 0, \\ b_{12}x_1 + (\lambda_i a_{23} + b_{23})x_3 + (a_{42}\lambda_i + b_{42})x_4 = 0, \end{gathered}$$

where λ_i $(i = 1, 2)$ are the roots of the quadratic

$$(38) \qquad a_{13}a_{42}\lambda^2 + (a_{13}b_{42} + a_{34}b_{12})\lambda + b_{12}b_{34} = 0$$

which determines the special complexes of the pencil.

The directrices do not lie in the osculating plane of C_u, or of C_v, but do intersect the axis $y\tau$ in the points

$$(a_{42}\lambda_i + b_{42},\ 0,\ 0,\ -b_{12}), \qquad (i = 1, 2).$$

If $I = J = 0$, the directrices meet the ray $\rho\sigma$ at the points

$$(0,\ \lambda_i a A^2,\ -B^2,\ 0),$$

where λ_i are the roots of the equation

$$2A^5\lambda^2 + A^2B^2\mathfrak{D}\lambda - 2B^5 = 0.$$

In this case the simultaneous invariant of the two complexes vanishes if, and only if, $\mathfrak{D} = 0$, so that the two complexes are in involution if, and only if, the parametric net is harmonic.

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A Certain General Type of Neumann Expansions and Expansions in Confluent Hypergeometric Functions.

By R. F. Graesser.

Introduction.

1. *Statement of the Problem.* The purpose of this paper is to obtain two expansion theories analogous to the one developed by Neumann and Gegenbauer for Bessel functions.* In the first division we deal with a certain general sequence of analytic functions of a complex variable z. Each element of this sequence is defined by a power series and all such series are assumed to converge in a single circle of finite radius. With one additional hypothesis, which will be found in § 2, it is possible to obtain an expansion of an analytic function $f(z)$ which will converge uniformly with respect to z in some finite region (Theorem II). An expansion of $f(z)$ is developed in terms of a sequence of polynomials in $1/z$ which are associated with the original sequence of analytic functions. An analogue to Laurent's expansion is obtained. The relation of these expansions to the ordinary Taylor and Laurent expansions of $f(z)$ is shown. Some properties of the elements of the sequence and of their associated polynomials in $1/z$ are found. Generalizations are made to functions of n complex variables.

In the second division a certain confluent hypergeometric equation is subjected to a transformation. Expansions in terms of a solution of the transformed equation are first discussed as a special case of the theory developed in division I. The results are then extended by further investigation.

The expansion theory, which is developed in detail in the present paper, revolves about our Theorems II and III. The latter are related to certain theorems of great generality due to G. D. Birkhoff.† The methods of the present paper are different, however, from those used by Birkhoff. Comparison of these theorems should also be made with certain results obtained in a paper by J. L. Walsh.*

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* See Watson, *Theory of Bessel Functions* (London, 1922), Chapters IX and XVI, where an excellent summary of these results is given as well as copious references.

† "Sur une généralisation de la Séries de Taylor," *Comptes Rendu des Séances de l'Académie des Sciences,* Vol. 164 (1917), pp. 942-945.

I. A CERTAIN GENERAL TYPE OF NEUMANN EXPANSIONS.

2. *The Sequence of Functions and the Hypotheses.* We take the general sequence of analytic functions obtained by allowing m to range over zero and the positive integers in the following definition of $F_m(z)$,

$$F_m(z) = \sum_{s=0}^{\infty} c_{ms} z^{m+s}, \quad c_{m0} = 1. \tag{1}$$

The power series defining each function of the above sequence is assumed to be convergent within some circle of radius R described about the origin as a center where R is independent of m. We have a well-known theorem which says:

"If $F(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ is a power series convergent in a circle described about the origin with radius R and if $N < R$ then the greatest value which $| F(z) |$ takes on on the circle of radius N described about the origin is at least as great as $| c_\nu | N^\nu$."

Designate the greatest value which $| F_m(z) |$ takes on on the circle of radius $N < R$ described about the origin by $M_N(m)$; then

$$M_N(m) \geqq | c_{ms} | N^{m+s}. \tag{2}$$

We will now make the further hypothesis that a constant M_N exists such that for all values of m we have

$$M_N \geqq M_N(m) N^{-m}.$$

3. *Expansion of z^n.* Proceeding formally to compute the coefficients γ_{nm}, $m = 0, 1, 2, \cdots$, of the expansion

$$z^n = \sum_{m=0}^{\infty} \gamma_{nm} F_m(z)$$

we insert in the right member the expansion of $F_m(z)$ and equate coefficients of like powers of z. We then find that the γ's satisfy the recurrence relation

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* *Transactions of the American Mathematical Society*, Vol. 26 (1924), pp. 155-170.

$$(3)\qquad \gamma_{n0}c_{0\nu}+\gamma_{n1}c_{1,\nu-1}+\gamma_{n2}c_{2,\nu-2}+\cdots+\gamma_{n,\nu-1}c_{\nu-1,1}+\gamma_{n\nu}=1,\quad \nu=n;$$
$$=0,\quad \nu\neq n;$$

from which we have $\gamma_{n\nu}=0$, $\nu=0, 1, 2, \cdots, (n-1)$; $\gamma_{nn}=1$ and

$$(4)\qquad \gamma_{n,n+\nu}=$$
$$(-)^{\nu+1}\begin{vmatrix} c_{n,1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{n,2} & c_{n+1,1} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n,\nu-2} & c_{n+1,\nu-3} & c_{n+2,\nu-4} & \cdots & \cdots & 1 & 0 \\ c_{n,\nu-1} & c_{n+1,\nu-2} & c_{n+2,\nu-3} & \cdots & \cdots & c_{n+\nu-2,1} & 1 \\ c_{n,\nu} & c_{n+1,\nu-1} & c_{n+2,\nu-2} & c_{n+3,\nu-3} & \cdots & c_{n+\nu-2,2} & c_{n+\nu-1,1} \end{vmatrix},\quad \nu>0.$$

We may then start the summation with n obtaining

$$(5)\qquad z^n=\sum_{\nu=0}^{\infty}\gamma_{n,n+\nu}F_{n+\nu}(z).$$

We shall show that the series in the right member is absolutely and uniformly convergent within and upon the circle $|z|=R_1$, where $R_1<N/(M_N+1)$, and that it is a valid expansion of z^n in the closed region bounded by this circle. To do this we need dominating expressions for $|F_{n+\nu}(z)|$ and $|\gamma_{n,n+\nu}|$.

Replacing $M_N(m)$ in (2) by its dominant we get

$$(6)\qquad |c_{ms}|\leq M_N N^{-s}.$$

Writing $n+\nu$ for m we have from the expansion for $F_m(z)$ the relation

$$(7)\qquad |F_{n+\nu}(z)|\leq |z|^{n+\nu}M_N\sum_{s=0}^{\infty}(|z|/N)^s=M_N N|z|^{n+\nu}/(N-|z|)$$

provided $|z|<N$.

When we omit the vanishing terms from (3) that recurrence relation becomes

$$\gamma_{nn}c_{n,\nu-n}+\gamma_{n,n+1}c_{n+1,\nu-n-1}+\cdots+\gamma_{n,\nu-1}c_{\nu-1,1}+\gamma_{n\nu}=1,\quad \nu=n;$$
$$=0,\quad \nu>n.$$

By giving ν the values $n, n+1, \cdots$ and employing the inequality (6) we have successively the following relations:

$$
\begin{aligned}
&| \gamma_{n,n} | = 1, \\
&| \gamma_{n,n+1} | \leq N^{-1} M_N, \\
&| \gamma_{n,n+2} | \leq N^{-2} M_N (M_N + 1), \\
&| \gamma_{n,n+3} | \leq N^{-3} M_N (M_N + 1)^2, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&| \gamma_{n,n+\nu} | \leq N^{-\nu} M_N (M_N + 1)^{\nu-1}.
\end{aligned}
$$

Hence

$$
(8) \qquad | \gamma_{n,n+\nu} | \leq N^{-\nu} (M_N + 1)^{\nu}.
$$

Making use of these two dominants we have

$$
| \gamma_{n,n+\nu} | \, | F_{n+\nu}(z) | \leq \frac{M_N N \, | z |^n}{N - | z |} \left[\frac{(M_N + 1) \, | z |}{N} \right]^{\nu}.
$$

The right member is the νth term of a convergent series if $| z | < N/(M_N+1)$ so that the series in (5) is absolutely convergent. If $| z | \leq R_1 < N/(M_N+1)$ we may choose R_2 such that $R_1 < R_2 < N/(M_N + 1)$; then we have the relation

$$
| \gamma_{n,n+\nu} | \, | F_{n+\nu}(z) | \leq \frac{M_N N R_2{}^n}{N - R_2} \left[\frac{(M_N + 1) R_2}{N} \right]^{\nu}.
$$

The right member is the νth term of a convergent series of positive constants. The absolute and uniform convergence of (5) follows from the test of Weierstrass provided that $| z | \leq R_1 < N/(M_N + 1)$.

By another theorem due to Weierstrass (cf. Osgood, *Lehrbuch der Funktionentheorie,* Vol. I, 4th Ed., Leipzig, 1923, p. 343) we may substitute for $F_{n+\nu}(z)$ from (1) in (5) and write the result in ascending powers of z. When this is done it is easily seen by the aid of the recurrence relations (3) on the γ's that the sum of the series in (5) is z^n.

4. *Expansion of* $1/(t - z)$. We have the expansion

$$
1/(t - z) = \sum_{n=0}^{\infty} z^n / t^{n+1}
$$

if $| z | < | t |$. We substitute for z^n its expansion from (5). Then we have

$$
(9) \qquad 1/(t - z) = \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \gamma_{n,n+\nu} F_{n+\nu}(z) / t^{n+1}.
$$

Proceeding formally we arrange the result according to ascending values of the subscripts of the F's and obtain the relation

$$(10) \qquad 1/(t-z) = \sum_{n=0}^{\infty} V_n(t)F_n(z), \quad \text{where } V_n(t) = \sum_{r=0}^{n} \gamma_{r,n}/t^{r+1}.$$

To justify this rearrangement it is sufficient to prove the absolute convergence of the series in (9). (Cf. Hobson, *Theory of Functions of a Real Variable* 1st Ed., Cambridge, 1907, p. 465). Making use of (7) and (8) we have

$$\frac{|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)|}{|t|^{n+1}} \leq \frac{M_N N\,|z|^n}{(N-|z|)\,|t|^{n+1}}\left[\frac{(M_N+1)\,|z|}{N}\right]^{\nu}$$

for each value of ν and

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{|\gamma_{n,n+\nu}|\,|F_{n+\nu}(z)|}{|t|^{n+1}} \leq \sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{M_N N\,|z|^n}{(N-|z|)\,|t|^{n+1}}\left[\frac{(M_N+1)\,|z|}{N}\right]^{\nu}.$$

The double series in the right member is convergent if $|z| < N/(M_N+1)$, and $|z| < |t|$. Consequently the double series in (9) is absolutely convergent in the open regions defined by the inequalities

$$|z| < N/(M_N+1), \quad |z| < |t|;$$

and (10) is valid in the same regions.

Suppose $|z| \leq R_1 < N/(M_N+1)$ and $|t| \geq R_4$ where $R_1 < R_4$. We can choose R_2 and R_3 such that $R_2 < N/(M_N+1)$ and $R_1 < R_2 < R_3 < R_4$. The series in (9), and therefore the double series obtained from the first equation in (10) by substitution for $V_n(t)$ its value in (10), is dominated by the convergent series of positive constants

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{M_N N R_2^n}{(N-R_2)R_3^{n+1}}\left[\frac{(M_N+1)R_2^n}{N}\right]^{\nu}.$$

Hence by the theorem of Weierstrass the double series is absolutely and uniformly convergent with respect to both z and t in the regions indicated. We may state the following theorem:

THEOREM I. *Let z and t be two complex variables such that*

$|z| \leq R_1 < N/(M_N+1)$ *and* $|t| \geq R_4$ *where* $R_1 < R_4$; *then the expansion*

$$1/(t-z) = \sum_{n=0}^{\infty} V_n(t)F_n(z), \quad \textit{where} \quad V_n(t) = \sum_{r=0}^{n} \gamma_{rn}/t^{r+1},$$

is valid. If t is a variable point upon or outside the circle $|z| = R_4$ and z is within or upon the circle $|z| = R_1$ then the expansion converges uniformly with respect to t and z.

In the remainder of the paper we shall use the symbols R_1, R_2, R_3, R_4 in a generic sense, the numbers denoted by them varying from section to section. In all cases we shall suppose that $R_1 < R_2 < R_3 < R_4$. Moreover we shall employ the symbol C_i to denote the circle $|z| = R_i$ for $i = 1, 2, 3, 4$.

5. *Expansion of $f(z)$ in Terms of $F_n(z)$.* By means of theorem I and Cauchy's integral formula we are able to expand any function $f(z)$, analytic and single-valued within and upon a circle C_4, the equation of C_4 being $|z| = R_4$; thus,

$$f(z) = (1/2\pi i) \int_{C_4} \frac{f(t)}{t - z}\, dt$$

$$= (1/2\pi i) \int_{C_4} \sum_{n=0}^{\infty} V_n(t) F_n(z) f(t)\, dt.$$

The expansion in the second member is uniformly convergent with respect to t and z under the conditions of theorem I and may therefore be integrated term by term. Hence

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z), \text{ where } a_n = (1/2\pi i) \int_{C_4} V_n(t) f(t)\, dt.$$

This expansion is valid and converges uniformly with respect to z if $|z| \leq R_1 < R_4$ and $R_1 < N/(M_N + 1)$. Summarizing we have the following theorem:

THEOREM II. *If $f(z)$ be a single-valued, analytic function of z within and upon C_4, the equation of C_4 being $|z| = R_4$, then the expansion,*

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z), \text{ where } a_n = (1/2\pi i) \int_{C_4} V_n(t) f(t)\, dt,$$

is valid and converges uniformly with respect to z if z is within or upon the circle $|z| = R_1$ where $R_1 < N/(M_N + 1)$ and $R_1 < R_4$.

6. *Expansion of $f(z)$ in Terms of $V_n(z)$.* We shall prove the following theorem:

THEOREM III. *If $f(z)$ be a single-valued, analytic function of z upon* *and outside of a circle C_1, the equation of C_1 being $|z| = R_1$, then the expansion*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n V_n(z), \text{ where } \alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt,$$

is valid and converges uniformly with respect to z if z is upon or outside the circle $|z| = R_4$ where $R_1 < N/(M_N + 1)$ and $R_1 < R_4$.

From theorem I we have

$$1/(t - z) = -\sum_{n=0}^{\infty} F_n(t) V_n(z);$$

this series converges uniformly with respect to z and t if

$$|t| \leq R_1 < N/(M_N + 1) \text{ and } |z| \geq R_4 \text{ where } R_1 < R_4.$$

By an argument analogous to that of the preceding section, we have

$$f(z) = -(1/2\pi i) \int_{C_1} \sum_{n=0}^{\infty} F_n(t) V_n(z) f(t)\, dt$$

where the integration is in the clockwise direction. If we take it in the counterclockwise direction we have

$$f(z) = (1/2\pi i) \int_{C_1} \sum_{n=0}^{\infty} F_n(t) V_n(z) f(t)\, dt.$$

Integrating term by term as is permissible we have

$$f(z) = \sum_{n=0}^{\infty} \alpha_n V_n(z), \text{where } \alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt,$$

from which the theorem follows.

7. *Analogue of the Laurent Expansions.* Let C_1, C_2, C_3, C_4 be, respectively, the circles $|z| = R_1$, $|z| = R_2$, $|z| = R_3$, $|z| = R_4$ where $R_1 < R_2 < R_3 < R_4$ and $R_3 < N/(M_N + 1)$. Let $f(z)$ be a single-valued, analytic function of z within and upon the boundary of the ring bounded by C_1C_4. If z is a point within or upon the boundary of the ring bounded by C_2C_3 we have by means of Cauchy's integral formula

$$f(z) = (1/2\pi i) \int_{C_4} \frac{f(t)}{t - z}\, dt + (1/2\pi i) \int_{C_1} \frac{f(t)}{z - t}\, dt$$

where both integrations are performed in the counterclockwise direction. With the aid of theorem I we obtain

$$f(z) = (1/2\pi i) \int_{C_4} [\sum_{n=0}^{\infty} V_n(t)F_n(z)]f(t)\,dt + (1/2\pi i) \times \int_{C_1} [\sum_{n=0}^{\infty} F_n(t)V_n(z)]f(t)\,dt.$$

We may integrate term by term so that

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) + \sum_{n=0}^{\infty} a_n' V_n(z),$$

where $a_n = (1/2\pi i) \int_{C_4} V_n(t)f(t)\,dt$ and $a_n' = (1/2\pi i) \int_{C_1} F_n(t)f(t)\,dt$.

This expansion is valid and converges uniformly with respect to z if z is within or upon the boundary of the ring C_2C_3.

THEOREM IV. *Let* C_1, C_2, C_3, C_4 *be, respectively, the circles* $|z| = R_1$, $|z| = R_2$, $|z| = R_3$, $|z| = R_4$ *where* $R_1 < R_2 < R_3 < R_4$ *and* $R_3 < N/(M_N+1)$. *If* $f(z)$ *is a single-valued, analytic function of* z *within and upon the boundary of the ring* C_1C_4, *then the expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) + \sum_{n=0}^{\infty} a_n' V_n(z),$$

where $a_n = (1/2\pi i) \int_{C_4} V_n(t)f(t)\,dt$ *and* $a_n' = (1/2\pi i) \int_{C_1} F_n(t)f(t)\,dt$,

is valid and converges uniformly with respect to z *when* z *is within or upon the boundary of the ring* C_2C_3.

8. *Relations to the Taylor and Laurent Expansions.* If $f(z)$ is analytic in the neighborhood of the origin let its Taylor expansion be

$$f(z) = \sum_{s=0}^{\infty} b_s z^s.$$

We substitute this expansion in the expression for the coefficients of the expansion in theorem II and also replace $V_n(t)$ by its value. We have

$$a_n = (1/2\pi i) \int_{C_4} V_n(t)f(t)\,dt$$

$$= (1/2\pi i) \int_{C_4} \left(\sum_{r=0}^{n} \gamma_{rn}/t^{r+1}\right) \left(\sum_{s=0}^{\infty} b_s t^s\right) dt$$

$$= (1/2\pi i) \int_{C_4} \sum_{r=0}^{n} \sum_{s=0}^{\infty} \gamma_{rn} b_s t^{s-r-1} dt.$$

Hence, by performing the integration,

$$a_n = \sum_{s=0}^{n} b_s \gamma_{sn}.$$

If $f(z)$ is analytic in the neighborhood of infinity its Taylor expansion takes the form

$$f(z) = \sum_{r=0}^{\infty} \beta_r z^{-r}.$$

We substitute in the expression for the coefficients in theorem III thus,

$$\alpha_n = (1/2\pi i) \int_{C_1} F_n(t) f(t)\, dt$$

$$= (1/2\pi i) \int_{C_1} \left(\sum_{s=0}^{\infty} c_{ns} t^{n+s}\right) \left(\sum_{r=0}^{\infty} \beta_r z^{-r}\right) dt$$

$$= (1/2\pi i) \int_{C_1} \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} c_{ns} \beta_r t^{n+s-r} dt.$$

Hence

$$\alpha_n = \sum_{s=0}^{\infty} c_{ns} \beta_{n+s+1}.$$

If $f(z)$ has the Laurent expansion

$$f(z) = \sum_{s=0}^{\infty} b_s z^s + \sum_{s=1}^{\infty} b_s' z^{-s}$$

valid in the ring C_1C_4 of theorem IV then the coefficients of the expansion of that theorem are related to these coefficients thus,

$$a_n = \sum_{s=0}^{n} b_s \gamma_{sn},$$

$$a_n' = \sum_{s=0}^{\infty} c_{ns} b'_{n+s}.$$

These relations may be useful in obtaining the expansions of theorems II, III, IV provided the Taylor and Laurent expansions are readily obtainable.

9. *Orthogonality Properties.* We shall establish the following:

$$\int_C F_n(z)F_m(z)\,dz = 0, \quad (m = n \text{ and } m \neq n),$$

where C is any closed contour in the region of analyticity of both $F_n(z)$ and $F_m(z)$.

$$\int_C V_n(z)V_m(z)\,dz = 0, \quad (m = n \text{ and } m \neq n),$$

where C is any closed contour in the finite plane not passing through the origin.

$$\int_C V_n(z)F_n(z)\,dz = 2\pi i k, \text{ (C enclosing and not passing through the origin)},$$
$$= 0, \text{ (C not enclosing nor passing through the origin)},$$

where C is a closed contour in the region of analyticity of $F_n(z)$, and k is the excess of the number of positive circuits over the number of negative circuits made by the contour about the origin.

$$\int_C V_n(z)F_m(z)\,dz = 0, \quad (m \neq n),$$

where C is any closed contour in the region of analyticity of $F_m(z)$ but not passing through the origin if $m < n$.

The value of the first integral follows at once from Cauchy's theorem. The product $V_n(z)\,V_m(z)$ is a polynomial in $1/z$ with $(1/z)^2$ as the lowest power. We have therefore the value of the second integral. The value of the third integral is obtained at once by the residue of the integrand at zero. If $m > n$ the product $V_n(z)\,F_m(z)$ is a power series in z so the value of the fourth integral is zero in that case. If $m < n$ we have from (1) and the expression for $V_n(z)$ in theorem II

$$\int_C V_n(z)F_m(z)\,dz = \int_C \left(\sum_{r=0}^{n}\gamma_{rn}/z^{r+1}\right)\left(\sum_{s=0}^{\infty}c_{ms}z^{m+s}\right)dz = 2\pi i\sum_{s=0}^{n-m}\gamma_{m+s,n}c_{ms}.$$

Writing $m + \nu$ for n the last summation becomes

$$\sum_{s=0}^{\nu}\gamma_{m+s,m+\nu}c_{ms}.$$

We then substitute for $\gamma_{m+s,m+\nu}$ from (4). When the determinant for $\gamma_{m,m+\nu}$ is expanded according to the elements of the first column and the result compared with the foregoing sum, the latter is seen to vanish.

The expansions of theorems II, III and IV are known to exist in some small regions under the conditions set forth in the theorems. The integral formulae just obtained enable us to readily determine the coefficients of the expansions. For example, the expansion of theorem II exists and converges uniformly with respect to z within and upon some small circle described about the origin. We may then multiply this expansion by $V_m(z)$ and integrate term by term in the right member. The integrations are in the positive direction. Also C lies in the circle of existence of the expansion and encloses the origin in a single circuit. We obtain

$$\int_C V_m(z)f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C V_m(z)F_n(z)\,dz.$$

By the last two integral formulae

$$\int_C V_m(z)f(z)\,dz = 2\pi i\, a_m;$$

this determines a_m.

10. *Results in the neighborhood of $z = a$.* By replacing z by $(z - a)$ and t by $(t - a)$ in the result stated in theorem I we obtain

$$1/[(t-a)-(z-a)] = \sum_{n=0}^{\infty} V_n(t-a)F_n(z-a).$$

It is readily seen that this expansion has the same properties in the neighborhood of a as are possessed by the expansion in theorem I in the neighborhood of the origin. By means of this result and Cauchy's integral formula we are able to expand any function $f(z)$, analytic and single-valued within and upon the boundary of a closed region bounded by C_4, the equation of C_4 being $|z - a| = R_4$. We have

$$f(z) = \frac{1}{2\pi i}\int_{C_4} \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_{C_4} \frac{f(t)}{(t-a)-(z-a)}\,dt$$

$$= (1/2\pi i)\int_{C_4} \sum_{n=0}^{\infty} V_n(t-a)F_n(z-a)f(t)\,dt$$

$$= \sum_{n=0}^{\infty} \bar{a}_n F_n(z-a), \text{ where } \bar{a}_n = (1/2\pi i)\int_{C_4} V_n(t-a)f(t)\,dt.$$

This last expansion is valid and converges uniformly with respect to z if $|z - a| \leq R_1 < R_4$ and $R_1 < N/(M_N + 1)$.

THEOREM V. *If $f(z)$ be a single-valued, analytic function of z within and upon C_4 whose equation is $|z-a| = R_4$, then the expansion.*

$$f(z) = \sum_{n=0}^{\infty} \bar{a}_n F_n(z-a), \text{ where } \bar{a}_n = (1/2\pi i) \int_{C_4} V_n(t-a) f(t)\,dt,$$

is valid and converges uniformly with respect to z if z is within or upon the circle $|z-a| = R_1$ where $R_1 < N/(M_N+1)$ and $R_1 < R_4$.

It is clear that all the results of § 6 to § 9 inclusive could be established for the neighborhood of $z = a$ as well as for the neighborhood of $z = 0$.

11. *Expansion of Functions of Several Variables.* The expansion theory obtained in the preceding sections can be easily extended for the expansion of functions of several variables in multiple series. As an illustration, an expansion similar to that of theorem II will be carried through for a function of two independent variables. From this special case the generalization to the case involving n variables will be readily seen.

We take two general sequences of functions of z_1 and z_2, respectively,

$$F_m^{(p)}(z_p) = \sum_{s=0}^{\infty} c_{ms}^{(p)} z_p^{m+s}, \quad c_{m0}^{(p)} = 1, \quad m = 0, 1, 2, \cdots, \quad p = 1, 2,$$

similar to the sequence defined in § 2 and place upon them similar hypotheses. The power series defining each function of the sequence $F_m^{(p)}(z_p)$ is assumed to be convergent within some circle of radius $R^{(p)}$ described about the origin as a center where $R^{(p)}$ is independent of m. By the theorem quoted in § 2 there is a greatest value $M_N^{(p)}(m)$ which $|F_m^{(p)}(z_p)|$ takes on on a circle of radius $N^{(p)} < R^{(p)}$ described about the origin and we have

$$M_N^{(p)}(m) \geq |c_{ms}^{(p)}| (N^{(p)})^{m+s}.$$

As before we then make the further assumption that there exists a finite $M_N^{(p)}$ such that for all values of m we have

$$M_N^{(p)} \geq M_N^{(p)}(m)(N^{(p)})^{-m}, \quad p = 1, 2.$$

The results of §§ 3, 4 follow as before for each sequence $F_m^{(p)}(z_p)$, $m = 0, 1, 2, \cdots$, $p = 1, 2$.

Let z_1, t_1 and z_2, t_2 be two pairs of complex variables in the z_1 and z_2 planes, respectively. Let them be restricted as follows:

$$\begin{cases} |z_1| \leq R_1^{(1)} < N^{(1)}/(M_N^{(1)}+1), & |t_1| \geq R_4^{(1)}, \quad R_1^{(1)} < R_4^{(1)}; \\ |z_2| \leq R_1^{(2)} < N^{(2)}/(M_N^{(2)}+1), & |t_2| \geqq R_4^{(2)}, \quad R_1^{(2)} < R_4^{(2)}. \end{cases}$$

Also we take

$$1/(t_1-z_1)=\sum_{m_1=0}^{\infty} V_{m_1}^{(1)}(t_1)F_{m_1}^{(1)}(z_1);\quad 1/(t_2-z_2)=\sum_{m_2=0}^{\infty} V_{m_2}^{(2)}(t_2)F_{m_2}^{(2)}(z_2),$$

where $V_{m_1}^{(1)}(t_1)$ and $V_{m_2}^{(2)}(t_2)$ are associated with $F_{m_1}^{(1)}(z_1)$ and $F_{m_2}^{(2)}(z_2)$, respectively, in the same manner as $V_m(t)$ is associated with $F_m(z)$. Allow t_1 and t_2 to be variable points outside or upon $C_4^{(1)}$ and $C_4^{(2)}$, respectively, where $C_4^{(1)}$ is the circle $|z_1|=R_4^{(1)}$ and $C_4^{(2)}$ the circle $|z_2|=R_4^{(2)}$, and z_1 and z_2 variable points, respectively, within or upon $C_1^{(1)}$ defined by $|z_1|=R_1^{(1)}$ and $C_1^{(2)}$ defined by $|z_2|=R_1^{(2)}$. Then the expansions for $1/(t_1-z_1)$ and $1/(t_2-z_2)$ converge uniformly with respect to t_1, z_1 and t_2, z_2, respectively. Also these expansions converge absolutely. On multiplying them together term by term we obtain the equation

$$1/(t_1-z_1)\cdot 1/(t_2-z_2)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2),$$

where the double series is absolutely and uniformly convergent. Let $f(z_1,z_2)$ be a function of the two independent, complex variables z_1 and z_2. Let it be analytic and single-valued in z_1 when z_1 is within or upon the circle $C_4^{(1)}$ and z_2 is held fixed within or upon the circle $C_4^{(2)}$. Likewise let it be analytic and single-valued in z_2 when z_2 is within or upon $C_4^{(2)}$ and z_1 is held fixed within or upon $C_4^{(1)}$. By Cauchy's generalized integral formula we have

$$f(z_1,z_2)=\frac{1}{(2\pi i)^2}\int_{C_4^{(1)}}\int_{C_4^{(2)}}\frac{f(t_1,t_2)}{(t_1-z_1)(t_2-z_2)}dt_2dt_1$$

$$=1/(2\pi i)^2\int_{C_4^{(1)}}\int_{C_4^{(2)}}\Big[\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2)\Big]$$

$$\times[f(t_1,t_2)/(t_1-z_1)(t_2-z_2)]dt_2dt_1.$$

Hence

$$f(z_1,z_2)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} A_{m_1m_2}F_{m_1}^{(1)}(z_1)F_{m_2}^{(2)}(z_2),$$

where $$A_{m_1m_2}=1/(2\pi i)^2\int_{C_4^{(1)}}\int_{C_4^{(2)}} f(t_1,t_2)V_{m_1}^{(1)}(t_1)V_{m_2}^{(2)}(t_2)\,dt_2dt_1.$$

This expansion is valid and converges uniformly with respect to z_1, z_2 when z_1, z_2 are, respectively, within or upon the circles $|z_1|=R_1^{(1)}$, $|z_2|=R_1^{(2)}$. Since the extension of this result to any number of variables is obvious, we shall state the following theorem for n variables:

THEOREM VI. *Let* $f(z_1, z_2, \cdots, z_n)$ *be a function of the independent, complex variables* $z_1, z_2, \cdots, z_n$. *Designate the circles* $|z_1| = R_4^{(1)}$, $|z_2| = R_4^{(2)}, \cdots, |z_n| = R_4^{(n)}$ *by* $C_4^{(1)}, C_4^{(2)}, \cdots, C_4^{(n)}$, *respectively. Let* $f(z_1, z_2, \cdots, z_n)$ *be analytic and single-valued in* z_1 *when* z_1 *is within or upon* $C_4^{(1)}$ *and* $z_2, z_3, \cdots, z_n$ *are held fixed within or upon* $C_4^{(2)}, C_4^{(3)} \cdots, C_4^{(n)}$, *respectively. Let it be similarly analytic and single-valued in* $z_2, z_3, \cdots, z_n$. *Then the expansion*

$$f(z_1, z_2, \cdots, z_n)$$
$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m_1 m_2 \ldots m_n} F_{m_1}^{(1)}(z_1) F_{m_2}^{(2)}(z_2) \cdots F_{m_n}^{(n)}(z_n),$$

where

$$A_{m_1 m_2 \ldots m_n} = 1/(2\pi i)^n \int_{C_4^{(1)}} \int_{C_4^{(2)}} \cdots$$
$$\int_{C_4^{(n)}} f(t_1, t_2 \cdots t_n) \; V_{m_1}^{(1)}(t_1) V_{m_2}^{(2)}(t_2) \cdots V_{m_n}^{(n)}(t_n) \, dt_n \cdots dt_2 dt_1,$$

is valid and converges uniformly with respect to $z_1, z_2, \cdots, z_n$ *when* $z_1, z_2, \cdots, z_n$ *are, respectively, within or upon the circles* $|z_1| = R_1^{(1)}$, $|z_2| = R_1^{(2)}, \cdots, |z_n| = R_1^{(n)}$ *where* $R_1^{(1)} < R_4^{(1)}, R_1^{(2)} < R_4^{(2)}, \cdots, R_1^{(n)} < R_4^{(n)}$ *and* $R_1^{(1)} < N^{(1)}/(M_N^{(1)} + 1)$, $R_1^{(2)} < N^{(2)}/(M_N^{(2)} + 1), \cdots$, $R_1^{(n)} < N^{(n)}/(M_N^{(n)} + 1)$.

12. *Further Results for Functions of Several Variables.* Results analogous to those obtained for analytic functions of a single variable in § 6 to § 10 inclusive can be obtained for functions of two independent variables or, in fact, of n independent variables. We shall not attempt to establish these results in detail. However, as an example, we shall work out the analogue of the Laurent expansion for the case of a function of two independent variables.

Let z_1, t_1 and z_2, t_2 be two pairs of independent, complex variables in the z_1 and z_2 planes, respectively. Let $C_1^{(1)}, C_2^{(1)} C_3^{(1)}, C_4^{(1)}$ be four circles in the z_1 plane concentric at the origin and with radii, respectively, $R_1^{(1)}, R_2^{(1)}, R_3^{(1)}, R_4^{(1)}$ where $R_1^{(1)} < R_2^{(1)} < R_3^{(1)} < R_4^{(1)}$. Also $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}$ be similarly defined in the z_2 plane. Let $f(z_1, z_2)$ be analytic and single-valued in z_1 when z_1 is within or upon the ring $C_1^{(1)} C_4^{(1)}$ and z_2 is held fixed within or upon the ring $C_1^{(2)} C_4^{(2)}$. Likewise let it be analytic and single-valued in z_2 when z_2 is within or upon $C_1^{(2)} C_4^{(2)}$ and z_1 is held fixed within or upon $C_1^{(1)} C_4^{(1)}$. Further let the R's be restricted by the relations

$$R_3^{(1)} < N^{(1)}/(M_N^{(1)} + 1), \; R_3^{(2)} < N^{(2)}/(M_N^{(2)} + 1).$$

As in § 11 we have the expansions

$$\left\{\begin{aligned} 1/(t_1-z_1) &= \sum_{m_1=0}^{\infty} V_{m_1}{}^{(1)}(t_1)F_{m_1}{}^{(1)}(z_1), \\ 1/(t_2-z_2) &= \sum_{m_2=0}^{\infty} V_{m_2}{}^{(2)}(t_2)F_{m_2}{}^{(2)}(z_2), \end{aligned}\right.$$

which are uniformly convergent with respect to t_1, z_1 and t_2, z_2 if t_1, t_2 are outside or upon $C_4{}^{(1)}$, $C_4{}^{(2)}$, respectively, and z_1, z_2 are within or upon $C_3{}^{(1)}$, $C_3{}^{(2)}$, respectively. Also we have the expansions

$$\left\{\begin{aligned} 1/(z_1-t_1) &= \sum_{m_1=0}^{\infty} V_{m_1}{}^{(1)}(z_1)F_{m_1}{}^{(1)}(t_1), \\ 1/(z_2-t_2) &= \sum_{m_2=0}^{\infty} V_{m_2}{}^{(2)}(z_2)F_{m_2}{}^{(2)}(t_2), \end{aligned}\right.$$

which are uniformly convergent with respect to t_1, z_1 and t_2, z_2 if t_1, t_2 are within or upon $C_1{}^{(1)}$, $C_1{}^{(2)}$, respectively, and z_1, z_2 are outside or upon $C_2{}^{(1)}$, $C_2{}^{(2)}$, respectively. These expansions converge absolutely. We treat the results of multiplying them together term by term as in § 11. Then by Cauchy's generalized integral formula we have (all integrations being in the counterclockwise direction)

$$f(z_1,z_2) = \frac{1}{2\pi i}\int_{C_4{}^{(1)}} \frac{f(t_1,z_2)}{(t_1-z_1)}dt_1 + \frac{1}{2\pi i}\int_{C_1{}^{(1)}} \frac{f(t_1,z_2)}{(z_1-t_1)}dt_1$$

and

$$f(t_1,z_2) = \frac{1}{2\pi i}\int_{C_4{}^{(2)}} \frac{f(t_1,t_2)}{(t_2-z_2)}dt_2 + \frac{1}{2\pi i}\int_{C_1{}^{(2)}} \frac{f(t_1,t_2)}{(z_2-t_2)}dt_2.$$

Substituting from the second in the first we have

$$\begin{aligned} f(z_1,z_2) = \frac{1}{(2\pi i)^2}\int_{C_4{}^{(1)}}\int_{C_4{}^{(2)}} \frac{f(t_1,t_2)}{(t_2-z_2)(t_1-z_1)}dt_2dt_1 \\ + \frac{1}{(2\pi i)^2}\int_{C_4{}^{(1)}}\int_{C_1{}^{(2)}} \frac{f(t_1,t_2)}{(z_2-t_2)(t_1-z_1)}dt_2dt_1 \\ + \frac{1}{(2\pi i)^2}\int_{C_1{}^{(1)}}\int_{C_4{}^{(2)}} \frac{f(t_1,t_2)}{(t_2-z_2)(z_1-t_1)}dt_2dt_1 \\ + \frac{1}{(2\pi i)^2}\int_{C_1{}^{(1)}}\int_{C_1{}^{(2)}} \frac{f(t_1,t_2)}{(z_2-t_2)(z_1-t_1)}dt_2dt_1. \end{aligned}$$

Using the expansions just obtained we have

$$f(z_1, z_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(1)}{}_{m_1 m_2} F_{m_1}{}^{(1)}(z_1) F_{m_2}{}^{(2)}(z_2)$$

$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(2)}{}_{m_1 m_2} F_{m_1}{}^{(1)}(z_1) V^{(2)}{}_{m_2}(z_2)$$

$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(3)}{}_{m_1 m_2} V_{m_1}{}^{(1)}(z_1) F_{m_2}{}^{(2)}(z_2)$$

$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A^{(4)}{}_{m_1 m_2} V_{m_1}{}^{(1)}(z_1) V_{m_2}{}^{(2)}(z_2),$$

where

$$A^{(1)}{}_{m_1 m_2} = \frac{1}{(2\pi i)^2} \int_{C_4{}^{(1)}} \int_{C_4{}^{(2)}} V_{m_1}{}^{(1)}(t_1) V_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(2)}{}_{m_1 m_2} = \frac{1}{(2\pi i)^2} \int_{C_4{}^{(1)}} \int_{C_1{}^{(2)}} V_{m_1}{}^{(1)}(t_1) F_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(3)}{}_{m_1 m_2} = \frac{1}{(2\pi i)^2} \int_{C_1{}^{(1)}} \int_{C_4{}^{(2)}} F_{m_1}{}^{(1)}(t_1) V_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1,$$

$$A^{(4)}{}_{m_1 m_2} = \frac{1}{(2\pi i)^2} \int_{C_1{}^{(1)}} \int_{C_1{}^{(2)}} F_{m_1}{}^{(1)}(t_1) F_{m_2}{}^{(2)}(t_2) f(t_1, t_2)\, dt_2 dt_1.$$

II. EXPANSIONS IN CONFLUENT HYPERGEOMETRIC FUNCTIONS.

13. *Certain Confluent Hypergeometric Functions Satisfying the Hypotheses in* § 2. We define $M_{km}(z)$ as follows,

$$M_{km}(z) = z^{\frac{1}{2}+m} e^{-z/2}$$

$$\times \left\{ 1 + \sum_{r=1}^{\infty} \frac{(\tfrac{1}{2} + m - k)(\tfrac{3}{2} + m - k) \cdots (r - \tfrac{1}{2} + m - k)}{(2m+1)(2m+2) \cdots (2m+r)} \frac{z^r}{r!} \right\},$$

where $2m$ is not a negative integer. Then $M_{km}(z)$ is an integral of

$$\text{(11)} \qquad \frac{d^2 y}{dz^2} + \left\{ -\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2} \right\} y = 0.$$

This equation is obtained from the confluent hypergeometric equation

$$\frac{d^2 u}{dz^2} + \frac{du}{dz} + \left\{ \frac{k}{z} - \frac{\tfrac{1}{4} - m^2}{z^2} \right\} u = 0$$

by means of the transformation $u = e^{-z/2}y$. The series for $M_{km}(z)$ converges for all finite values of z*. We shall write

$$\overline{M}_{km}(z) = z^{-\frac{1}{2}}e^{z/2}M_{km}(z)$$

and deal with expansions in terms of $\overline{M}_{km}(z)$ allowing m to range over zero and the positive integers.

For the circle of radius R mentioned in § 2 we may take any circle described about the origin. Then, using $M_N(m)$, N and M_N in the same senses as in § 2 we have

$$M_N(m) \leq N^m \times \left\{ 1 + \sum_{r=1}^{\infty} \frac{|\tfrac{1}{2} + m - k|\,|\tfrac{3}{2} + m - k| \cdots |r - \tfrac{1}{2} + m - k|}{(2m+1)\,(2m+2)\cdots(2m+r)} \frac{N^r}{r!} \right\}.$$

The rth term in the summation on the right has r factors of the form

$$|\alpha - \tfrac{1}{2} + m - k|/(2m + \alpha), \quad \alpha = 1, 2, \cdots, r.$$

Now

$$|\alpha - \tfrac{1}{2} + m - k|/(2m + \alpha) \leq (m + \alpha + |k - \tfrac{1}{2}|)/(m + \alpha + m).$$

Hence there exists a constant L such that

$$L \geq |\alpha - \tfrac{1}{2} + m - k|/(2m + \alpha) \tag{12}$$

independently of m and α and for any fixed k. We then have

$$M_N(m) \leq N^m \{ 1 + \sum_{r=1}^{\infty} (LN)^r/r! \} = N^m e^{LN},$$

and M_N may be taken

$$M_N = e^{LN}.$$

The sequence of functions $\overline{M}_{km}(z)$, $m = 0, 1, 2, \cdots$ then satisfies the hypotheses of § 2 and the theorems of the first division apply to it. The expansions there obtained converge in some small regions. We now proceed to extend those regions by further investigation.

14. *Expansion of* $1/(t - z)$. Now we have an expansion of the form

$$z^n = \sum_{\nu=0}^{\infty} c_{n+\nu}\, \overline{M}_{k,n+\nu}(z).$$

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* See Whittaker and Watson, *A Course of Modern Analysis* (3d Ed., Cambridge, 1920), Chapter XVI, for discussion of solutions of (11) and further references.

By inserting in the right member the expansion of $\overline{M}_{km}(z)$ and equating the coefficients of like powers of z, we find that the c's satisfy the recurrence relations

$$\sum_{m=0}^{r-1} \frac{(\tfrac{1}{2}+m-k)(\tfrac{3}{2}+m-k)\cdots(r-\tfrac{1}{2}-k)}{(2m+1)(2m+2)\cdots(m+r)\ \ (r-m)\,!}\, c_m + c_r = 1,\quad r = n,$$
$$= 0,\quad r \neq n;$$

from which it follows that

$$c_n = 1 \text{ and } c_{n+\nu} = (-)^{\nu}\,\frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{(2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)\ \ \nu\,!}.$$

Making use of (12) we have

$$|\,\overline{M}_{km}(z)\,| \leq |\,z\,|^{m}\, e^{L|z|}.$$

The expansion for z^n is then dominated by the series

$$e^{L|z|}\,|\,z\,|^{n} + \sum_{\nu=1}^{\infty} \left|\, \frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{(2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)\ \ \nu\,!} \,\right| \times |\,z\,|^{n+\nu} e^{L|z|}.$$

By the ratio test it is seen that this series converges for all finite values of z and the same is true of the series dominated. It follows that the series for z^n is absolutely and uniformly convergent in any finite closed region. Therefore since the series represents z^n in the neighborhood of zero it represents z^n everywhere. Now from the equation

$$1/(t-z) = \sum_{n=0}^{\infty} z^{n}/t^{n+1}, \quad |\,z\,| < |\,t\,|,$$

we have

$$(13)\qquad 1/(t-z) = \sum_{n=0}^{\infty} 1/t^{n+1}\,\{\,\overline{M}_{kn}(z)$$
$$+ \sum_{\nu=1}^{\infty} (-)^{\nu}\,\frac{(n+\tfrac{1}{2}-k)(n+\tfrac{3}{2}-k)\cdots(n+\nu-\tfrac{1}{2}-k)}{\nu\,!\ \ (2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)}\,\overline{M}_{k,n+\nu}(z)\}.$$

This series is dominated by

$$\sum_{n=0}^{\infty} e^{L|z|}/|\,t\,|^{n+1}\,\{\,|\,z\,|^{n}$$
$$+ \sum_{\nu=1}^{\infty} \frac{|\,n+\tfrac{1}{2}-k\,|\ \ |\,n+\tfrac{3}{2}-k\,|\ \cdots\ |\,n+\nu-\tfrac{1}{2}-k\,|}{\nu\,!\ \ (2n+\nu)(2n+\nu+1)\cdots(2n+2\nu-1)}\,|\,z\,|^{n+\nu}\}.$$

By an argument similar to that used in establishing (12) we have

$$|\,n+\alpha-\tfrac{1}{2}-k\,|/(2n+2\alpha-1) \leq L', \quad \alpha = 1, 2, \cdots, \nu;$$

and

$$(14) \quad \sum_{n=0}^{\infty} e^{L|z|}/|\,t\,|^{n+1} \;|\,z\,|^{n} \sum_{\nu=0}^{\infty} (L'|\,z\,|)^{\nu}/\nu! \;= e^{(L+L')|z|} \sum_{n=0}^{\infty} |\,z\,|^{n}/|\,t\,|^{n+1}$$

which is convergent if $|\,z\,| < |\,t\,|$ and consequently (13) is absolutely convergent if $|\,z\,| < |\,t\,|$. By an argument similar to that of § 4 we may rearrange (13) obtaining

$$1/(t-z) = \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z), \text{ where } V_{ko}(t) = 1/t$$

$$\text{and } V_{kn}(t) = \sum_{r=0}^{n-1} (-)^{n+r} \frac{(r+\tfrac{1}{2}-k)(r+\tfrac{3}{2}-k)\cdots(n-\tfrac{1}{2}-k)}{(n-r)!\,(n+r)(n+r+1)\cdots(2n-1)t^{r+1}} + \frac{1}{t^{n+1}}, \; n \geqq 1.$$

We may state the following:

If z and t be two complex variables such that $|\,z\,| \leq R_1$ and $|\,t\,| \geq R_4$ where $R_1 < R_4$ then the expansion

$$(15) \qquad 1/(t-z) = \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z), \text{ where } V_{k0} = 1/t$$

$$\text{and} \quad \mathrm{V}_{kn}(t) = \sum_{r=0}^{n-1} (-)^{n+r} \frac{(r+\tfrac{1}{2}-k)(r+\tfrac{3}{2}-k)\cdots(n-\tfrac{1}{2}+k)}{(n-r)!\,(n+r)(n+r+1)\cdots(2n-1)t^{r+1}} + \frac{1}{t^{n+1}}, \quad n \geq 1,$$

is valid and converges uniformly with respect to t and z.

15. *The differential Equation Satisfied by $V_{kn}(t)$.* By means of the transformation $M_{kn}(z) = e^{-z/2}z^{\frac{1}{2}}\overline{M}_{kn}(z)$ applied to the differential equation (11) we find that $\overline{M}_{kn}(z)$ satisfies the equation

$$z^2 \frac{d^2u}{dz^2} + (z-z^2)\frac{du}{dz} + \{(k-\tfrac{1}{2})z - n^2\}u = 0.$$

We differentiate both members of the equation (15) with respect to z and multiply through by the corresponding coefficients from the above differential equation. Then using that equation we obtain

$$2z^2/(t-z)^3 + (z-z^2)/(t-z)^2 + (k-\tfrac{1}{2})/(t-z) = \sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z)\cdot n^2.$$

Expanding the numerators of the terms of the left member in powers of $(t-z)$ and simplifying, we have

$$2t^2/(t-z)^3-(3t+t^2)/(t-z)^2+[(k+3/2)t+1]/(t-z)-k-1/2$$
$$=\sum_{n=0}^{\infty} V_{kn}(t)\overline{M}_{kn}(z)\cdot n^2.$$

Again differentiating both members of the equation (15), this time with respect to t, and making use of the last equation we find a differential equation satisfied by $V_{kn}(t)$ to be

$$t^2\,\frac{d^2y}{dt^2}+(t^2+3t)\,\frac{dy}{dt}+[(k+3/2)t+(1-n^2)]y$$
$$=(1/2+k)\,\frac{(1/2-k)(3/2-k)\cdots(n-1/2-k)}{n!\;n(n+1)\cdots(2n-1)}.$$

16. *Expansions in Terms of* $V_{kn}(z)$ *and* $\overline{M}_{kn}(z)$. By an argument similar to that in §§ 5, 6, 7 we can establish the following theorems:

THEOREM VII. *If* $f(z)$ *be a single-valued, analytic function of* z *within and upon* C_4, *the equation of* C_4 *being* $|z|=R_4$, *then the expansion*

$$f(z)=\sum_{n=0}^{\infty} a_{kn}\overline{M}_{kn}(z),\ \textit{where}\ \ a_{kn}=(1/2\pi i)\int_{C_4} V_{kn}(t)f(t)\,dt,$$

is valid and converges uniformly with respect to z *if* z *is within or upon the circle* $|z|=R_1$, *where* $R_1<R_4$.

COROLLARY. *If* $f(z)$ *is analytic throughout the finite plane then the foregoing expansion for* $f(z)$ *is valid throughout the finite plane.*

THEOREM VIII. *If* $f(z)$ *be a single-valued, analytic function of* z *upon and outside* C_1, *the equation of* C_1 *being* $|z|=R_1$, *then the expansion*

$$f(z)=\sum_{n=0}^{\infty} \alpha_{kn}V_{kn}(z),\ \textit{where}\ \ \alpha_{kn}=(1/2\pi i)\int_{C_1} \overline{M}_{kn}(t)f(t)\,dt,$$

is valid and converges uniformly with respect to z *if* z *is outside or upon the circle* $|z|=R_4$ *where* $R_1<R_4$.

THEOREM IX. *Let* C_1, C_2, C_3, C_4 *be, respectively, the circles* $|z|=R_1$,

$|z| = R_2$, $|z| = R_3$, $|z| = R_4$ where $R_1 < R_2 < R_3 < R_4$. If $f(z)$ be a single-valued, analytic function of z within and upon the boundary of the ring C_1C_4 then the expansion

$$f(z) = \sum_{n=0}^{\infty} a_{kn}\overline{M}_{kn}(z) + \sum_{n=0}^{\infty} a'_{kn}V_{kn}(z),$$

where $a_{kn} = (1/2\pi i) \int_{C_4} V_{kn}(t)f(t)\,dt$ *and* $a'_{kn} = (1/2\pi i) \int_{C_1} \overline{M}_{kn}(t)f(t)\,dt,$

is valid and converges uniformly with respect to z when z is within or upon the boundary of the ring C_2C_3.

We can establish the relationships between the coefficients of these expansions and those of the Taylor and Laurent expansions of $f(z)$. Orthogonality properties for $V_{kn}(z)$ and $\overline{M}_{kn}(z)$ similar to those in § 9 can also be proven. Results may be obtained in the neighborhood of $z = a$. The theory may be extended to the expansions of functions of n variables in multiple series.

A Modern Presentation of Grassmann's Tensor Analysis.

BY HELEN BARTON.

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1. *Introduction.* Since Einstein advanced his Theory of Relativity in 1916, there has been a considerable increase in interest in what is known as Tensor Analysis. Mathematicians and physicists alike are realizing the power of this method of treatment; especially since it has led to the association of physical quantities which hitherto had seemed unrelated. The idea of a tensor, however, far antedates Einstein, it being introduced by Grassmann in 1844.*

In this work of Grassmann's *Die Ausdehnungslehre,* the author gives a rather exhaustive algebraic treatment of these quantities, which he termed "extensive Grösse," but his ideas were often abstruse and vague, so that the work has never been used nor appreciated very much by mathematicians. The main part of this paper will be devoted to the development of explicit expressions for these "extensive Grösse," and for various combinations of them. It is hoped by this more definite treatment of the subject that Grassmann's work will be simplified and thereby become of greater value to the mathematical world.

One method of defining a tensor is by means of its components and the transformation of these components under a change of coordinates. If we form n functions of n ordered numbers in the x space

$$\phi^1(x^1, \cdots, x^n), \quad \phi^2(x^1, \cdots, x^n), \cdots, \quad \phi^n(x^1, \cdots, x^n),$$

arbitrary except that the ratios ϕ^1/x^1, $\phi^2/x^2, \cdots, \phi^n/x^n$ are of the same physical dimensions, and then form another set of n functions of the corresponding y's in the y space,

$$\psi^1(y^1, \cdots, y^n), \quad \psi^2(y^1, \cdots, y^n), \cdots, \quad \psi^n(y^1, \cdots, y^n)$$

which are defined by the equation

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* H. Grassmann, *Die Ausdehnungslehre,* Werke, Bd. 1, Leipzig, 1896.

$$\psi^r = \phi^\alpha \, \partial y^r / \partial x^\alpha \, {}^*$$

then these n functions $\phi^1, \phi^2, \cdots, \phi^n$ are the n components of a contravariant tensor of rank one as presented in the x space, and $\psi^1, \psi^2, \cdots, \psi^n$ are the n components of the same contravariant tensor as presented in the y space.

Similarly, if we start with n functions of the x coordinates, $\phi_1, \phi_2, \cdots,$ ϕ_n which are now subjected to the condition that each product $\phi_r x^r$ is of the same physical dimensions, and then form the corresponding functions of the y coordinates $\psi_1, \psi_2, \cdots, \psi_n$ where $\psi_r = \phi_\alpha \, \partial x^\alpha / \partial y^r$ then $\phi_1, \phi_2, \cdots, \phi_n$ are the n components of a covariant tensor of rank one as presented in the x space, and $\psi_1, \psi_2, \cdots, \psi_n$ are the n components of the same covariant tensor as presented in the y space.

Other tensors of each of these types, as well as combinations of the two, but of higher rank are defined in a similar way; but without going into further detail, it is evident that tensors, as viewed from this angle, are composite quantities, and their components presented in one space bear a certain definite relation to the components in another space.

Grassmann had a very different point of view. He was not interested in the presentation of these "extensive Grösse" with reference to any system of cordinates, nor in their transformations, but his work dealt with the various operations which could be applied to these quantities. To him, an "extensive Grösse" was a quantity built up from simple quantities which have both direction and magnitude. These "extensive Grösse," therefore, are defined by their parts or components in certain given directions. It follows from the definition that two "Grösse" are equal if the corresponding components of the two are equal.

Grassmann made no reference as to whether these "extensive Grösse" were contravariantly or covariantly presented, and the reason for this is clear. In Euclidean space, where the coordinates are orthogonal, we have the fundamental relation

$$\begin{aligned}(ds)^2 &= (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2 \\ &= (dy^1)^2 + (dy^2)^2 + \cdots + (dy^n)^2.\end{aligned}$$

If linear transformations only are considered, they are of the type $x^r = c_\alpha{}^r y^\alpha$, $y^r = c_r{}^\alpha x^\alpha$ and there is no distinction between the covariant and the contravariant presentation of a tensor.

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* A Greek letter in an expression plays the rôle of an umbral symbol, i. e. it indicates that the expression is a summation of terms, each formed by assigning the n numerical values in succession to the umbral symbol.

2. *Outer Product of Tensors.* We know from ordinary Vector Analysis that two vectors may be combined into a vector or outer product, to give a new vector of the same type but of rank two, and its components are given by a two row determinant. Thus the r_1r_2 component of the outer product of the vectors a and b is given by

$$[ab]^{r_1r_2} = \begin{vmatrix} a^{r_1} & a^{r_2} \\ b^{r_1} & b^{r_2} \end{vmatrix}$$

but this might also be written in the form

$$A^{r_1r_2} = [ab]^{r_1r_2} = \begin{bmatrix} r_1 & r_2 \\ \alpha_1 & \alpha_2 \end{bmatrix} a^{\alpha_1} b^{\alpha_2}.*$$

The symbol here used $\begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix}$ is known as the *generalized Kronecker symbol.*

The symbol is, as its name indicates, a generalization of the ordinary Kronecker symbol and we shall represent it in the form $\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix}$ where m is any integer $1, 2, \cdots, n$ and $r_1, r_2, \cdots, r_m$ and $s_1, s_2, \cdots, s_m$ may take independently any of the n integral values. By definition, the symbol has but three distinct values:

$$\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix} = 0$$

if $r_1, r_2, \cdots, r_m$ and $s_1, s_2, \cdots, s_m$ are not arrangements of the same set of m distinct integers;

$$\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix} = \pm 1$$

according as it takes an even or odd number of interchanges to bring $r_1, r_2, \cdots, r_m$ into the same arrangement as $s_1, s_2, \cdots s_m$.

It should be noted further that this symbol is itself a tensor, mixed in type—covariant of rank m and contravariant of rank m.† Furthermore, this mixed tensor is non-metric, that is, it holds for all space, in which a point is merely a set of n ordered numbers.

If, however, we modify the symbol by fixing the order of one of the sets of labels, for example, $\begin{bmatrix} 1, & 2, & \cdots, & n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$, as we do in defining the complement of a tensor, it should be noted that it is no longer a mixed tensor of rank $2m$, but if it is multiplied by $\pm g^{1/2}$ it becomes a covariant tensor of rank m, where g is a determinant whose elements are the components of a covariant tensor of rank 2,—obtained from the metrical form

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* F. D. Murnaghan, *American Mathematical Monthly*, Vol. 32 (1925), pp. 233-241.

† F. D. Murnaghan, *Bulletin of the American Mathematical Society*, Vol. 31 (1925), pp. 323-329.

$$(ds)^2 = g_{\alpha\beta} x^\alpha x^\beta.$$

In normal space, where $g_{rs} = 0$ if $r \neq s$ and $g_{rr} = 1$, the determinant $g = 1$. Hence, since Grassmann was dealing, in effect, with points which were expressed in normal coordinates, the factor $g^{1/2}$ does not appear.

This symbol therefore lends itself to the formation of an outer product of any number of these vectors or tensors of rank one, thereby producing a tensor of higher rank,—equal to the number of these vectors contained in the product. But we need not restrict ourselves to the outer product of tensors of rank one. We may form the outer product of several tensors, each of different rank, and so obtain a tensor of still higher rank.

Thus if A is a tensor of rank p and B is a tensor of the same type as A and of rank q, and $p + q \leqq n$ (where n is the dimension of the space), then $[AB]^{r_1 \cdots r_p s_1 \cdots s_q}$ may be defined as

$$C^{r_1 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_1 r_2 \cdots r_p & s_1 s_2 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p & \beta_1 \beta_2 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

This is the explicit expression for the $r_1 \cdots r_p s_1 \cdots s_q$ component of the outer product of A and B, and it is readily seen that in this case C is a tensor of rank $p + q$.

Furthermore, it should be noted that C, the tensor obtained from the outer product of two or more tensors is alternating. For

$$[AB]^{r_1 r_2 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_1 r_2 \cdots r_p \, s_1 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p \, \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

$$[AB]^{r_2 r_1 \cdots r_p s_1 \cdots s_q} = \begin{bmatrix} r_2 r_1 \cdots r_p \, s_1 \cdots s_q \\ \alpha_1 \alpha_2 \cdots \alpha_p \, \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

It is evident that all odd permutations in the first case will be even in the second, and vice versa; hence, from the definition of the generalized Kronecker symbol,

$$[AB]^{r_1 r_2 \cdots r_p s_1 \cdots s_q} = - [AB]^{r_2 r_1 \cdots r_p s_1 \cdots s_q}.$$

Furthermore, if A and B are each alternating, then many terms in the summation become identical and we may remove the numerical factor $1/p!\,q!$. Since, in this paper, we shall confine our attention to the case of alternating tensors, we shall write as our explicit expression for the $r_1 r_2 \cdots r_p s_1 \cdots s_q$ component of the outer product of the alternating tensors A and B

$$(1) \qquad [AB]^{r_1 \cdots r_p s_1 \cdots s_q} = \frac{1}{p!\,q!} \begin{bmatrix} r_1 \cdots r_p\, s_1 \cdots s_q \\ \alpha_1 \cdots \alpha_p\, \beta_1 \cdots \beta_q \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_q}.$$

It is evident that this definition has no meaning if $p + q > n$, for in that case, the generalized Kronecker symbol vanishes identically.

With these definitions as a starting point, we shall proceed to prove a few theorems relating to the outer product of tensors.

We shall first prove that

$$[AB] = (-1)^{pr}[BA]$$

where A is a tensor of rank p, B is a tensor of rank r, and $p + r < n$.

By definition

$$[AB]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 l_2 \cdots l_p\, l_{p+1} \cdots l_{p+r} \\ \alpha_1 \alpha_2 \cdots \alpha_p\, \alpha_{p+1} \cdots \alpha_{p+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\alpha_{p+1} \cdots \alpha_{p+r}}$$

$$[BA]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 l_2 \cdots l_r\, l_{r+1} \cdots l_{r+p} \\ \alpha_1 \alpha_2 \cdots \alpha_r\, \alpha_{r+1} \cdots \alpha_{r+p} \end{bmatrix} B^{\alpha_1 \cdots \alpha_r} A^{\alpha_{r+1} \cdots \alpha_{p+r}}.$$

In order to compare similar terms, let us rewrite $[BA]$ in a different manner.

$$[BA]^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & \cdot & l_{p+r} \\ \alpha_{p+1} \cdots \alpha_{p+r} & \alpha_1 \cdots \alpha_p \end{bmatrix} B^{\alpha_{p+1} \cdots \alpha_{p+r}} A^{\alpha_1 \cdots \alpha_p}.$$

In order to determine the sign, we must rearrange the α's in the symbol so as to make them agree with those in the sign factor of $[AB]$. To do this, α_1 must be moved over r positions to bring it to the first position, i. e. the sign will be $(-1)^r$; similarly it will take r interchanges to bring α_2 to the second position, etc. Therefore, the final sign, when all p of the α's have reached their proper positions will be $(-1)^{rp}$. Hence

$$(2) \qquad [AB] = (-1)^{pr}[BA].$$

From this it follows that if C is a tensor of rank s, and D is a tensor of rank t, then

$$\begin{aligned} [ABCD] &= (-1)^{st}\,[ABDC] = (-1)^{st+rt}\,[ADBC] \\ &= (-1)^{st+rt+pt}\,[DABC] = (-1)^{st+rt+pt+rs+ps+pr}\,[DCBA]. \end{aligned}$$

We shall now prove that

$$[b_1 b_2 \cdots b_m] = \text{determinant of } \alpha\text{'s} \cdot [a_1 a_2 \cdots a_m]$$

where $b_1, b_2, \cdots, b_m$ is a series of simple "extensive Grösse" which are linearly dependent upon m other simple "extensive Grösse" $a_1, a_2, \cdots, a_m$; i. e.

$$b_1 = \alpha_1^{\rho_1} a_{\rho_1}, \quad b_2 = \alpha_2^{\rho_2} a_{\rho_2} \cdots b_m = \alpha_m^{\rho_m} a_{\rho_m} .$$

Any component σ_1 of b_1 may be represented as $b_1^{\sigma_1} = \alpha_1^{\rho_1} a_{\rho_1}^{\sigma_1}$. By definition

$$[b_1 b_2 \cdots b_m]^{r_1 r_2 \cdots r_m} = \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} b_1^{\sigma_1} b_2^{\sigma_2} \cdots b_m^{\sigma_m}$$

$$= \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \alpha_1^{\rho_1} a_{\rho_1}^{\sigma_1} \cdot \alpha_2^{\rho_2} a_{\rho_2}^{\sigma_2} \cdots \alpha_m^{\rho_m} a_{\rho_m}^{\sigma_m}$$

$$= \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \alpha_1^{\rho_1} \alpha_2^{\rho_2} \cdots \alpha_m^{\rho_m} \cdot a_{\rho_1}^{\sigma_1} a_{\rho_2}^{\sigma_2} \cdots\cdots a_{\rho_m}^{\sigma_m}$$

$$= (1/m!) \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \begin{bmatrix} \lambda_1 \lambda_2 \cdots \lambda_m \\ 1\ 2 \cdots m \end{bmatrix} \alpha_{\lambda_1}^{\rho_1} \alpha_{\lambda_2}^{\rho_2} \cdots \alpha_{\lambda_m}^{\rho_m} \cdot a_{\rho_1}^{\sigma_1} a_{\rho_2}^{\sigma_2} \cdots a_{\rho_m}^{\sigma_m}$$ [*]

$$= (1/m!) \begin{bmatrix} r_1 \cdots r_m \\ \sigma_1 \cdots \sigma_m \end{bmatrix} \overset{\rho_1 \cdots \rho_m}{\underset{1 \cdots m}{D(\alpha)}} \quad a_{\rho_1}^{\sigma_1} a_{\rho_2}^{\sigma_2} \cdots\cdots a_{\rho_m}^{\sigma_m}$$

Hence

$$(3) \qquad [b_1 \cdots b_m]^{r_1 \cdots r_m} = \overset{1 \cdots m}{\underset{1 \cdots m}{D(\alpha)}} \quad \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} a_1^{\sigma_1} a_2^{\sigma_2} \cdots a_m^{\sigma_m}$$

$$= \text{Determinant of } \alpha\text{'s } [a_1 a_2 \cdots a_m].$$ [†]

So far, we have been concerned only with the outer product of tensors, the sum of whose ranks is less than or at most equal to n. But before proceeding to the case where this sum is greater than n, we shall define a new quantity known as the complement of a tensor.

If A is a tensor of rank p, then the complement of A is a tensor B of rank $n - p$ (represented as $|A \equiv B$). Any component of the complement of A is explicitly defined as

$$(4) \qquad (|A)_{s_1 \cdots s_{n-p}} = B_{s_1 \cdots s_{n-p}} = (1/p!) \begin{bmatrix} 1 & \cdots & \cdots & n \\ s_1 \cdots s_{n-p} & \rho_1 & \cdots & \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p}.$$ [‡]

Before proceeding to the next proof, we shall introduce at this point several lemmas, concerning the generalized Kronecker symbol, as they will be needed in the following proofs.

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[*] Formula 2.3 of Professor Murnaghan's paper, *American Mathematical Monthly.*

[†] Formula 3.4 of Professor Murnaghan's paper, *American Mathematical Monthly.*

[‡] If A is a contravariant tensor of rank p, the complement of A is covariant of rank $n - p$. The shift of labels from above to below is used to indicate the change of type of tensor.

LEMMA I.

$$(5)\quad \begin{bmatrix} r_1 \cdot \cdot \cdot \cdot \cdot \cdot r_m \\ s_1 \cdot \cdot \cdot s_g\, \alpha_1 \cdot \cdot \cdot \alpha_{m-g} \end{bmatrix} \begin{bmatrix} \alpha_1 \cdot \cdot \cdot \alpha_{m-g} \\ p_1 \cdot \cdot \cdot p_{m-g} \end{bmatrix} = (m-g)\,! \begin{bmatrix} r_1 \cdot \cdot \cdot \cdot \cdot \cdot r_m \\ s_1 \cdot \cdot \cdot s_g\, p_1 \cdot \cdot \cdot p_{m-g} \end{bmatrix}$$

From the second factor on the left-hand side, it is evident that in order that the symbol shall not be zero, $\alpha_1 \cdot \cdot \cdot \alpha_{m-g}$ must be some arrangement of the quantities $p_1, p_2, \cdot \cdot \cdot, p_{m-g}$. Any inversion in a particular set of values assigned to $\alpha_1, \alpha_2, \cdot \cdot \cdot, \alpha_{m-g}$ in the one factor will cause a similar inversion in the other factor; hence, their product will have the same sign as if the arrangements of $\alpha_1 \cdot \cdot \cdot \alpha_{m-g}$ and $p_1 \cdot \cdot \cdot p_{m-g}$ are identical. Since there are $(m-g)\,!$ such arrangements of $\alpha_1, \alpha_2, \cdot \cdot \cdot, \alpha_{m-g}$ we must introduce the factor $(m-g)\,!$ on the right.

LEMMA II.

$$(6)\quad \begin{bmatrix} r_1 \cdot \cdot \cdot \cdot \cdot \cdot r_m \\ s_1 \cdot \cdot \cdot s_g\, \alpha_1 \cdot \cdot \cdot \alpha_{m-g} \end{bmatrix} \begin{bmatrix} b_1 \cdot \cdot \cdot b_g\, \alpha_1 \cdot \cdot \cdot \alpha_{m-g} \\ r_1 \cdot \cdot \cdot \cdot \cdot \cdot r_m \end{bmatrix} = (m-g)\,! \begin{bmatrix} b_1 \cdot \cdot \cdot b_g \\ s_1 \cdot \cdot \cdot s_g \end{bmatrix}.$$

It is evident that if the symbol is not to be zero, that $s_1, s_2, \cdot \cdot \cdot, s_g$ is contained in $r_1, \cdot \cdot \cdot, r_m$; also $b_1, \cdot \cdot \cdot, b_g$ is contained in $r_1, \cdot \cdot \cdot, r_m$; therefore $\alpha_1, \cdot \cdot \cdot, \alpha_{m-g}$ may assume only such values of $r_1, \cdot \cdot \cdot, r_m$ as are not contained in $s_1, \cdot \cdot \cdot, s_g$. Consequently, $b_1, \cdot \cdot \cdot, b_g$ must contain only those values of $r, \cdot \cdot \cdot, r_m$ not assumed by $\alpha_1, \cdot \cdot \cdot, \alpha_{m-g}$, i. e. those contained in $s_1, \cdot \cdot \cdot, s_g$. Since there are $(m-g)\,!$ such arrangements of $\alpha_1, \cdot \cdot \cdot, \alpha_{m-g}$ the truth of (6) follows.

LEMMA III.

$$(7)\quad \begin{bmatrix} r_1 \cdot \cdot \cdot r_m \\ \alpha_1 \cdot \cdot \cdot \alpha_m \end{bmatrix} A^{\alpha_1 \cdot \cdot \cdot \alpha_m} = \frac{m\,!}{g\,!\,(m-g)\,!} \begin{bmatrix} r_1 \cdot \cdot \cdot r_g \\ \alpha_1 \cdot \cdot \cdot \alpha_g \end{bmatrix} \times \begin{bmatrix} r_{g+1} \cdot \cdot \cdot r_m \\ \alpha_{g+1} \cdot \cdot \cdot \alpha_m \end{bmatrix} A^{\alpha_1 \cdot \cdot \cdot \alpha_m}$$

$$\begin{bmatrix} r_1 \cdot \cdot \cdot r_m \\ \alpha_1 \cdot \cdot \cdot \alpha_m \end{bmatrix} A^{\alpha_1 \cdot \cdot \cdot \alpha_m} = m\,!\, A^{r_1 \cdot \cdot \cdot r_m}$$

$$\begin{bmatrix} r_1 \cdot \cdot \cdot r_g \\ \alpha_1 \cdot \cdot \cdot \alpha_g \end{bmatrix} \begin{bmatrix} r_{g+1} \cdot \cdot \cdot r_m \\ \alpha_{g+1} \cdot \cdot \cdot \alpha_m \end{bmatrix} A^{\alpha_1 \cdot \cdot \cdot \alpha_m} = g\,!\,(m-g)\,!\, A^{r_1 \cdot \cdot \cdot r_m}$$

Both of these statements follow from the alternating character of these tensors.

Therefore the truth of (7) follows.

LEMMA IV.

(8) $$\begin{bmatrix} r_1 \cdots r_m \, \lambda_1 \cdots \lambda_r \\ \alpha_1 \cdots \cdots \alpha_{m+r} \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \lambda_1 \cdots \lambda_r \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}$$

$$= \frac{(m+r)!}{m!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_{m+1} \cdots \alpha_{m+r} \, \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}.$$

From (7) we have

$$\begin{bmatrix} r_1 \cdots r_m \, \lambda_1 \cdots \lambda_r \\ \alpha_1 \cdots \alpha_m \, \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} = \frac{(m+r)!}{m!\,r!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \times \begin{bmatrix} \lambda_1 \cdots \cdots \lambda_r \\ \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}}.$$

Substituting this in the left-hand member of (8) we obtain

$$\frac{(m+r)!}{m!\,r!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} \lambda_1 \cdots \cdots \lambda_r \\ \alpha_{m+1} \cdots \alpha_{m+r} \end{bmatrix} \times \begin{bmatrix} 1 \cdots \cdots n \\ \lambda_1 \cdots \lambda_r \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}.$$

$$= \frac{(m+r)!}{m!} \begin{bmatrix} r_1 \cdots r_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_{m+1} \cdots \alpha_{m+r} \, \beta_1 \cdots \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_{m+r}} B^{\beta_1 \cdots \beta_{n-r}}. \qquad \text{(by 5)}$$

It is readily shown that *the complement of the complement of a tensor is equal to ± the original tensor.*

If A is a tensor of rank p, then $B = | A$ is a tensor of rank $n-p$, and $C = | B$ is a tensor of rank p. We shall prove therefore that

$$|| A = \pm A.$$

By definition,

$$C^{p_1 \cdots p_p} = \frac{1}{(n-p)!} \begin{bmatrix} p_1 \cdots p_p \, \alpha_1 \cdots \alpha_{n-p} \\ 1 \cdots \cdots n \end{bmatrix} [\,| B]_{\alpha_1 \cdots \alpha_{n-p}}$$

$$= \frac{1}{(n-p)!\,p!} \begin{bmatrix} p_1 \cdots p_p \, \alpha_1 \cdots \alpha_{n-p} \\ 1 \cdots \cdots n \end{bmatrix} \times \begin{bmatrix} 1 \cdots \cdots n \\ \alpha_1 \cdots \alpha_{n-p} \, \rho_1 \cdots \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p} \qquad \text{(by 4)}$$

$$= \frac{(-1)^{p(n-p)}}{p!} \begin{bmatrix} p_1 \cdots p_p \\ \rho_1 \cdots \rho_p \end{bmatrix} A^{\rho_1 \cdots \rho_p} \qquad \text{(by 6)}$$

$$= (-1)^{p(n-p)} A^{p_1 \cdots p_p}$$

(9) $$\therefore \quad || A = (-1)^{p(n-p)} A.$$

If A is of order p and B is of order r, and $p + r > n$, then as we noted above, our former definition of the outer product of two tensors (when $p + r < n$) would have no meaning. Hence, we define the outer product of AB, when $p + r > n$ as a tensor of rank $p + r - n$ and its explicit expression is found by taking the outer product of the complements of A and of B, and taking the complement of this outer product i. e. $|\,[|\,A \cdot |\,B] = [AB]$ by definition, when $p + r > n$. $|\,A$ is a tensor of rank $n - p$; $|\,B$ is a tensor of rank $n - r$; $[|\,A \cdot |\,B]$ is a tensor of rank $2n - p - r < n$; $|\,[\,|\,A \cdot |\,B]$ is a tensor of rank $p + r - n < n$.

To find the explicit expression for this outer product, we proceed directly.

$$[|\,A \cdot |\,B]_{l_1 \cdots l_{2n-p-r}} = \frac{1}{p!\,r!\,(n-p)!\,(n-r)!} \begin{bmatrix} \mu_1 & \cdots & \mu_{n-p} & \tau_1 & \cdots & \tau_{n-r} \\ l_1 & \cdot & \cdot & \cdot & \cdot & l_{2n-p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \mu_1 & \cdots & \mu_{n-p} & \alpha_1 & \cdots & & \alpha_p \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \tau_1 & \cdots & \tau_{n-r} & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

Taking the complement of this, we obtain

$$|\,[\,|\,A \cdot |\,B]_{l_1 \cdots l_{2n-p-r}} = C^{m_1 \cdots m_{p+r-n}}$$

$$= \frac{1}{p!\,r!\,(n-p)!\,(n-r)!\,(2n-p-r)!}$$

$$\begin{bmatrix} m_1 & \cdots & m_{p+r-n} & \rho_1 & \cdots & \rho_{2n-p-r} \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} \begin{bmatrix} \mu_1 & \cdots & \mu_{n-p} & \tau_1 & \cdots & \tau_{n-r} \\ \rho_1 & \cdot & \cdot & \cdot & & \rho_{2n-p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \mu_1 & \cdots & \mu_{n-p} & \alpha_1 & \cdots & & \alpha_p \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \tau_1 & \cdots & \tau_{n-r} & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

$$\doteq \frac{(-1)^{(n-p)(p+r-n)}}{p!\,r!\,(n-r)!} \begin{bmatrix} m_1 & \cdots & m_{p+r-n} & \tau_1 & \cdots & \tau_{n-r} \\ \alpha_1 & \cdot & \cdot & \cdot & \cdot & \alpha_p \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \tau_1 & \cdots & \tau_{n-r} & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

(by 5 and 6).

$$(10) \qquad = \frac{(-1)^{(n-p)(r-1)}}{r!\,(n-r)!\,(p+r-n)!} \begin{bmatrix} m_1 & \cdots & m_{p+r-n} \\ \alpha_1 & \cdots & \alpha_{p+r-n} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \alpha_{p+r-n+1} & \cdots & \alpha_p & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

By means of this expression, it is readily shown that *the complement of the outer product of two tensors is equal to ± the outer product of their complements,* i. e.,

$$|\,[AB] = \pm [|\,A \cdot |\,B]$$

where A is a tensor of rank p, B is a tensor of rank r.

Case I. $p + r < n$.

$| [AB]$ is of rank $n - p - r$.

$[| A \cdot | B]$ is of rank $2n - p - r - n = n - p - r < n$.

$$| [AB] = | C^{l_1 \cdots l_{p+r}}$$

$$C^{l_1 \cdots l_{p+r}} = \frac{1}{p!\,r!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & l_{p+r} \\ \alpha_1 \cdots & \alpha_p & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

$$| C^{l_1 \cdots l_{p+r}} = D_{s_1 \cdots s_{n-p-r}}$$

$$= \frac{1}{p!\,r!\,(p+r)!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ s_1 & \cdots & s_{n-p-r} & \lambda_1 & \cdots & & \lambda_{p+r} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_1 & \cdot & \cdot & \cdot & \cdot & \lambda_{p+r} \\ \alpha_1 \cdots & \alpha_p & \beta_1 & \cdots & & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

$$= \frac{1}{p!\,r!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ s_1 & \cdots & s_{n-p-r} & \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_r \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r}$$

(by 5).

To find $[| A \cdot | B]$ we must use the expression for the outer product of two tensors, the sum of whose ranks is greater than n.

$$[| A \cdot | B]_{s_1 \cdots s_{n-p-r}} = \frac{(-1)^{p(n-p-r)}}{(n-p-r)!\,(n-r)!\,r!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p-r} \\ s_1 \cdots s_{n-p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_{n-p-r+1} & \cdots & \alpha_{n-p} & \beta_1 & \cdots & \beta_{n-r} \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} (| A)_{\alpha_1 \cdots \alpha_{n-p}} (| B)_{\beta_1 \cdots \beta_{n-r}}$$

$$= \frac{(-1)^{p(n-p-r)}}{(n-p-r)!\,(n-r)!\,r!^2\,p!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p-r} \\ s_1 \cdots s_{n-p-r} \end{bmatrix} \begin{bmatrix} \alpha_{n-p-r+1} & \cdots & \alpha_{n-p} & \beta_1 & \cdots & \beta_{n-r} \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \alpha_1 & \cdots & \alpha_{n-p} & \gamma_1 & \cdots & \gamma_p \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \beta_1 & \cdots & \beta_{n-r} & \delta_1 & \cdots & \delta_r \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r}$$

$$= \frac{(-1)^{(p+r)(n-1)}}{r!\,p!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ s_1 & \cdots & s_{n-p-r} & \gamma_1 & \cdots & \gamma_p & \delta_1 & \cdots & \delta_r \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r}$$

(by 5, 6, 7)

$$\therefore \; | [AB] = (-1)^{(p+r)(n-1)} [| A \cdot | B].$$

Case II. $p + r > n$.

$[| A \cdot | B]$ is a tensor of rank $2n - p - r < n$.

$| [AB]$ is a tensor of rank $2n - p - r$.

$| [AB] = | [| A \cdot | B]$ by definition.

$$\therefore \quad | [AB] = || [| A \cdot | B]. \quad = (-1)^{(2n-p-r)(n-1)} [| A \cdot | B] \qquad \text{(by 9)}$$

$$(11) \qquad = (-1)^{(p+r)(n-1)} [| A \cdot | B].$$

Hence the theorem holds when $p + r < n$ and $p + r > n$.

We shall next prove $[A(BC)] = [ABC]$ where A, B, C are tensors of rank p, r, s respectively.

Case I. $p + r + s < n$.

$$[A(BC)]^{l_1 \cdots l_{p+r+s}} = \frac{1}{(r+s)!\,p!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_{r+s} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} [BC]^{\beta_1 \cdots}$$

$$= \frac{1}{(r+s)!\,p!\,r!\,s!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \beta_1 & \cdots & \beta_{r+s} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta_1 & \cdot & \cdot & \cdot & \beta_{r+s} \\ \gamma_1 & \cdots & \gamma_r & \delta_1 & \cdots \delta_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\gamma_1 \cdots \gamma_r} C^{\delta_1 \cdots \delta_s}$$

$$= \frac{1}{p!\,r!\,s!} \begin{bmatrix} l_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & l_{p+r+s} \\ \alpha_1 & \cdots & \alpha_p & \gamma_1 & \cdots & \gamma_r & \delta_1 & \cdots & \delta_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\gamma_1 \cdots \gamma_r} C^{\delta_1 \cdots \delta_s}$$

$$= [ABC]^{l_1 \cdots l_{p+r+s}}.$$

Hence

$$(12) \qquad [A(BC)] = [ABC].$$

Case II. $p + r + s > 2n$.

Let $A' = | A$, $B' = | B$, $C' = | C$. Then A' is a tensor of rank $n - p$; B' is a tensor of rank $n - r$; C' is a tensor of rank $n - s$. $[A'B'C']$ is a tensor of rank $3n - p - r - s < n$ and therefore comes under Case I.

$$\therefore \quad [A'(B'C')]^{l_1 \cdots l_{3n-p-r-s}} = [A'B'C']^{l_1 \cdots l_{3n-p-r-s}}$$

$| [A'B'C'] = (-1)^{(3n-p-r-s)(n-1)} [| A' \cdot | B' \cdot | C']$. But $A' = | A$ and

$$| A' = || A = (-1)^{p(n-1)} A.$$

$$\therefore \quad | [A'B'C'] = (-1)^{(3n-p-r-s+p+r+s)(n-1)} [ABC] = (-1)^{3n(n-1)} [ABC]$$
$$= [ABC].$$

Similarly $| [A'(B'C')] = (-1)^{(3n-p-r-s)(n-1)} [| A' \cdot | (B'C')]$

$$= (-1)^{3n(n-1)} [A(BC)] = [A(BC)].$$

Since $[A'(B'C')] = [A'B'C']$,

then $[A(BC)] = [ABC]$.

The explicit expression for $[ABC]$ when $p+r+s>2n$ may be derived as follows:

$$
\begin{aligned}
&|\,[A'B'C'] = [ABC]^{s_1 \cdots s_{p+r+s-2n}} \\
&= \frac{1}{(3n-p-r-s)!\,(n-r)!\,(n-s)!\,(n-p)!\,r!\,p!\,s!}\begin{bmatrix} s_1 \cdots s_{p+r+s-2n}\,\lambda_1 \cdots \lambda_{3n-p-r-s} \\ 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \end{bmatrix} \\
&\times \begin{bmatrix} \epsilon_1 \cdots \epsilon_{n-p}\,\delta_1 \cdots \delta_{n-r}\,\sigma_1 \cdots \sigma_{n-s} \\ \lambda_1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \lambda_{3n-p-r-s} \end{bmatrix}\begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad n \\ \epsilon_1 \cdots \epsilon_{n-p}\,\alpha_1 \cdots \alpha_p \end{bmatrix}\begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad n \\ \delta_1 \cdots \delta_{n-r}\,\beta_1 \cdots \beta_r \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \sigma_1 \cdots \sigma_{n-s}\,\gamma_1 \cdots \gamma_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r} C^{\gamma_1 \cdots \gamma_s} \\
&= \frac{(-1)^{(p+r+s-2n)(n-p)}}{(p+r+s-2n)!\,(n-r)!\,(n-s)!\,r!\,s!}\begin{bmatrix} s_1 \cdots s_{p+r+s-2n} \\ \alpha_1 \cdots \alpha_{p+r+s-2n} \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_{p+r+s-2n+1} \cdots \alpha_{p+s-n}\,\beta_1 \cdots \beta_r \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_{p+s-n+1} \cdots \alpha_p\,\gamma_1 \cdots \gamma_s \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B^{\beta_1 \cdots \beta_r} C^{\gamma_1 \cdots \gamma_s}
\end{aligned}
\tag{12A}
$$

We shall further prove that

$$[AB \cdot AC] = \pm [ABC]A. \tag{13}$$

where A, B, C are tensors of rank p, r, s respectively.

Case I. $p+r+s=n$.

$[|\,AB \cdot |\,AC]$ is a tensor of rank $s+r$ or $n-p$,

$|\,[|\,AB \cdot |\,AC]$ is a tensor of rank p.

$$
\begin{aligned}
[AB \cdot AC]^{m_1 \cdots m_p} &= \frac{(-1)^{ps}}{p!\,(p+s)!\,r!}\begin{bmatrix} m_1 \cdots m_p \\ \alpha_1 \cdots \alpha_p \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_{p+1} \cdots \alpha_{p+r}\,\beta_1 \cdots \beta_{p+s} \end{bmatrix} [AB]^{\alpha_1 \cdots \alpha_{p+r}} [AB]^{\beta_1 \cdots \beta_{p+s}} \\
&= \frac{(-1)^{ps}}{p!^3\,(p+s)!\,r!^2\,s!}\begin{bmatrix} m_1 \cdots m_p \\ \alpha_1 \cdots \alpha_p \end{bmatrix}\begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_{p+1} \cdots \alpha_{p+r}\,\beta_1 \cdots \beta_{p+s} \end{bmatrix} \\
&\times \begin{bmatrix} \alpha_1 \quad \cdot \quad \cdot \quad \cdot \quad \alpha_{p+r} \\ \gamma_1 \cdots \gamma_p\,\delta_1 \cdots \delta_r \end{bmatrix}\begin{bmatrix} \beta_1 \quad \cdot \quad \cdot \quad \cdot \quad \beta_{p+s} \\ \mu_1 \cdots \mu_p\,\rho_1 \cdots \rho_s \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} B^{\delta_1 \cdots \delta_r} A^{\mu_1 \cdots \mu_p} C^{\rho_1 \cdots \rho_s} \\
&= \frac{(-1)^{p(s+r)}}{p!^2\,r!\,s!}\begin{bmatrix} m_1 \cdots m_p \\ \gamma_1 \cdots \gamma_p \end{bmatrix}\begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \mu_1 \cdots \mu_p\,\delta_1 \cdots \delta_r\,\rho_1 \cdots \rho_s \end{bmatrix} \\
&\qquad \times A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\rho_1 \cdots \rho_s}
\end{aligned}
$$

$$= \frac{(-1)^{p(s+r)}}{p!} \begin{bmatrix} m_1 \cdots m_p \\ \gamma_1 \cdots \gamma_p \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} \cdot$$

$$\times \frac{1}{p!\,r!\,s!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \mu_1 & \cdots & \mu_p & \delta_1 & \cdots & \delta_r & \rho_1 & \cdots & \rho_s \end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\rho_1 \cdots \rho_s}$$

$$(13A) \qquad = (-1)^{p(s+r)} [ABC]\, A^{m_1 \cdots m_p}.$$

Case II. $p + r + s = 2n$.

Let $A' = | A$, $B' = | B$, $C' = | C$. Then $[A'B'C']$ is a tensor of rank $3n - p - r - s = n$. Hence we can apply the theorem for Case I.

$$\therefore \quad [A'B' \cdot A'C'] = (-1)^{(n-p)(n-1)} \; [A'B'C'] A'$$

$$\begin{aligned} | \, [A'B' \cdot A'C'] &= (-1)^{(2n-p)(n-1)} \, [| A' \cdot | B' \cdot | A' \cdot | C'] \\ &= (-1)^{(2n+p+r+s)(n-1)} [AB \cdot AC] \\ &= [AB \cdot AC]. \end{aligned}$$

Therefore to find the expression for $[AB \cdot AC]$ when $p + r + s = 2n$ we shall find the complement of $[A'B' \cdot A'C']$

$$[A'B' \cdot A'C']_{l_1 \cdots l_{n-p}} = \frac{(-1)^{p(n-1)}}{(n-p)!^2 (n-r)!(n-s)!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p} \\ l_1 \cdots l_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \rho_1 & \cdots & \rho_{n-p} & \sigma_1 & \cdots & \sigma_{n-r} & \epsilon_1 & \cdots & \epsilon_{n-s} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix}$$

$$\times A'_{\alpha_1 \cdots \alpha_{n-p}} \; A'_{\rho_1 \cdots \rho_{n-p}} \; B'_{\sigma_1 \cdots \sigma_{n-r}} \; C'_{\epsilon_1 \cdots \epsilon_{n-s}} \qquad \text{(Case I)}$$

$$= \frac{(-1)^{p(n-1)}}{p!^2 r!s!(n-p)!^2 (n-r)!(n-s)!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p} \\ l_1 \cdots l_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \rho_1 & \cdots & \rho_{n-p} & \sigma_1 & \cdots & \sigma_{n-r} & \epsilon_1 & \cdots & \epsilon_{n-s} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \alpha_1 & \cdots & \alpha_{n-p} & \gamma_1 & \cdots & \gamma_p \end{bmatrix}.$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \rho_1 & \cdots & \rho_{n-p} & \mu_1 & \cdots & \mu_p \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \sigma_1 & \cdots & \sigma_{n-r} & \delta_1 & \cdots & \delta_r \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \epsilon_1 & \cdots & \epsilon_{n-s} & \nu_1 & \cdots & \nu_s \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

Taking the complement of this, we obtain

$$| \, [A'B' \cdot A'C'] = [AB \cdot AC]^{m_1 \cdots m_p}$$

$$= \frac{(-1)^{p(n-1)}}{p!^2 r!s!(n-p)!^3 (n-r)!(n-s)!} \begin{bmatrix} m_1 & \cdots & m_p & \lambda_1 & \cdots & \lambda_{n-p} \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p} \\ \lambda_1 \cdots \lambda_{n-p} \end{bmatrix} \begin{bmatrix} \rho_1 \cdots \rho_{n-p}\, \sigma_1 \cdots \sigma_{n-r}\, \epsilon_1 \cdots \epsilon_{n-s} \\ 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_1 \cdots \alpha_{n-p}\, \gamma_1 \cdots \gamma_p \end{bmatrix} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \rho_1 \cdots \rho_{n-p}\, \mu_1 \cdots \mu_p \end{bmatrix} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \sigma_1 \cdots \sigma_{n-r}\, \delta_1 \cdots \delta_r \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \epsilon_1 \cdots \epsilon_{n-s}\, \nu_1 \cdots \nu_s \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= \frac{1}{p!} \begin{bmatrix} m_1 \cdots m_p \\ \gamma_1 \cdots \gamma_p \end{bmatrix} A^{\gamma_1 \cdots \gamma_p} \cdot \frac{1}{r!s!(n-r)!(n-s)!}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \mu_1 \cdots \mu_{n-r}\, \delta_1 \cdots \delta_r \end{bmatrix} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \mu_{n-r+1} \cdots \mu_p\, \nu_1 \cdots \nu_s \end{bmatrix}$$

$$\times A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= A^{m_1 \cdots m_p} [\dot{A}BC]. \tag{13B}$$

In order to show that the above expression is equal to $[ABC]$ when $p + r + s = 2n$, we shall first find $[A'B'C']$ whose rank $= n$.

$$[A'B'C'] = \frac{1}{(n-r)!(n-p)!(n-s)!} \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p}\, \beta_1 \cdots \beta_{n-r}\, \rho_1 \cdots \rho_{n-s} \\ 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \end{bmatrix}$$

$$\times A'_{\alpha_1 \cdots \alpha_{n-p}} \; B'_{\beta_1 \cdots \beta_{n-r}} \; C'_{\rho_1 \cdots \rho_{n-s}}$$

But the complement of $[A'B'C']$ is $[ABC]$; i. e.

$$|\,[A'B'C'] = (-1)^{n(n-1)} [\,|A' \cdot |B' \cdot |C'] = [ABC]$$

$$\therefore \quad |\,[A'B'C'] = \frac{1}{n!(n-r)!(n-p)!(n-s)!p!r!s!} \begin{bmatrix} \sigma_1 \cdots \sigma_n \\ 1 \cdots n \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_1 \cdots \alpha_{n-p}\, \beta_1 \cdots \beta_{n-r}\, \rho_1 \cdots \rho_{n-s} \\ 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \end{bmatrix} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \alpha_1 \cdots \alpha_{n-p}\, \mu_1 \cdots \mu_p \end{bmatrix}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \beta_1 \cdots \beta_{n-r}\, \delta_1 \cdots \delta_r \end{bmatrix} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \rho_1 \cdots \rho_{n-s}\, \nu_1 \cdots \nu_s \end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

$$= \frac{1}{(n-r)!(n-s)!r!s!} \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \mu_1 \cdots \mu_{n-r}\, \delta_1 \cdots \delta_r \end{bmatrix} \tag{13C}$$

$$\times \begin{bmatrix} 1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad n \\ \mu_{n-r+1} \cdots \mu_p\, \nu_1 \cdots \nu_s \end{bmatrix} A^{\mu_1 \cdots \mu_p} B^{\delta_1 \cdots \delta_r} C^{\nu_1 \cdots \nu_s}$$

3. *Inner Product of Tensors.* In the study of simple vectors, we consider two varieties of products,—outer or vector products and inner or scalar products. So here, we form not only outer products of tensors, but also inner products.

The inner product of two "extensive Grösse" A, B, as defined by Grassmann, is the outer product of A into the complement of B; i.e. the inner product of A into B equals $[A \mid B]$. We note here that if A is of rank p and B is of rank r, and $p > r$, then the rank of $[A \mid B]$ is equal to $p - r$; but if $p < r$, then the rank of $[A \mid B]$ is $n + p - r$. In order to keep the two cases more nearly similar, we shall modify the above definition as follows:

If $p \geqq r$, the inner product of A into B is defined as $[A \mid B]$.
If $p < r$, the inner product of A into B is defined as $\mid [A \mid B]$.

The rank in the second case is now $r - p$.

We shall now proceed to derive the explicit expressions for the inner product of two tensors A and B, in the following cases:

I. The ranks of the two tensors are equal.
II. The rank of A is greater than rank of B.
III. The rank of A is less than rank of B.

Case I. A is a tensor of rank p, B is a tensor of rank r and $p = r$. Then the rank of $[A \cdot \mid B]$ is zero.

$$[A \mid B]^{s_1 \cdots s_n} = \frac{1}{p!p!(n-p)!} \begin{bmatrix} s_1 & \cdot & \cdot & \cdot & \cdot & s_n \\ \alpha_1 & \cdots & \alpha_p & \lambda_1 & \cdots & \lambda_{n-p} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda_1 & \cdots & \lambda_{n-p} & \beta_1 & \cdots & \beta_p \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_p}$$

$$= \frac{(-1)^{p(n-p)}}{p!\,p!} \begin{bmatrix} \beta_1 & \cdots & \beta_p \\ \alpha_1 & \cdots & \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \cdots \beta_p}$$

$$(14) \qquad = \frac{(-1)^{p(n-1)}}{p!} A^{\alpha_1 \cdots \alpha_p} B_{\alpha_1 \cdots \alpha_p}$$

Similarly

$$[B \mid A]_{s_1 \cdots s_n} = \frac{1}{p!\,p!(n-p)!} \begin{bmatrix} \beta_1 & \cdots & \beta_p & \mu_1 & \cdots & \mu_{n-p} \\ s_1 & \cdot & \cdot & \cdot & \cdot & s_n \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ \mu_1 & \cdots & \mu_{n-p} & \alpha_1 & \cdots & \alpha_p \end{bmatrix} B_{\beta_1 \cdots \beta_p} A^{\alpha_1 \cdots \alpha_p}$$

$$= \frac{(-1)^{p(n-1)}}{p!} A^{\alpha_1 \cdots \alpha_p} B_{\alpha_1 \cdots \alpha_p} .$$

Hence $[A \mid B] = [B \mid A]$ when $p = r$.

It is evident from (14) that Grassmann's definition of the inner product of two tensors is in accord with the usual method of finding the inner product of two tensors, that of multiplying scalarly the contravariant presentation of one into the covariant presentation of the other.

Case II. $p > r$.

By definition, the inner product of A into B is $[A \mid B]$. It is evident that the sum of the ranks of A and of $\mid B$ is $p + n - r$ which is greater than n, hence we must use formula 10 in writing the explicit expression for $[A \mid B]$. Thus

$$[A \mid B]^{m_1 \cdots m_{p-r}} = \frac{(-1)^{(p-r)(n-p)}}{(p-r)!(n-r)!r!} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & n \\ \alpha_{p-r+1} & \cdots & \alpha_p & \beta_1 & \cdots & & \beta_{n-r} \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} [\mid B]^{\beta_1 \cdots \beta_{n-r}}$$

$$(15) \qquad = \frac{(-1)^{(p-r)(n-p)+r(n-1)}}{(p-r)!r!^2} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & \gamma_r \\ \alpha_{p-r+1} & \cdots & \alpha_p \end{bmatrix}$$

$$\times A^{\alpha_1 \cdots \alpha_p} B_{\gamma_1 \ldots \gamma_r}.$$

Case III. $p < r$.

By definition, the inner product of A into B in this case is $\mid [A \cdot \mid B]$. The rank of this tensor is $r - p$.

$$[A \mid B]^{s_1 \cdots s_{n+p-r}} = \frac{1}{p!(n-r)!r!} \begin{bmatrix} s_1 & \cdot & \cdot & \cdot & & s_{n+p-r} \\ \alpha_1 & \cdots & \alpha_p & \mu_1 & \cdots & \mu_{n-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \mu_1 & \cdots & \mu_{n-r} & \beta_1 & \cdots & \beta_r \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \ldots \beta_r}.$$

Taking the complement of this, we obtain

$$\mid [A \mid B] = C_{l_1 \ldots l_{r-p}}$$

$$= \frac{1}{(n+p-r)!p!r!(n-r)!} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & n \\ l_1 & \cdots & l_{r-p} & \sigma_1 & \cdots & \sigma_{n+p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \sigma_1 & \cdot & \cdot & \cdot & & \sigma_{n+p-r} \\ \alpha_1 & \cdots & \alpha_p & \mu_1 & \cdots & \mu_{n-r} \end{bmatrix} \begin{bmatrix} \mu_1 & \cdots & \mu_{n-r} & \beta_1 & \cdots & \beta_r \\ 1 & \cdot & \cdot & \cdot & \cdot & n \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \ldots \beta_r}.$$

$$(16) \qquad = \frac{(-1)^{r(n-1)}}{(r-p)!p!^2} \begin{bmatrix} \beta_1 & \cdots & \beta_{r-p} \\ l_1 & \cdots & l_{r-p} \end{bmatrix} \begin{bmatrix} \beta_{r-p+1} & \cdots & \beta_r \\ \alpha_1 & \cdots & \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \ldots \beta_r}.$$

Let us now compare the inner product of A into B with that of B into A.

I. When $p > r$.

$$[A \mid B]^{m_1 \cdots m_{p-r}} = \frac{(-1)^{(p-r)(n-p)+r(n-1)}}{(p-r)!r!^2} \begin{bmatrix} m_1 \cdots m_{p-r} \\ \alpha_1 \cdots \alpha_{p-r} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta_1 & \cdots & \beta_r \\ \alpha_{p-r+1} & \cdots & \alpha_p \end{bmatrix} A^{\alpha_1 \cdots \alpha_p} B_{\beta_1 \ldots \beta_r}.$$

Now $[B \mid A]$ will fall under Case III, therefore by 16, we may write at once

$$[B \mid A]_{m_1 \ldots m_{p-r}} = \frac{(-1)^{p(n-1)}}{(p-r)!\,r!^2} \begin{bmatrix} \alpha_1 & \cdots & \alpha_{p-r} \\ m_1 & \cdots & m_{p-r} \end{bmatrix} \times \begin{bmatrix} \alpha_{p-r+1} & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_r \end{bmatrix} A_{\alpha_1 \ldots \alpha_p} \; B^{\beta_1 \ldots \beta_r}.$$

The two expressions are of different types, but otherwise, they are identical except for sign. However, if we multiply $[B \mid A]$ by $(-1)^{r(p-1)}$, the two are equal, since the exponents of (-1) are congruent, modulus 2.

II. When $p < r$, the same relation holds, since in either case, it results in a comparison of formulas 15 and 16. Hence the inner product of A into $B = (-1)^{r(p-1)}$ inner product of B into A, provided we do not distinguish between contravariant and covariant.

4. *Conclusion.* In the preceding pages, we have endeavored to give a modern presentation of Grassmann's *Die Ausdehnungslehre* with special emphasis on the development of the explicit expressions for various combinations of tensors.

Having defined, by means of the generalized Kronecker symbol, the outer product of tensors, the sum of whose rank $< n$, and also the complement of a tensor, we have derived the explicit expression for the outer product of tensors, the sum of whose ranks $> n$,

From these expressions as a basis, various relations between the outer products of tensors are readily derived.

We have also derived expressions for the inner product of two tensors, of ranks p and q, in the cases where $p = r$, $p > r$ and $p < r$.*

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* Since writing the above paper, we have read with interest a paper by C. L. E. Moore, "Grassmannian Geometry in Riemannian Space," *Journal of Mathematics and Physics* (Massachusetts Institute of Technology), Vol. V, No. 4 (June, 1926), which bears directly on this subject.